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Topology and its Applications 116 (2001) 185–198

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

When the maximum ring of quotients of $C(X)$ is uniformly complete

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Received 27 January 1999; received in revised form 21 March 2001

Abstract

A Tychonoff space X such that the maximum ring of quotients of $C(X)$ is uniformly complete is called a uniform quotients space. It is shown that this condition is equivalent to the Dedekind–MacNeille completion of $C(X)$ being a ring of quotients of $C(X)$, in the sense of Utumi. A compact metric space is a uniform quotients space precisely when it has a dense set of isolated points. Extremally disconnected spaces and almost P-spaces are uniform quotients spaces. Also characterized are the compact spaces of dense constancies which are uniform quotients spaces.

AMS classification: Primary 54H10; 06F25; 13B30, Secondary 54D35; 54G05

Keywords: Maximum ring of quotients; Maximum domain; Oscillation; Essential hull; Co-σ-boundary; E.d. point

1. Introduction

During recent work on rings of quotients of rings of functions, we became interested in when the maximum ring of quotients of the ring $C(X)$ of all continuous real-valued functions on a topological space X is complete relative to uniform convergence. It emerged that this situation has been investigated by Veksler [17]; his Theorems 2 and 4 are very similar to our main theorem, Theorem 2.6. We recognize Veksler's earlier accomplishment, although, as far as we are aware, no proofs of the results in [17] have appeared. In any event, this paper covers new ground.

All spaces considered in this article are Tychonoff. Recall that a topological space X is said to be *Tychonoff* if it is Hausdorff and for each closed set K and each $p \in X$ not in K,

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0166-8641/01/$ – see front matter
PII: S0166-8641(01)00128-6

there is a continuous real-valued function f such that $f(p) = 1$ and $f(K) = \{0\}$. For all undefined terms involving rings of continuous functions, we refer the reader to [9]. [19] will be our reference for general topology.

We shall be dealing with rings of quotients of $C(X)$. "Ring of quotient" here means "Utumi ring of quotient". For basic facts concerning rings of quotients we recommend [11] and for rings of quotients of rings of functions [8]. All rings arising in this article are assumed to be commutative and to possess an identity. We shall refer to a certain amount of language and terminology involving lattice-ordered groups and f-rings; for background on these subjects the reader is referred to [4].

Lastly in this introduction, we wish to thank the referee, who helped us improve the presentation in Section 2, by pointing out how to streamline the exposition.

2. Characterizations of uniform quotients spaces

We begin with our basic definition:

Definition 2.1. The space X is a *uniform quotients space* if the complete ring of quotients $Q(X)$ of $C(X)$ is uniformly complete.

Now some comments to clarify terminology.

Definition and Remarks 2.2. (a) As explained in [8], $Q(X)$ can be obtained by forming the following direct limit. Let $\mathcal{G}(X)$ denote the family of all dense open sets of X. We consider all rings $C(U)$ ranging over all $U \in \mathcal{G}(X)$. These rings are assembled into a direct limit by considering the restriction maps $C(U_1) \to C(U_2)$ (with $U_2 \subseteq U_1$) as bonding maps. Then

$$Q(X) = \varinjlim C(U) \quad (U \in \mathcal{G}(X)).$$

Of some importance for our purposes is the companion notion of the essential hull, or *metric completion*, as it is referred to in [8]. Consider now the family $\mathcal{G}_\delta(X)$ of all dense $\mathcal{G}_\delta$-sets of X. Let βX denote the Stone–Čech compactification of X. Put

$$C(X)^e \equiv \varinjlim C(U) \quad (U \in \mathcal{G}_\delta(\beta X)),$$

again with restriction maps as bonding maps. As is shown in [8], Theorem 4.6, $C(X)^e$ is the uniform completion of $Q(X)$.

As to the concept of the essential hull, we review the work already in [5]. Assume here that all ℓ-groups are archimedean. When G is an ℓ-subgroup of H, we say that H is an *essential extension* of G if, for each convex ℓ-subgroup K of H, $K \cap H = \{0\}$ implies that $K = \{0\}$. The *essential hull* of G is the maximum essential extension of G. G is *essentially closed* if it has no proper essential extensions. In [5] Conrad describes the essentially closed ℓ-groups fully. We urge the reader to refer to this source; however, we shall not have any occasion to use these descriptions. For our purposes the one given above for $C(X)$ suffices.

(b) Let us denote the Dedekind–MacNeille completion of $C(X)$ by $C(X)^c$. For an account of Dedekind–MacNeille completeness we refer the reader to [4]. An archimedean lattice-ordered group G is said to be *Dedekind–MacNeille complete* if every subset of G with an upper bound has a least upper bound.

Similarly, $C(X)^l$ stands for the lateral completion of $C(X)$. A lattice-ordered group G is *laterally complete* if every set of pairwise disjoint elements has a supremum. It is shown in [1] that, for any Tychonoff space X, $Q(X) = C(X)^l$. In greater generality, this is reprised in [13]. Let us amplify a bit: $f \in C(X)^e$ belongs to $Q(X)$ if there is a dense open set V of X so that for each $p \in V$ there is a neighborhood $U_p \subseteq V$ of p and a function $f_p \in C(X)$ so that $f(x) = f_p(x)$, for each $x \in U_p$. This justifies the expression that the functions in $Q(X)$ are "locally" functions in $C(X)$.

