UNIQUE a-CLOSURE FOR SOME ℓ-GROUPS OF RATIONAL VALUED FUNCTIONS

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Abstract. Usually, an abelian ℓ-group, even an archimedean ℓ-group, has a relatively large infinity of distinct a-closures. Here, we find a reasonably large class with unique and perfectly describable a-closure, the class of archimedean ℓ-groups with weak unit which are "$\mathbb{Q}$-convex". ($\mathbb{Q}$ is the group of rationals.) Any $C(X,\mathbb{Q})$ is $\mathbb{Q}$-convex and its unique a-closure is the Alexandroff algebra of functions on X defined from the clopen sets; this is sometimes $C(X)$.

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Introduction

A *lattice-ordered group* (or *ℓ-group* for short) is a group $(G,+)$ with a partial order that is a lattice (infimum and supremum are denote by $\wedge$ and $\vee$, respectively) such that the ordering is compatible with the group operation. That is, for all $g,h,k\in G$ with $g\leqslant h$ we have $g+k\leqslant h+k$. The set of positive elements of G is written as G^+; note that the additive identity is an element of this set.

Elements $g,h\in G^+$ are *archimedean equivalent* (or *a-equivalent*), denoted $g\sim_a h$, if there exist natural numbers n, m for which $g\leqslant nh$ and $h\leqslant mg$. If G is an ℓ-subgroup of H then H is an *a-extension of G* if every positive element of H is a-equivalent to a positive element of G. We write $G\leqslant_a H$ in this case. The divisible hull of an abelian ℓ-group is an a-extension, for example. If G has no proper

a-extension, then G is *a-closed*. By Holland's Embedding Theorem, a-closures exist (see [7]); however, a-closures are not necessarily unique (see [4]).

Throughout, we use $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ to represent the naturals, rationals and reals, respectively.

Over the past 30 years, several researchers have sought a-closures in various classes of ℓ-groups. Recently, the authors of [6] sought a-closures via valuation mappings of an ℓ-group onto a distributive lattice. Also, in [14] the authors considered a class of ℓ-groups that generalizes the class of hyperarchimedean ℓ-groups (see also [5]) and determined the a-closures of these groups. In particular, they explicitly describe the a-closures of $C(X, \mathbb{Z})$, the ring of continuous integer-valued functions on X. In the present article we are interested in determining a-extensions and a-closures of certain more general objects in the category, $\mathbf{W}$, of archimedean ℓ-groups with weak unit.

In this section we introduce standard concepts needed throughout the paper.

The ℓ-group G is *archimedean* if whenever $0 \leqslant g \leqslant nh$ for all $n \in \mathbb{N}$, then $g = 0$. All archimedean ℓ-groups are necessarily abelian. This is explained in [7].

An element $u \in G^+$ is a *weak order unit* if $u \wedge g = 0$ implies $g = 0$. $\mathbf{W}$ denotes the category whose objects are the archimedean ℓ-groups with designated weak order unit and whose morphisms are the lattice-preserving group homomorphisms that also preserve the unit. (G, u) denotes an object in $\mathbf{W}$.

Recall that an ℓ-subgroup $K \leqslant G$ is *convex* if $0 \leqslant g \leqslant k \in K$ implies that $g \in K$. Let (G, u) be a $\mathbf{W}$-object. By Zorn's Lemma, there exist convex ℓ-subgroups of G that are maximal with respect to not containing u. We let YG denote the set of these. In the hull-kernel topology, YG is a compact Hausdorff space. Define

$$D(YG) = \{f\colon\ YG \to \mathbb{R} \cup \{\pm\infty\}\colon\ f \text{ is continuous and } f^{-1}\mathbb{R} \subseteq YG \text{ is dense}\}.$$

Though $D(YG)$ is rarely a group under pointwise addition, it is known that G may be mapped bijectively, via an ℓ-group isomorphism, onto an ℓ-group $\hat{G}$ of $D(YG)$, which maps u to the constant function 1 and so that the elements of $\hat{G}$ separate the points of YG. This representation is unique: If $G \cong \tilde{G} \leqslant D(X)$ is an ℓ-isomorphism with X compact Hausdorff and $\tilde{u} = 1$, then there is a continuous surjection $\tau\colon\ X \to YG$ such that $\tilde{g} = \hat{g} \circ \tau$ for each $g \in G$; moreover, $\tilde{G}$ separates the points of X if and only if τ is a homeomorphism. We identify G with its image $\hat{G}$. This representation is the "Yosida Embedding" (see [21] and [16]) and YG is called the *Yosida space* of G.

We now turn to topological considerations and to $C(X)$, the ℓ-group of real-valued continuous functions on the space X with the pointwise ordering. See [9] for details.

We assume that all spaces are Tychonoff, that is, completely regular and Hausdorff. βX denotes the Stone-Čech compactification of X, and we note that the Yosida space of $C(X)$ is homeomorphic to βX. $C^*(X)$ is the ℓ-subgroup containing the bounded elements of $C(X)$. There is a natural isomorphism between $C^*(X)$ and $C(\beta X)$, given by extension (and inversely, restriction) of functions to βX (inversely, to X). Whenever $C(X) = C^*(X)$, we call X *pseudocompact*.

Recall that a space is called *zero-dimensional* if it has a base of clopen sets and that every zero-dimensional space has a maximal zero-dimensional compactification called the *Banaschewski compactification* (see [20]) denoted by $\beta_0 X$. The space $\beta_0 X$ is homeomorphic to the Yosida spaces of $C(X, \mathbb{Z})$ and $C(X, \mathbb{Q})$ and the map β_0 is the compact zero-dimensional reflection. When $\beta X = \beta_0 X$, the space βX is zero-dimensional and we call X *strongly zero-dimensional*.

2. Unique a-closure and convex ℓ-groups

Let (G, u) be in $\mathbf{W}$ and $g \in G$. The *zeroset* of g is $Z(g) = \{p \in YG\colon\ g(p) = 0\}$ and the *cozeroset* of g is $YG \setminus Z(g)$. We use $\mathscr{Z}G$ to denote the set of all zerosets of G.

Theorem 2.1. *Let (G, u) be in $\mathbf{W}$. If $G \leqslant_a H$ then G majorizes H (that is, for every $h \in H^+$ there exists $g \in G^+$ such that $h \leqslant g$); u is a weak unit in H, $Y(G, u) = Y(H, u)$ and in the Yosida representation $G \leqslant H \leqslant D(Y(G, u))$ and $\mathscr{Z}H = \mathscr{Z}G$.*

Proof. Let (G, u) be in $\mathbf{W}$ and assume that $G \leqslant_a H$. That G majorizes H follows directly from the definition of a-extension. If there is $h \in H^+$ such that $u \wedge h = 0$, then for any $g \in G$ such that $g \sim_a h$, we have that $u \wedge g = 0$. Hence $g = 0$ and $0 \leqslant h \leqslant mg = 0$ for some m and therefore, $h = 0$. It follows from Theorem 2.1 of [4] that $Y(G, u) = Y(H, u)$; hence, $G \leqslant H \leqslant D(Y(G, u))$ and $\mathscr{Z}H = \mathscr{Z}G$. □

For $g \in G$, let $g^+ = g \vee 0$ and $g^- = (-g) \vee 0$. Then $g = g^+ - g^-$ and we define $|g| = g^+ + g^-$.

Definition 2.2. Let (G, u) be in $\mathbf{W}$.

(a) $G^c = \{f \in D(YG)\colon\ |f| \leqslant g \text{ for some } g \in G\}$.

(b) From [2]: G is *convex* if $G = G^c$.

G^c is usually not an ℓ-group, as we discuss shortly.

Corollary 2.3. *In* $\mathbf{W}$*:*

(a) *If $G \leqslant_a H$ then $H \subseteq G^c$.*

(b) *If G is convex, then G is a-closed.*

(c) *If G^c is an ℓ-group and if $G \leqslant_a G^c$ then G^c is the unique a-closure of G.*

(d) *If H is convex and $G \leqslant_a H$, then H is the unique a-closure of G.*

P r o o f. It is clear that Theorem 2.1 implies statements (a) and (b) which together imply (c). To verify (d), note that $G \leqslant_a H$ implies $H \subseteq G^c$ by (a). But also, $G^c \subseteq H^c = H$. Thus, $G^c = H$, and (c) applies. □

The statement of Corollary 2.3 (c) and (d) present us with the following two versions of the same questions, which the sequel examines.

Question 2.4. Let G be an archimedean ℓ-group.
1. (a) For which G is G^c an ℓ-group?
   (b) For which G is G^c an ℓ-group and $G \leqslant_a G^c$?
2. For convex H, what **W**-subobjects G have $G \leqslant_a H$?

The following compendium from the literature illustrates what the class of convex ℓ-groups encompasses. Recall that an *f-ring* is a subdirect product of totally ordered rings, [3].