(c) It is due to Conrad that $C(X)^e = (C(X)^c)^l$, and to Bernau that the two completions l and c may be applied in either order (see [5]).

(d) To amplify slightly the comment in (b), recall, for any archimedean f-ring A, the following description of $Q(A)$, the maximum ring of quotients of A:

$$Q(A) = q(A)^l = q\left(A^l\right) \quad \text{(see [13])}.$$

Note: $q(A)$ stands for the classical ring of quotients, consisting of formal fractions with denominators which are not divisors of zero, under the standard arithmetic of fractions.

(e) It should be evident then, from the preceding comments, that X is a uniform quotients space precisely when $C(X)^e = Q(X)$, if and only if $C(X)^c \leqslant C(X)^l$.

Much of the discussion to follow will turn on the concept of the maximum domain of a continuous function. We note that in this respect our terminology is different from that of [8].

Definition and Remarks 2.3. (a) Suppose that X is a space, and let us refer to a subset $S \subseteq X$ for which there is a $g \in C(S)$ which admits no continuous extension to any point of $X \setminus S$ as a *maximum domain of* g. If a bounded g exists we say that S *is the maximum domain of a bounded function*. As is shown in Theorem 3.10(1) of [8], every maximum domain of X is necessarily a dense G_δ-set. Indeed, as pointed out in [8, Theorem 3.10(2)]:

If X is locally connected then every dense subset which is a countable intersection of cozerosets of X is a maximum domain of a bounded function.

As we will see in Remark 3.11, the above is false if one replaces "locally connected" with "connected".

We shall return to Theorem 3.10 of [8] in addressing the problem of which metric spaces are uniform quotients spaces.

(b) Closely associated with maximum domains are the sets whose complements are countable unions of boundaries. We explain. By a *boundary* we mean a set B of the form $B = \mathrm{cl}_X V \setminus V$, where V is a regular open set. Note that if B is a boundary, then $X \setminus B$ is a maximum domain, namely of the following characteristic function. Let V be a regular open set with boundary B, and consider the characteristic function χ on $X \setminus B$ of the set V.

The task before us is to show that any dense subset whose complement is a countable union of boundaries is also a maximum domain.

Veksler (in [17]) refers to any set which is contained in a countable union of boundaries as a set of *category* $\frac{1}{2}$; we shall not use this terminology. Instead, let us call a countable union of boundaries a *σ-boundary*; furthermore, any G_δ-set which is the complement of a σ-boundary will be called a *co-σ-boundary*.

Finally, X is said to be *weakly Baire* if every co-σ-boundary is dense.

The following characterization builds on a familiar result for Baire spaces. We are pleased to thank the referee for it. We use $\mathcal{RO}(X)$ to denote the boolean algebra of regular open subsets of X.

Lemma 2.4. *For a space X, the following are equivalent.*

(a) *X is a weakly Baire space.*

(b) *For any compactification Y of X and any co-σ-boundary T of Y, $X \cap T$ is dense in X.*

(c) *X has a compactification Y such that $X \cap T$ is dense in X for any co-σ-boundary T of Y.*

Proof. (a) $\Rightarrow$ (b): Suppose that Y is a compactification of X and X is weakly Baire. Let $T \subseteq Y$ be of the form $T = \bigcap_{n<w} X\backslash B_n$, where $B_n = \mathrm{cl}_Y V_n \backslash V_n$, with V_n regular open in Y. Put $W_n = Y\backslash \mathrm{cl}_Y V_n$. Now trace on X: let $A_n = X \cap V_n$ and $B_n = X \cap W_n$. Observe that these are regular open in X and complementary in $\mathcal{RO}(X)$. Finally,

$$X \cap T = \bigcap_{n<\omega} (A_n \cup B_n),$$

which is a co-σ-boundary of X, and therefore dense.

As (b) obviously implies (c), we prove that (c) implies (a). Suppose that S is a co-σ-boundary of X. Thus, there exist regular open sets of X, $(A_n)_{n<\omega}$ and $(B_n)_{n<\omega}$ such that $A_n \cap B_n = \emptyset$, $A_n \cup B_n$ is dense in X, for each $n < \omega$, and $S = \bigcap_{n<\omega} A_n \cup B_n$. Now, if Y is some compactification of X satisfying (c), note that tracing $U \mapsto X \cap U$ defines an isomorphism of the boolean algebra $\mathcal{RO}(Y)$ onto $\mathcal{RO}(X)$. Let V_n and W_n be regular open sets of Y which trace onto A_n and B_n, respectively, and put

$$T \equiv \bigcap_{n<\omega} (V_n \cup W_n).$$

As $X \cap T = S$ and T is a co-σ-boundary of Y, we have that S is dense in X, proving that X is weakly Baire. □

Next, let us see how the previous lemma pays off.