Theorem 2.5. *For the following classes of* **W**-*objects, for each n, the class (n) is contained the class $(n+1)$.*
(1) *Rings of continuous functions, $C(X)$.*
(2) *Alexandroff algebras: ℓ-subalgebras of $\mathbb{R}^X$ containing 1 that are closed under uniform convergence and inversion (see § 5 below).*
(3) **W**-*objects closed under countable composition.*
(4) *Archimedean f-rings with identity, that are divisible and uniformly complete.*
(5) *Convex* **W**-*objects.*

P r o o f. That (1) $\subseteq$ (2) is clear; (2) $\subseteq$ (3) $\subseteq$ (4) can be found in [18]; and (4) $\subseteq$ (5) is in [17]. (One has to recognize that the representation in [17] and [18] of an f-algebra is the Yosida representation of the underlying **W**-object). □

As a class of study, "convex" was introduced in [2], and there shown to be monoreflective in **W**: for each (G,u) there is a group cG such that $G \leqslant cG$ with cG convex such that each $\varphi\colon G \to H$ in **W** with H convex has a unique extension $c\varphi\colon cG \to H$ in **W**. Usually, YcG is much larger than YG, but it is easy to see that if G^c is an ℓ-group then $G^c = cG$.

Remark 2.6. (a) Recall that $V \in YG$ is *real* if $G/V \hookrightarrow \mathbb{R}$ and $\mathscr{R}G \subseteq YG$ denotes the set of all such points. Let $G|_{\mathscr{R}G} = \{g|_{\mathscr{R}G}\colon g \in G\}$. In Theorem 2.1 and Corollary 2.3 (a), suppose that $\bigcap \mathscr{R}G = (0)$, so that $G|_{\mathscr{R}G} \subseteq C(\mathscr{R}G)$ is a representation of G; then $G^c|_{\mathscr{R}G} \subseteq C(\mathscr{R}G)$ also and $G \leqslant_a H$ implies that $H \subseteq C(\mathscr{R}G)$. Within the category **W**, this sharpens an observation in Example 6.2 of [4].

(b) By Theorem 2.5, $C(X)$ is convex for any X. Here's another proof: The Yosida embedding of $C(X)$ is given by $\{\beta f \in D(\beta X)\colon f \in C(X)\}$, therefore, $C(X)^c =$

$C(X)$. Thus, by Corollary 2.3 (b), $C(X)$ is convex. This improves Example 6.2 of [4] in which Conrad shows that $C(X)$ is a-closed.

(c) If G is hyperarchimedean, then the converse of Theorem 2.1 holds (see [13]), but the converse fails in general. Let $\alpha\mathbb{N}$ be the one-point compactification of $\mathbb{N}$ and let $G \leqslant C(\alpha\mathbb{N})$ be given by $g \in G$ if and only if there exist $r, s \in \mathbb{R}$ such that eventually $g(n) = r + s/n$. Then $\mathscr{X}G = \mathscr{X}C(\alpha\mathbb{N})$, though G is not a-extended by $C(\alpha\mathbb{N})$ since $f(n) = \mathrm{e}^{-1/n} \in C(\alpha\mathbb{N})$ has no a-equivalent element in G.

3. Relatively convex ℓ-groups

Definition 3.1. Let A be a subgroup of $\mathbb{R}$ containing 1 and (G, u) in $\mathbf{W}$.

(a) For a compact Hausdorff space X, let

$$D_A(X) = \{f \in D(X) \colon\ f(p) \in \mathbb{R} \Rightarrow f(p) \in A\}.$$

(b) G is *A-convex* if for $f \in D_A(YG)$, $|f| \leqslant g \in G$ implies $f \in G$. When $A \neq \mathbb{R}$, we assume that YG is zero-dimensional.

(c) $W_A G = G \cap D_A(YG)$.

Note that an A-convex group is Z-convex. In fact, we are really only interested in Z- and $\mathbb{Q}$-convex objects.

In this section, we show that G is A-convex if and only if G^c is a convex ℓ-group for which $YG^c = YG$ is zero-dimensional and

$$W_A G = W_A G^c \leqslant G \leqslant G^c.$$

This relates the two queries in Question 2.4 and addresses Question 2.4.1 (a). We also note the rarity of $W_Z G \leqslant_a G$.

In the next section we show that $W_{\mathbb{Q}} G \leqslant_a G$ for convex groups G. Thus, $\mathbb{Q}$-convex is the answer to Question 2.4.

Remark 3.2. The operator W_Z is studied in [15], there denoted $\mathbf{W}_s$. It is a coreflection of $\mathbf{W}$ onto the full subcategory whose objects satisfy $G = W_Z G$ (called *singular*). The situation with W_A is analogous, but we won't pursue that here. Note that Z-convexity is an extension of the *singularly convex* condition in [14].