Lemma 2.5. *Any dense co-σ-boundary S of the space X is the maximum domain of a bounded function. Conversely, every maximum domain of a bounded function is dense and contains a co-σ-boundary.*

Proof. Recall the notion of the oscillation of a function. Assume that $g \in C(Y)$, $Y \subseteq X$. For each $x \in X$,

$$\operatorname{osc} g(x) = \inf_U \bigl[\sup g(Y \cap U) - \inf g(Y \cap U)\bigr],$$

the infimum being taken over all the open neighborhoods U of x. As shown in the proof of Theorem 3.10(2) of [8], if Y is a maximum domain of g, then

$$Y = \bigl\{p \in X\colon \operatorname{osc} g(p) = 0\bigr\}.$$

Now suppose that S is dense and that $X \setminus S = \bigcup_{n<\omega} B_n$, where each B_n is a boundary. As remarked in 2.3(b), each $X \setminus B_n$ is the maximum domain of a bounded function f_n with oscillation 1 at each point of B_n. Then defining

$$f = \sum_{n<\omega} \frac{f_n}{3^n},$$

we get an $f \in C^*(S)$ which has positive oscillation at each point of the complement, proving that S is the maximum domain of f.

Conversely, suppose that $S \subseteq X$ is the maximum domain of $f \in C(S)$. It has already been noted in 2.3(a) that S must be dense. Next, for each rational number q define

$$A_q \equiv \bigcup \bigl\{V\colon V \text{ is open in } X,\ f(S \cap V) \subseteq (-\infty, q]\bigr\},$$

and

$$C_q \equiv \bigcup \bigl\{V\colon V \text{ is open in } X,\ f(S \cap V) \subseteq (q, \infty)\bigr\}.$$

Then, for each $q \in \mathbb{Q}$, A_q and C_q are disjoint open sets with dense union.

$$A_q = \operatorname{int}_X \operatorname{cl}_X f^{-1}(-\infty, q],$$

and therefore regular open. Put $B_q = X \setminus \operatorname{cl}_X A_q$, and observe that $C_q \subseteq B_q$. We claim that $\bigcap_{q\in\mathbb{Q}} (A_q \cup B_q) \subseteq S$, which completes the proof.

To that end, note that if $x \notin S$, then, since f has positive oscillation at x, there is a rational number q such that

$$\sup_U \inf f(S \cap U) < q < \inf_U \sup f(S \cap U),$$

with U ranging over all the open neighborhoods of x in X. (Note that the leftmost expression above could be $-\infty$, and the rightmost ∞.) Thus, every neighborhood U of x meets both A_q and B_q, which means that $x \notin A_q \cup B_q$, as A_q and B_q are disjoint. □

Here is the main theorem. Recall the standard notation: when A is a ring of functions on a space, then A^* stands for the subring of all bounded functions in A. It is condition (g) in Theorem 2.6 which connects with Veksler's "sets-of-category $\frac{1}{2}$" formulation.

Theorem 2.6. *For any Tychonoff space X, the following are equivalent.*

(a) *X is a uniform quotients space.*

(b) $C(X)^l = C(X)^e$.

(b′) $(C(X)^e)^* = (C(X)^l)^*$.

(c) *$C(X)^e$ is a ring of quotients of $C(X)$.*
(d) *$C(X)^c$ is a ring of quotients of $C(X)$.*
(e) *For each dense G_δ-set U of βX $C(U)$ is a ring of quotients of $C(X)$.*
(f) *Every maximum domain of βX has dense interior.*
(f′) *Every maximum domain of βX of a bounded function has dense interior.*
(g) *Every co-σ-boundary of X has dense interior.*

Proof. That (a) implies (b) is a restatement of (e) in 2.2. Thus, it should also be clear that (b) $\Rightarrow$ (c). Any subring of a ring of quotients of $C(X)$—extending $C(X)$—is a ring of quotients of $C(X)$, meaning that (c) $\Rightarrow$ (d). (b) $\Rightarrow$ (b′) is obvious, and the reverse follows by applying the lateral completion; let's explain. First, $(C(X)^e)^*$, being a convex ℓ-subgroup of a Dedekind–MacNeille complete ℓ-group, is also Dedekind–MacNeille complete. This means that $((C(X)^e)^*)^l = ((C(X)^e)^*)^e = C(X)^e$. On the other hand,

$$\big((C(X)^l)^*\big)^l = \big(Q(X)^*\big)^l = Q(X) = C(X)^l.$$

This shows that (b′)$\Rightarrow$ (b).

Assume (d). From basic principles (see [11]) it should be clear that if B is a ring of quotients of A, then B and A have the same complete ring of quotients. Therefore, as $C(X)^c$ is a ring of quotients of $C(X)$,

$$C(X)^e = \big(C(X)^c\big)^l = C(X)^l,$$

proving (a).

Next, any direct limit of rings of quotients of A is a ring of quotients of A, so (a) and (e) are equivalent because of the definition of $C(X)^e$. Finally, from the definition of a maximum domain it is clear that (b) and (f) are equivalent. Just as clearly, (b′) is equivalent to (f′).

Finally, we proceed to show the equivalence of (a) through (f′) with (g).