Proposition 3.3. *G^c is an ℓ-group (hence it is convex and $YG^c = YG$) if and only if $\beta g^{-1}\mathbb{R} = YG$ for each $g \in G$.*

P r o o f. $\Rightarrow$: Suppose that $g^{-1}\mathbb{R}$ is not C^*-embedded (without loss of generality, we may take $g \in G^+$), say $f \in C^*(g^{-1}\mathbb{R})$ fails to extend over YG. Choose $m \geqslant |h|$ and define $h(x) = f(x) + g(x)$ if $x \in g^{-1}\mathbb{R}$ and $f(x) = +\infty$ if $x \notin g^{-1}\mathbb{R}$. Then $|h| \leqslant g + m \in G$, so that $h \in G^c$. But $h - g \notin D(YG)$, so G^c is not closed under addition.

$\Leftarrow$: The lattice operations are inherited from $D(YG)$. Suppose that $f_i \in D(YG)$ with $|f_i| \leqslant g_i \in G^+$ for $i = 1, 2$. Then $f_i^{-1}\mathbb{R} \supseteq g_i^{-1}\mathbb{R}$ so that

$$f_1 + f_2 \in C(g_1^{-1}\mathbb{R} \cap g_2^{-1}\mathbb{R}).$$

Since $g_1^{-1}\mathbb{R} \cap g_2^{-1}\mathbb{R} = (|g_1| + |g_2|)^{-1}\mathbb{R}$ and we assume that this set is C^*-embedded, we have the extension to $h \in D(YG)$ and $|h| \leqslant g_1 + g_2$ (since that holds on the dense set $g_1^{-1}\mathbb{R} \cap g_2^{-1}\mathbb{R}$). Thus $h \in G^c$ and $h = f_1 + f_2$ in G^c. □

Proposition 3.4. *Suppose that G is Z-convex.*

(a) *If $g \in G^+$ and there is $0 < r \in \mathbb{R}$ such that $g(x) > 0$ implies $g(x) \geqslant r$, then there is $f \in W_Z G$ such that $f \sim_a g$.*

(b) *For all $g \in G$, there exists $f \in W_Z G$ such that $f^{-1}\mathbb{R} = g^{-1}\mathbb{R}$.*

(c) *For all $g \in G$, $\beta g^{-1}\mathbb{R} = YG$.*

(d) *G^c is a convex ℓ-group with $YG^c = YG$.*

P r o o f. The definition of Z-convex includes the assumption that YG is zero-dimensional, so any $g^{-1}\mathbb{R}$ is zero-dimensional and Lindelöf, thus strongly zero-dimensional. See [9] and [20].

(a) Without loss of generality, $r \geqslant 3$. For every $n \geqslant 3$, choose a clopen set U_n with $g^{-1}[n-1, n+1] \subseteq U_n \subseteq g^{-1}(n-2, n+2)$ so that

$$n - 2 \leqslant \bigwedge_n g|_{U_n} \leqslant \bigvee_n g|_{U_n} \leqslant n + 2.$$

Let $V_n = U_n \setminus \bigcup\limits_{j<n} U_j$. Then the functions $g|_{V_n}$ retain the preceding inequalities and $g^{-1}\mathbb{R} = Z(g) \bigsqcup_n V_n$. Clearly, this set is open.

Now define $f \in D_Z(YG)$ by $f|_{V_n} = n - 2$, $f|_{YG - g^{-1}\mathbb{R}} = +\infty$ and $f|_{Z(g)} = 0$. So then $f \leqslant g$ on $g^{-1}\mathbb{R}$ and hence, $f \leqslant g$. Also

$$g|_{V_n} \leqslant n + 2 = (n-2) + 4 = f|_{V_n} + 4.$$

Then $g \leqslant f + 4 \leqslant 5f$, since $1 \leqslant f$.

(b) Apply (a) to $|g| \vee 3$ to get f.

(c) Since $g^{-1}\mathbb{R}$ is strongly zero-dimensional, it suffices to demonstrate that any $h \in C(g^{-1}\mathbb{R}, \{0,1\})^+$ extends over YG. See [9] and [20]. By (b), we can assume that $g \in W_Z G$. Define $f \in D_Z(YG)$ by $f(x) = g(x) + h(x)$ if $x \in g^{-1}\mathbb{R}$ and $f(x) = +\infty$, otherwise. Then $f \leqslant g + 2 \in G$. Since G is Z-convex, $f \in G$. Thus, $g - f \in G$ and this is the desired extension of h.

(d) By (c) and Proposition 3.3. □

Theorem 3.5. *Let A be a proper subgroup of $\mathbb{R}$ containing* 1.

(a) *If H is convex with YH zero-dimensional, then $W_A H$ is A-convex and $H = (W_A H)^c$.*

(b) *If G is A-convex, then G^c is a convex ℓ-group with YG^c zero-dimensional and $W_A(G^c) \leqslant G$.*

Proof. (a) If YH is zero-dimensional, then $C(YH, Z)$ separates points of YH. Since $C(YH, Z) \leqslant W_Z H \leqslant W_A H$, the group $W_Z H$ also separates points of YH and thus $YW_A H = YH$. Now suppose that H is convex and $f \in D_A(YW_A H)$, such that $|f| \leqslant g \in W_A H$ for some g. Then $f \in D(YH)$ and $|f| \leqslant g \in H$. Since H is convex, $f \in H$. Since also $f \in D_A(YH)$, we have that $f \in W_A H$ and, hence, $W_A H$ is convex.

We know that $W_A H \subseteq H$ and so $(W_A H)^c \subseteq H$ since H is convex. For the reverse, $H^+ \subseteq (W_Z H)^c$ by Proposition 3.4 (a); so $H \subseteq (W_Z H)^c$ since the larger set is an ℓ-group by the above and by Proposition 3.4 (d). Since we have the containment $(W_Z H)^c \subseteq (W_A H)^c$, the proof is complete.

(b) Assume that G is A-convex. Since $Z \leqslant A$, G is Z-convex, so Proposition 3.4 (d) applies. Let $f \in W_A G^c$, that is, $f \in D_A(YG^c)$ and $|f| \leqslant g \in G^c$. Thus, $|f| \leqslant g \leqslant g' \in G$. Since G is A-convex, $f \in G$. □

Corollary 3.6. *Let A be a proper subgroup of $\mathbb{R}$ containing* 1.

(a) *The following are equivalent:*
  - (a_1) *H is convex with YH zero-dimensional.*
  - (a_2) *$H = G^c$ for some A-convex G.*
  - (a_3) *$H = G^c$ for a unique A-convex G with $G = W_A G$, namely $G = W_A H$.*

(b) *The following are equivalent:*
  - (b_1) *G is A-convex.*
  - (b_2) *$W_A H \leqslant G \leqslant H$ for some convex H with YH zero-dimensional; such an H is unique, namely $H = G^c$.*

Proof. (a_3) ⇒ (a_2) is clear and (a_2) ⇒ (a_1) by Theorem 3.5 (b).

(a_1) $\Rightarrow$ (a_3): We know that $H = (W_A H)^c$ by Theorem 3.5 (a). If also $H = G^c$ for some A-convex $G = W_A G$, then

$$W_A H = W_A(G^c) \leqslant G \leqslant W_A G \leqslant W_A(G^c),$$

using Theorem 3.5 (b) and the fact that $G \leqslant G^c$ implies that $W_A G \leqslant W_A G^c$.

(b_1) $\Rightarrow$ (b_2): Assume that G is A-convex. By Theorem 3.5 (a), if H satisfies (b_2) then $H = G^c$ and by Theorem 3.5 (b) G^c does satisfy (b_2).

(b_2) $\Rightarrow$ (b_1): Suppose that G and H satisfy (b_2). Then $YG = YH$ and if $f \in D_A(YG)$ with $|f| \leqslant g \in G$ then $f \in W_A H$ so $f \in G$. Thus, G is A-convex. □

Remark 3.7. (a) Proposition 3.3 is the content of Remark 2.6 (e) in [2], where no proof was given.

(b) Proposition 3.4 is related to a lemma in [2].

(c) A **W**-object (G, u) for which every $g \in G^+$ satisfies the hypothesis of Proposition 3.4 (a) is called *bounded away.* So we have shown that when G is Z-convex and bounded away, $W_Z G \leqslant_a G$. This is closely related to Corollary 4.5 of [14].

In Proposition 3.4 (a), the bounded away condition can not be dropped: Let X be a compact and zero-dimensional space, then $C(X)$ is Z-convex. However, $W_Z C(X) = C(X, Z)$ and $C(X, Z) \leqslant_a C(X)$ if and only if X is finite. (See [13].)

(d) In fact, for H convex, $W_Z H \leqslant_a H$ if and only if YH is finite (whence $H \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$): sufficiency is easy to show, so let's show necessity. If H is convex, then $H^* = C(YH)$ and if $W_Z H \leqslant_a H$, then

$$C(YH, Z) = W_Z H^* = (W_Z H)^* \leqslant_a H^* = C(YH)$$

and we have the situation of the above. So YH is finite.