Suppose that X is a uniform quotients space, and S is a co-σ-boundary of X. As in the proof of Lemma 2.4, there is a co-σ-boundary $T \subseteq \beta X$ such that $S = X \cap T$. According to Lemma 2.5, T is the maximum domain of a bounded function f. Since $f \in C(X)^e = Q(X)$, there is a dense open set U of X such that $f|_U \in C(U)$. This says that $U \subseteq S$, and so S has dense interior.

Conversely, suppose that every co-σ-booundary has dense interior. Pick $g \in C(X)^e$ with maximum domain $V \subseteq \beta X$. By Lemma 2.5, V contains a co-σ-boundary T, which traces on X to a co-σ-boundary $X \cap T$, which has dense interior U, by assumption. Thus, $f|_U \in C(U)$, and so $f \in Q(X)$, proving that X is a uniform quotients space. □

We highlight two corollaries of Theorem 2.6. The first requires no further commentary.

Corollary 2.7. *Every uniform quotients space is weakly Baire.*

It should be clear that (a) and (d) of the next corollary are equivalent. The others are not difficult to pin down and are left to the reader.

Corollary 2.8. *The following are equivalent for a space X:*

(a) *X is a uniform quotients space.*
(b) *X has a compactification which is a uniform quotients space.*
(c) *Every compactification of X is a uniform quotients space.*
(d) *βX is a uniform quotients space.*

Maximum domains are related to the notion of extreme disconnectivity at a point, a concept due to van Douwen [7]. We conclude the section with a discussion of this matter.

Definition 2.9. Recall that $x \in X$ is an *e.d. point* (or as van Douwen defined it, that X is *extremally disconnected at x*) if x is not in the closures of two disjoint open sets. Evidently, if A and B are complementary sets in $\mathcal{RO}(X)$ then $A \cup B$ must contain all the e.d. points of X.

Here are two ways to describe the set of all e.d. points.

Proposition 2.10. *In any space X the following describe the set of all e.d. points, $ed(X)$:*

(a) $\bigcap\{S\colon$ *S is a co-σ-boundary*$\}$.
(b) $\bigcap\{S\colon$ *S is the maximum domain of a bounded function*$\}$.

Proof. We have already indicated that $ed(X)$ must be contained in the set in (a). Lemma 2.5 tells us that the set in (a) is contained in the one in (b). Finally, if $p \in X$ is not an e.d. point then it lies in the closures of both of the disjoint open sets U and V, and without loss of generality, one may assume that U and V are complementary in $\mathcal{RO}(X)$. Thus, p lies on a boundary and hence outside the maximum domain of a bounded function. □

The above proposition describes the intersection of all the maximum domains of bounded functions. Let us now remove the assumption of boundedness. Recall that $p \in X$ is a P-point if every function $f \in C(X)$ is constant on a neighborhood of p.

Proposition 2.11. *The intersection of all the maximum domains of X is the set of all e.d. points which are P-points.*

Proof. In view of Proposition 2.10, the named intersection is a subset of $ed(X)$. Now if $p \in X$ is not a P-point there is a function $g \in C(X)$ which vanishes at p, but is not identically zero on any neighborhood of p. Define $f \in C(\mathrm{coz}(g))$ by $h(x) = g(x)^{-1}$. Extend h to its maximum domain S, and call the extension $\tilde{h}$. The reader may easily verify that $\tilde{h}$ cannot be defined at p; that is, $p \notin S$, proving that the intersection described in the proposition consists of e.d. points which are P-points.

To conclude this proof, we verify that any maximum domain contains all the e.d. points which are P-points. Suppose that $p \in X$ is both an e.d. point and a P-point, and let S be the maximum domain of $f \in C(S)$. Owing to Proposition 2.10, we may as well assume that f is unbounded, and then, without loss of generality, that $f \geqslant 1$. Consider $g = f^{-1} \in C^*(S)$.

Invoking Proposition 2.10 again, we have that g may be extended to p, which implies that f too can be extended to p. This means that $p \in S$, and the proof is complete. □

3. Applications

Remarks 3.1. Let us give some examples of uniform quotients spaces.

(i) Recall that X is an *almost P-space* if every nonempty G_δ-set has nonempty interior. It is well known (see [12]) that if X is an almost P-space then every dense G_δ-set has dense interior. Thus, if X is a compact almost P-space then (f) in Theorem 2.6 tells us that X is a uniform quotients space.

Indeed, the condition that every dense G_δ-set has dense interior is less stringent than assuming the space is an almost P-space; consider, for example, the one-point compactification $\alpha\omega$ of the discrete naturals ω. Let us say that X is a *Q_{G_δ}-space* if it has this property. So, any Q_{G_δ}-space is a uniform quotients space, but the converse is not true. We give an example at the conclusion of this section (Example 3.12).

(ii) If X has a dense subset D of isolated points then D is the least dense open subset of X, whence every dense G_δ-set has dense interior, and it is clear that X is a uniform quotients space.