(e) Proposition 3.4 shows that Z-convex answers Question 2.4.1 (a), while Corollary 3.6 and Remark (d) above show that Z-convex fails to answer Question 2.4.1 (b), equivalently, the condition $W_Z H = G$ fails to answer Question 2.4.2.

4. The main theorem

We now replace Z by $\mathbb{Q}$.

Theorem 4.1. *In* **W***:*

(a) *If H is convex with YH zero-dimensional, then $W_{\mathbb{Q}}H \leqslant_a H$ and H is the unique a-closure of $W_{\mathbb{Q}}H$.*

(b) *If G is $\mathbb{Q}$-convex, then $G \leqslant_a G^c$, so G^c is the unique a-closure of G.*

(c) *If H is $\mathbb{Q}$-convex, then $W_{\mathbb{Q}}H \leqslant_a H$.*

Proof. By § 3, (a) and (b) two are the same statement, so we prove (a). Statement (c) is a direct consequence of (a) and (b).

Let H be convex with YH zero-dimensional and $h \in H^+$. Choose a clopen set $U \subseteq YH$ with $h^{-1}[0, \frac{1}{2}] \subseteq U \subseteq h^{-1}[0, 1)$. Let $h_1(p) = h(p)$ if $p \in U$, $h_1(p) = 0$ otherwise and let $h_2(p) = h(p)$ if $p \notin U$ and $h_2(p) = 0$ if $p \in U$. Since U is clopen, $h_1, h_2 \in D(YH)$ and since $0 \leqslant h_1, h_2 \leqslant h$, and H is convex, $h_1, h_2 \in H$. It suffices to find $g_1, g_2 \in W_{\mathbb{Q}}H^+$ with $g_i \sim_a h_i$ when $i = 1, 2$ and then $g_1 + g_2 \sim_a h_1 + h_2 = h$.

Now $h_2(p) > 0$ implies that $h_2(p) \geqslant \frac{1}{2}$. So by Proposition 3.4 (a), there is $g_2 \in W_Z H$ with $g_2 \sim_a h_2$.

For $i = 1$: since H is convex, $H^* = C(YH)$. We finish by using the following Lemma (with $f = h_1$). □

Lemma 4.2. *If X is compact and zero-dimensional and $f \in C(X)$ such that $0 \leqslant f \leqslant 1$, then there is $g \in C(X, \mathbb{Q})$ with $g \sim_a f$.*

Proof. By induction, choose clopen sets $K_0 \supseteq K_2 \supseteq \ldots$ as follows: $K_0 = X$ and for each n,

$$f^{-1}[0, 1/2^{n+1}] \subseteq K_{n+1} \subseteq K_n \cap f^{-1}[0, 1/2^n).$$

Then we see that $Z(f) = \bigcap_n K_n$,

$$1/2^{n+1} \leqslant f|_{K_n \setminus K_{n+1}} \leqslant 1/2^{n-1}$$

and $\operatorname{coz}(f) = \bigcup_n (K_n \setminus K_{n+1})$. Define $g \in C(X, \mathbb{Q})$ by $g(x) = 0$ when $x \in Z(f)$ and $g(x) = 1/2^{n+1}$ when $x \in K_n \setminus K_{n+1}$. Then $g \leqslant f$ and $f \leqslant 4g$. Thus, $g \sim_a f$. □

5. Alexandroff algebras and $C(X, \mathbb{Q})$

Throughout, we assume that X is zero-dimensional; otherwise, $C(X, \mathbb{Q})$ may be too small.

Theorem 5.1. *Suppose X is zero-dimensional.*

(a) *Each $g \in C(X,\mathbb{Q})$ has an extension $\hat{g} \in D(\beta_0 X)$, and $\{\hat{g}\colon\ g \in C(X,\mathbb{Q})\}$ is the Yosida representation. In particular, $YC(X,\mathbb{Q}) = \beta_0 X$.*

(b) *$C(X,\mathbb{Q})$ is $\mathbb{Q}$-convex and so has a unique a-closure $C(X,\mathbb{Q})^c$.*

(c) *$W_{\mathbb{Q}}C(X) = W_{\mathbb{Q}}C(X,\mathbb{Q})$ and $W_{\mathbb{Q}}C(X) \leqslant_a C(X,\mathbb{Q})$.*