(iii) Recall that X is *extremally disconnected* if the closure of every open set of X is open. The reader should also keep in mind the Stone–Nakano Theorem (see [9, 3N.6]): *X is extremally disconnected if and only if $C(X)$ is Dedekind–MacNeille complete*. Vacuously then, from (d) in Theorem 2.6, any extremally disconnected space is also a uniform quotients space.

(iv) It follows from the fact that for any dense open set U, $C(U)$ is a ring of quotients of $C(X)$, that if X is a uniform quotients space then so is U. The converse is also true.

(v) Generalizing (ii) and (iii) a bit, suppose that X has a dense open subspace Y which is a uniform quotients space, in its own right. Then, since for each dense open set V of X, $V \cap Y$ is dense and open in Y, it follows that $Q(Y) = Q(X)$, which is uniformly complete by Theorem 2.6. This shows that X is a uniform quotients space. In particular, if X has a dense open extremally disconnected subspace, then X is a uniform quotients space.

(vi) Suppose that X is a uniform quotients space; then any open subset of X is a uniform quotients space. For suppose that U is open. Consider a dense G_δ-set G of U, and a function $f \in C(G)$. Then one can define a function $f' \in C(G \cup (X \backslash \mathrm{cl}_X(U)))$, by extending f to be zero on $X \backslash \mathrm{cl}_X(U)$. As $G \cup (X \backslash \mathrm{cl}_X(U))$ is a dense G_δ-set of X—which is assumed to be a uniform quotients space—it follows that there is a dense open set V of X on which f' is continuously definable. Restricting to $V \cap U$, which is dense and open in U, we have shown that $Q(U) = C(U)^e$, whence U is a uniform quotients space. From this it easily follows that if X is a uniform quotients space then so is any regular closed subset.

(vii) Recall that X is said to be a *Baire space* if any countable intersection of dense open sets is dense. The content of the Baire Category Theorem is that any complete metric space is a Baire space. It is also well known that every compact space is Baire. As is observed in [8], Theorem 4.6, if X is a Baire space then $C(X)^e$ may be constructed as the direct limit of the $C(U)$, over all the dense G_δ-sets of X. With this in mind consider $\mathbb{R}$, the real line with the usual metric topology. Then the subspace Y of all irrational numbers is a dense G_δ-set without interior, and by Theorem 3.10(2) of [8] it is also a maximum domain. Hence, $\mathbb{R}$ is not a uniform quotients space.

As we have seen (Remark 3.1(ii)), if X has a dense subset of isolated points then it is a uniform quotients space. We now generalize to a larger class of spaces.

Definition and Remarks 3.2. (a) Recall that a space X is said to be a *DC-space* (or, more elaborately, a *space of dense constancies*) if for each $f \in C(X)$ there is a family of pairwise disjoint open sets $\{U_i\colon i \in I\}$ whose union is dense in X, such that $f|_{U_i}$ is constant. If X also has a π-base consisting of clopen sets then the family $\{U_i\colon i \in I\}$ can be chosen so that each U_i is clopen. In this event we say that X is a *Specker* space. (Note: a *π-base* for the open sets of a topology is a collection of open sets $\mathcal{B}$ such that for any open set U of X there is a $V \in \mathcal{B}$ contained in U.) For additional information on *DC*-spaces and Specker spaces the reader is referred to [2,14,3]. Note that X is Specker if and only if $C(X)$ is contained in the maximum ring of quotients of the subalgebra $S(X)$ of $C(X)$ generated by the idempotents of $C(X)$. Thus, X is Specker precisely when $Q(S(X)) = Q(X)$.

A Tychonoff space X is Specker (respectively a *DC*-space) if and only if βX is Specker (respectively a *DC*-space) [14, l.ll(b)] and [3, Proposition 3.6]). We stress that a Specker space need not be zero-dimensional, but it does have a π-base consisting of clopen sets. Any space with a dense set of isolated points is Specker.

(b) An interesting question surrounding compact Specker spaces has to do with their absolutes. Recall that a continuous surjection $f : Y \to X$ of compact spaces is said to be *irreducible* if X is not the image of any proper closed subset of Y. We consider pairs (E, e), such that $e : E \to X$ is irreducible and E is compact and extremally disconnected. Such a pair always exists: one takes for E the Stone dual of the boolean algebra of all regular closed sets of X, which is complete, making E extremally disconnected. For $e : E \to X$ one takes the map which associates with each ultrafilter α the unique point of intersection in $\bigcap \alpha$. For details the reader is referred to [18, Chapter 10]. If (E', e') is also a pair as above, then there is a homeomorphism $h : E \to E'$ such that $e' \cdot h = e$. This characterizes the so-called *absolute* of X. We denote the absolute *space* of X by *EX*.

The knowledgeable reader will recognize that the irreducible surjection $e : EX \to X$ is intimately connected with the embedding of $C(X)$ in $C(X)^e$. In this context this matter need not concern us, however.

Note that if $f : Y \to X$ is an irreducible surjection, and Y is a *DC*-space then so is X [3, Proposition 3.9(1)]. In particular, if X is a compact space with a clopen π-base and *EX* is a Specker space, then so is X. The converse was examined in [2], and it emerged that the answer sometimes is no. On the other hand, there is the following characterization in [14, Theorem 2.6]:

Assume that X has a clopen π-base. Then EX is Specker if and only if $Q(S(X)) = S(X)^e$.

For example (see [2, 2.9]), if X is a compact Specker space which satisfies the *countable chain condition*—i.e., every set of pairwise disjoint open sets is countable—then *EX* is Specker.

One note should be added here, regarding the result quoted above. The assumption that X has a clopen π-base means that $S(X)$ is essential in $C(X)$, and, thus that $S(X)^e = C(X)^e$.

(c) What we wish to do is characterize the compact *DC*-spaces which are also uniform quotients spaces. To accomplish this we recall the notion (from [3]) of a *DC*-algebra; the appropriate context is that of an f-algebra.

First, let A be a real commutative algebra with identity; we say that A is an *f-algebra* if A is a lattice-ordered algebra and a vector lattice—that is, $0 \leqslant r \in \mathbb{R}$ and $0 \leqslant a \in A$ together imply that $0 \leqslant ra$—and if $a \wedge b = 0$ and $c \geqslant 0$ imply that $ac \wedge b = 0$. Now suppose that A is an f-algebra. It is called a *DC*-algebra—or, more elaborately, an *algebra of dense constancies* – if for each $0 \neq a \in A$ and each $0 < x \in A$, for which $ax \neq 0$, there exists $ab \in A$, with $0 < 6 \leqslant x$, and a nonzero real number r such that $ab = rb$. As is observed in [3, 2.7], X is a *DC*-space precisely when $C(X)$ is a *DC*-algebra. The observation which is crucial to our purposes here is Theorem 2.11, from [3], which we now quote:

> If A is a DC-algebra then so is $Q(A)$, the maximum ring of quotients of A.
> Moreover, $Q(A)$ is the largest extension of A which is a DC -algebra and in which A is order-dense.

Incidentally, if A is an ℓ-subalgebra of the f-algebra B, we say that A is *order-dense* in B if for each $0 < b \in B$ there is an $a \in A$ such that $0 < a \leqslant b$.

There is more, in Corollary 2.12 [3]: *if A is a DC-algebra, then $Q(A)$ is also the lateral completion and maximum ring of quotients of the subalgebra generated by its idempotents*.

(d) Before proceeding, a comment about e.d. points: if X is compact then it can be shown that $p \in X$ is an e.d. point precisely when it has one preimage under the canonical $e : EX \to X$ from the absolute of X,

Putting (b) and (c) in the above remarks together we get the characterization we had in mind.

Proposition 3.3. *Suppose that X is a compact DC-space. Then EX is Specker if and only if X is a uniform quotients space.*

Proof. Assume first that *EX* is Specker; then we have, by 3.2(b) that $Q(S(EX)) = S(EX)^e$. Since $C(X)$ is a *DC*-algebra, 3.2(c) tells us that $Q(X)$ is the maximum ring of quotients of the subalgebra B generated by its idempotents. But it is well known that the maximal ideal space of a complete ring of quotients is extremally disconnected; see [11, §2.4]. Indeed, it is the Stone dual of the boolean algebra of all annihilators. In our context this observation translates into the fact that $B = S(EX)$. Conclusion: $Q(X) = Q(S(EX))$, which is uniformly complete, whence X is a uniform quotients space.

Conversely, if X is a uniform quotients space, then owing to 3.2(c), applied to $C(X)$, $Q(X)$ is a DC-algebra, whence its subring of bounded functions—that is to say, $C(EX)$, is a DC-algebra as well; see [3, Proposition 2.8]. But this says that EX is a Specker space. □

Let us return to Theorem 3.10 of [8], quoted in 2.3(a). The following is a simple application.

Proposition 3.4. *No locally connected Baire space having a countable dense set of nonisolated G_δ-points is a uniform quotients space.*

Proof. If S is a countable dense subset consisting of G_δ-points then $X\backslash S$ is a G_δ-set which is dense—because there are no isolated points—and it has no interior. Since $X\backslash S$ is the maximum domain of a function $f \in C(X\backslash S)$, one cannot define f on a dense open set. It follows that X is not a uniform quotients space. □

We have the following corollary to Proposition 3.4, for totally ordered spaces (with respect to the interval topology).

Corollary 3.5. *Suppose that X is a compact totally ordered space. If X is a uniform quotients space then X has no nontrivial connected separable intervals. Thus, if X is also locally connected, then X is a DC-space with Specker absolute.*

Proof. First observe that a totally ordered space with the interval topology which is connected is also locally connected. Next, any interval in X is a uniform quotients space if X is, by Remark 3.1(vi). But by Proposition 3.4, such an interval cannot be connected and separable. The final claim is a consequence of Proposition 3.6 in [2], coupled with Proposition 3.9(1) in [3]—or the comment in 3.2(b)—which insures that if EX is Specker then X is a DC-space. □

The following example shows that a nontrivial compact totally ordered uniform quotients space can be connected.