(d) *$W_{\mathbb{Q}}C(X) = C(X,\mathbb{Q})$ if and only if X is pseudocompact.*

P r o o f. (a) Consider the commutative diagram of continuous functions:

$$\begin{array}{ccc}
X & \hookrightarrow & \beta_0 X \\
\downarrow g & & \downarrow \beta_0 g \\
\mathbb{Q} & \hookrightarrow & \beta_0\mathbb{Q} = \beta\mathbb{Q} \\
\downarrow & & \downarrow f \\
\mathbb{R} & \hookrightarrow & \mathbb{R} \cup \{\pm\infty\}
\end{array}$$

in which $\beta_0 g$ exists with $\beta_0 g|_X = g$, because β_0 is the reflection functor to compact zero-dimensional spaces. Since $\mathbb{Q}$ is strongly zero-dimensional, we have that $\beta_0\mathbb{Q} = \beta\mathbb{Q}$. Then f is the extension of the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R} \subseteq \mathbb{R}\cup\{\pm\infty\}$, and $\hat{g} = f \circ \beta_0 g \in D(\beta_0 X)$.

We have $C(X,\mathbb{Q}) \supseteq C^*(X,Z) \cong C(\beta_0 X, Z)$, and the last separates the points of $\beta_0 X$, thus so does $\{\hat{g}\colon\ g \in C(X,\mathbb{Q})\}$ hence this is the Yosida representation.

(b) Let $f \in D_{\mathbb{Q}}(\beta_0 X)$ and $|f| \leqslant \hat{g}$, where $g \in C(X,\mathbb{Q})$. Then $f|_X \in C(X,\mathbb{Q})$ and $\widehat{f|_X} = f$. Thus, $C(X,\mathbb{Q})$ is $\mathbb{Q}$-convex. Then $C(X,\mathbb{Q})^c$ is the unique a-closure by Theorem 4.1.

(c) Since $C(X,\mathbb{Q}) \leqslant C(X)$ we have $W_{\mathbb{Q}}C(X,\mathbb{Q}) \leqslant W_{\mathbb{Q}}C(X)$. For the reverse, let $f \in W_{\mathbb{Q}}C(X)$. This means that $f = \beta g$ for $f|_X = g \in C(X)$ and for $p \in \beta X$, whenever $f(p) \in \mathbb{R}$ necessarily means that $f(p) \in \mathbb{Q}$. Thus $g \in C(X,\mathbb{Q})$. We have $f = \beta g = \hat{g}\circ\varphi$, where $\varphi\colon\ \beta X \to \beta_0 X$ is the canonical map. Then whenever $\hat{g}(q) \in \mathbb{R}$, we necessarily have that $\hat{g}(q) \in \mathbb{Q}$ for all $q \in \beta_0 X$.

That $W_{\mathbb{Q}}C(X) \leqslant_a C(X,\mathbb{Q})$ follows from (b) and Theorem 4.1.

(d) By (c), having $W_{\mathbb{Q}}C(X) = C(X,\mathbb{Q})$ is equivalent to having the inclusion $W_{\mathbb{Q}}C(X) \supseteq C(X,\mathbb{Q})$, which means that for $f \in D(\beta_0 X)$, $f(X) \subseteq \mathbb{Q}$ implies $f|_{f^{-1}\mathbb{R}} \subseteq \mathbb{Q}$.

Suppose that X is pseudocompact, $f \in D(\beta_0 X)$ and $f(X) \subseteq \mathbb{Q}$. Then $f(X)$ is a pseudocompact subset of $\mathbb{Q}$, hence compact, so $f^{-1}\mathbb{R} = \beta_0 X$ and $f(\beta_0 X) = f(X) \subseteq \mathbb{Q}$.

Suppose that X is not pseudocompact. Then, since X is zero-dimensional, $X = \bigcup\limits_n U_n$ for nonempty pairwise disjoint clopen sets U_n. Let $x_n \to r$ in $\mathbb{R}$ with $x_n \in \mathbb{Q}$

and $r \notin \mathbb{Q}$, and define $g \in C(X, \mathbb{Q})$ by $g|_{U_n} = x_n$. Then the extension $\hat{g} \in D(\beta_0 X)$ must have $\hat{g}(p) = r$ for some p therefore $g \notin W_{\mathbb{Q}} C(X)$. □

Corollary 5.2. *$C(X, \mathbb{Q})^c = C(X)$ (equivalently, $C(X, \mathbb{Q}) \leqslant_a C(X)$) if and only if X is strongly zero-dimensional.*

We now describe $C(X, \mathbb{Q})^c$, in general.