Example 3.6. Let $\mathbb{H}$ be the Dedekind cut completion of an η_1-set, with first and last element adjoined. It is well known (see [9, Chapter 13]) that $\mathbb{H}$ is compact and connected. As explained in Example 3.9 in [2], $\mathbb{H}$ is an almost P-space, and, hence, a uniform quotients space. Then, owing to Theorem 2.8 in [14], we also get that $\mathbb{H}$ is a DC-space. We are fully in the context of Corollary 3.5, and so $\mathbb{H}$ has no nontrivial separable connected intervals, while $E\mathbb{H}$ is Specker.

The next two lemmas will be used to characterize compact metrizable uniform quotients spaces. They are interesting in their own right. The second is, doubtless, folklore.

Lemma 3.7. *Suppose that $f : Y \to X$ is an irreducible continuous surjection of compact spaces. If X is a uniform quotients space then so is Y.*

Proof. If $f: Y \to X$ is irreducible the canonical $e: EX \to X$ from the absolute of X factors through Y. From this it follows that $C(Y)$ is intermediate to $C(X)$ and $C(EX)$, which is the Dedekind–MacNeille completion of $C(X)$ (see [8, 4.11], or else [6, 3.10(b)]). Since $C(EX)$ is a ring of quotients of $C(X)$, it is also a ring of quotients of $C(Y)$. □

Lemma 3.8. *If X is a compact metric space without isolated points then it is the image of the Cantor set under an irreducible map.*

Proof. It is well known that such a space is a continuous image of the Cantor set C. Suppose that $g: C \to X$ is such a continuous surjection. By Gleason's Lemma [18, 10.40] there is a closed subspace C' of C so that the restriction $g|_{C'}$ is irreducible. But then C' is compact zero-dimensional and without isolated points. Thus, C' is homeomorphic to C. □

Finally, the characterization of compact metrizable uniform quotients spaces.

Theorem 3.9. *Suppose X is a compact metric space. Then X is a uniform quotients space if and only if there is a dense subset of isolated points.*

Proof. The sufficiency has already been noted, in Remark 3.1(ii).

Now suppose that X is a compact metrizable uniform quotients space. We establish that X must have an isolated point. Suppose this is not so; then X is a continuous irreducible image of the Cantor set C, by Lemma 3.8. According to Lemma 3.7 then, C is a uniform quotients space. So we proceed to show that C cannot be a uniform quotients space, thus arriving at a contradiction, so that we may then conclude that X must have an isolated point if it is a uniform quotients space.

To say that $p \in C$ is an *endpoint* of C means that it is an endpoint of one of the omitted intervals in the traditional construction of the Cantor set. Also, for purposes of this proof, let us refer to a function $f: U \to I$, from an open interval $U = (a, b)$, with $a, b \in C$, of the Cantor set into the closed unit interval, as a *topologist's sine curve* (abbr. *ts*-curve) if it is a continuous function defined on U with oscillation 1 at both a and b. Since every point p of the Cantor set is the limit of endpoints of C, it should be clear that given any nontrivial interval $U = (a, b)$ in C, there exists a *ts*-curve on U. Now suppose that V is a dense open subset of C; express it as a disjoint union of countably many open intervals $V_n (n < \omega)$. Choose, for each $n < \omega$, a *ts*-curve f_n on V_n, and let $f: V \to I$ be the function which agrees with f_n on V_n. It is then easy to see that V is a maximum domain for f. All of which proves that each dense open set is the maximum domain of a bounded function which has oscillation 1 at each of the points of the complement. As argued in the proof of Lemma 2.5, it follows that every dense G_δ-set is also a maximum domain of a bounded function. However, the endpoints of C form a countable dense set, so that the complement is a dense G_δ-set without interior, proving that C is not a uniform quotients space.

We continue, knowing that there must be an isolated point in X. Let N be the set of all isolated points, and assume that N is not dense; by Remark 3.1(vi) $X \backslash \mathrm{cl}_X N$ is a uniform quotients space. Now $X\backslash \mathrm{cl}_X N$ is an open subset of a compact metric space; hence, by Baire's Theorem [19, 23.5] it is a Baire space. We may therefore apply Proposition 3.4 and deduce that $X\backslash \mathrm{cl}_X N$ contains an isolated point, which is, necessarily, an isolated point of X; this is a contradiction, and it follows that N is dense in X, which concludes the proof. □

Remark 3.10. We wish to suggest that what makes Theorem 3.9 tick is the fact that a compact space is metrizable if and only if it has countable weight. (Note: the *weight* of a space is the least cardinality of a base for the open sets.)

(a) For example, countable π-weight does not suffice. (A space X has *countable π-weight* if there is a countable π-base for X.) To illustrate, the absolute EC of the Cantor set C has countable π-weight—as the π-weight is preserved by passage to an irreducible preimage—and has no isolated points, but as it is extremally disconnected, EC is a uniform quotients space. Since a space with countable π-weight is also separable, we see that a compact uniform quotients space which is separable need not have isolated points.

(b) Nor is it likely that other countability stipulations on a uniform quotients space will force the existence of isolated points. Let us assume the existence of a *Suslin line*; that is, a connected totally ordered space which satisfies the countable chain condition (abbr. *ccc*) but is not separable. For our purposes it is enough to remind the reader that the existence of a Suslin line depends on one's set theoretic assumptions; see [10] and [16]. We also refer the reader to Mary Ellen Rudin's monograph [15]. As explained in [15, p. 15], if a Suslin line exists, then one can also find one, say L, which is compact, connected and has no nontrivial separable intervals. Applying Proposition 3.6 of [2], L is a *DC*-space with Specker absolute, whence a uniform quotients space. L is also first countable, hereditarily *ccc*—every subspace has *ccc*—and even hereditarily Lindelöf.

It is an open question whether an example can be constructed, *within ZFC*, of a compact uniform quotients space which is hereditarily Lindelöf and has no isolated points.

Remark 3.11. Recall that X is called a *quasi F-space* if each dense cozeroset is C^*-embedded in X. Quasi F-spaces received much attention in connection with order convergence in [6]; the concept was actually introduced earlier in a paper of Henriksen. The following should be clear from Theorem 3.10(2) of [8]:

If X is a locally connected quasi P-space, then X is an almost F-space.

To illustrate then, that Theorem 3.10(2) from [8] is false for connected spaces, as promised in 2.3(a), one requires a compact connected quasi F-space which is not an almost P-space.

Here's how to get one: as before, $\mathbb{H}$ stands for the Dedekind cut completion of an η_1 set, with top and bottom element adjoined. Y is the countable disjoint union of copies of $\mathbb{H}$, with their respective top elements identified to a point. (The reader familiar with the Fréchet–Urysohn Fan could think of this space as the result of knitting together copies of $\mathbb{H}$ into a fan.) It is not hard to see that Y is still a quasi F-space—essentially because the identified point from $\mathbb{H}$ is a P-point. Y is clearly connected and σ-compact, although not compact. This implies that βY is a connected quasi F-space which is not an almost P-space, as Y is a proper dense cozeroset in it.

Finally, the example promised in Remark 3.1(i), of a uniform quotients space which is not a Q_{G_δ}-space:

Example 3.12. Consider the absolute EC of the Cantor set C; let $e : EC \to C$ stand for the canonical irreducible map. Take a countable dense subset S of C, and let $V = EC \setminus e^{-1}(S)$; then V is a dense G_δ-set in EC; (it is well known that, under an irreducible map, the inverse image of a dense subset is dense.) However, by the same token V also has no interior. So EC is not a Q_{G_δ}-space. On the other hand, EC is a uniform quotients space.

References

[1] M. Anderson, P. Conrad, The hulls of $C(Y)$, Rocky Mountain J. 12 (1) (1982) 7–22.
[2] A. Bella, A.W. Hager, J. Martinez, S. Woodward, H. Zhou, Specker spaces and their absolutes, I, Topology Appl. 72 (1996) 259–271.
[3] A. Bella, J. Martinez, S. Woodward, Algebras and spaces of dense constancies, Czech. Math. J., to appear.
[4] A. Bigard, K. Keimel, S. Wolfenstein, Groupes et Anneaux Réticulés, Lecture Notes in Math., Vol. 608, Springer, Berlin, 1977.
[5] P.F. Conrad, The essential closure of an archimedean lattice-ordered group, Duke Math. J. 38 (1971) 151–160.
[6] F. Dashiell, A. Hager, M. Henriksen, Order-Cauchy completions of rings and vector lattices of continuous functions, Canad. J. Math. XXXII (3) (1980) 657–685.
[7] E.K. van Douwen, Remote Points, Dissertationes Math., Vol. CLXXXVII, Polish Sci. Publ., Warsaw, 1981.
[8] N. Fine, L. Gillman, J. Lambek, Rings of Quotients of Rings of Functions, McGill Univ. Press, Montréal, Quebec, 1965.
[9] L. Gillman, M. Jerison, Rings of Continuous Functions, Graduate Texts in Math., Vol. 43, Springer, Berlin, 1976.
[10] T. Jech, Non-provability of Souslin's hypothesis, Comment. Math. Univ. Carolinae 8 (1967) 291–305.
[11] J. Lambek, Lectures on Rings and Modules, Chelsea Publ. Co., New York, 1976.
[12] R. Levy, Almost P-spaces, Canad. J. Math. 29 (1977) 284–288.
[13] J. Martinez, The maximal ring of quotients of an f-ring, Algebra Universalis 33 (1995) 355–369.
[14] J. Martinez, S. Woodward, Specker spaces and their absolutes, II, Algebra Universalis 33 (1996) 333–341.
[15] M.E. Rudin, Lectures on Set Theoretic Topology, CBMS Regional Conf. Ser. in Math., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.
[16] R.M. Solovay, S. Tennenbaum, Iterated Cohen extensions and Souslin's problem, Ann. of Math. (2) 94 (1971) 201–245.
[17] A.I. Veksler, On Banach and Dedekind completeness of spaces of continuous functions, vector lattices and maximal rings of quotients, Soviet. Math. Dokl. 12 (1) (1971) 15–18.
[18] R.C. Walker, The Stone–Čech Compactification, Ergeb. Math., Vol. 83, Springer, Berlin, 1974.
[19] S. Willard, General Topology, Addison-Wesley, Reading, MA, 1970.