If X is zero-dimensional, let $\operatorname{clop}(X)$ be the Boolean algebra of clopen sets of X Then for $U \in \operatorname{clop}(X)$, the map $U \mapsto \operatorname{cl} U \in \operatorname{clop}(\beta_0 X)$ is its Stone representation.

Define $(\operatorname{clop}(X))_\sigma = \{\bigcup_n U_n \colon U_n \in \operatorname{clop}(X)\}$. Clearly, $(\operatorname{clop}(X))_\sigma \subseteq \operatorname{coz}(X)$, with equality if and only if X is strongly zero-dimensional. In fact,

$$(\operatorname{clop}(X))_\sigma = \{K \cap X \colon K \in \operatorname{coz}(\beta_0 X)\}.$$

Define $A(X) = \{f \in \mathbb{R}^X \colon f^{-1}K \in (\operatorname{clop}(X))_\sigma$ for $K \subseteq \mathbb{R}$ open$\}$. Then $A(X)$ is a **W**-object and $A(X) \leqslant C(X)$ with equality if and only if X is strongly zero-dimensional. See § 7 of [10] for a discussion.

Theorem 5.3.

(a) *$A(X)$ is of the type in Theorem 2.5. (2), thus is convex.*

(b) *$C(X, \mathbb{Q}) \leqslant A(X)$ and for each $f \in A(X)$ there is a sequence of functions $\{g_n\}_{n=1}^\infty \in C(X, \mathbb{Q})$ such that $g_n \to f$ uniformly on X.*

(c) *Each $f \in A(X)$ has an extension $\hat{f} \in D(\beta_0 X)$ and $\{\hat{f} \colon f \in A(X)\}$ is the Yosida representation. In particular, $Y A(X) = \beta_0 X$.*

(d) *$W_{\mathbb{Q}} A(X) = W_{\mathbb{Q}} C(X) \leqslant C(X, \mathbb{Q}) \leqslant A(X)$.*

(e) *$A(X) = C(X, \mathbb{Q})^c$, that is, $A(X)$ is the unique a-closure of $C(X, \mathbb{Q})$.*

P r o o f. (a) This is easily verified, or one may see § 7 of [10].

(b) Let $g \in C(X, \mathbb{Q})$ and let A be an open set in $\mathbb{R}$. Since $\mathbb{Q}$ is strongly zero-dimensional, $A \cap \mathbb{Q} = \bigcup_n U_n$ for clopen sets $U_n \in \mathbb{Q}$. Thus, we can write $g^{-1}A = \bigcup_n g^{-1}U_n \in (\operatorname{clop}(X))_\sigma$.

Let $f \in A(X)$ and $\varepsilon > 0$. Let $\mathscr{A}$ be a countable cover of $\mathbb{R}$ by open intervals of length less than ε. So, for $A \in \mathscr{A}$, $f^{-1}A = \bigcup_n U(n, A)$ for clopen $U(n, A)$ and $\mathscr{U} = \{U(n, A) \colon A \in \mathscr{A}, n \in \mathbb{N}, U(n, A) \neq \emptyset\}$ is a countable cover of X by clopen sets. We re-index the sets as $\mathscr{U} = \{U_n\}$ and disjointify: $V_n = U_n \setminus \bigcup_{i<n} U_i$. Let $\mathscr{V} = \{V_n\}_n$.

For each $A \in \mathscr{A}$, choose $r_A \in A \cap \mathbb{Q}$. Let $g = \sum_n \{r_A \chi_{V_n} \colon V_n \in \mathscr{V}\}$, where χ_{V_n} is the characteristic function of V_n. Then $g \in C(X, \mathbb{Q})$ and $|g(x) - f(x)| < \varepsilon$ for each $x \in X$.

(c) The extensions $\hat{f}$ exist by Theorem 5.1 (a) and the fact that a uniform limit of extendible functions is extendible. These extensions separate the points, since the extensions $\hat{g}$ for $g \in C(X, \mathbb{Q})$ do. The rest follows from this.

(d) This follows from Theorem 5.1 (c) and from $C(X, \mathbb{Q}) \leqslant A(X) \leqslant C(X)$.

(e) By (a), (d) and § 4. □

Remark 5.4. (a) Theorem 5.3 (a), (b), and (c) are implicit in § 7 of [11].

(b) From a more general perspective, $(\mathrm{clop}(X))_\sigma$ is an example of what is called a *cozero field*, $A(X)$ is its associated *Alexandroff algebra*, and Theorem 2.5 (2) is a characterization of such things. One may see [10], [11], [12] and the original references therein to Hausdorff, Lebesgue and A. D. Alexandroff.

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