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ON THE LATTICE OF $z$-IDEALS OF A COMMUTATIVE RING

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This article is dedicated to Professor Aleˇs Pultr on his 80th birthday.

Abstract. We prove that the lattice of $z$-ideals of a commutative ring with identity is a coherent frame. We characterize when it is a Yosida frame, and when it satisfies some projectability properties. We also characterize Hilbert rings in terms of ideals that arise naturally in this study. A ring with zero Jacobson radical is shown to be feebly Baer precisely when its frame of $z$-ideals is feebly projectable. Denote by $\text{ZId}(A)$ the frame of $z$-ideals of a ring $A$. We show that the assignment $A \mapsto \text{ZId}(A)$ is the object part of a functor $\text{CRng} \to \text{CohFrm}$, where $\text{CRng}$ designates the category whose objects are commutative rings with identity and whose morphisms are the ring homomorphisms that contract $z$-ideals to $z$-ideals.

1. Introduction

Throughout the paper, by “ring” we mean a commutative ring with identity $1 \neq 0$. Almost all our rings are reduced, which is to say they have no nonzero nilpotent elements. The history of $z$-ideals in rings is easy to record. The first usage of the term “$z$-ideal” was by Kohls [13] in the study of rings of continuous functions. He observed that they could be characterized purely algebraically. Modifying that characterization, it was Mason [16] who initiated the study of $z$-ideals of commutative rings in earnest.

The interest in lattices of $z$-ideals started with the paper of Martínez and Zenk [15], in which they prove that the lattice of $z$-ideals of the ring $C(X)$ is a frame. They actually proved that it is a coherently normal Yosida frame. In her doctoral thesis [9], the first-named author of the present article extended this result of Martínez and Zenk to lattices of $z$-ideals of the ring $RL$ of continuous real-valued functions on a completely regular frame. This was further extended by Dube [5] to lattices of $z$-ideals of an $f$-ring with bounded inversion.

The present paper (which started as a question raised at the Ordered Algebraic Structures meeting held at Louisiana State University in 2016) significantly improves the earlier results. It is organized as follows.

We recall in Section 2 the necessary background, and we fix notation. In Section 3, we prove that the lattice, $\text{ZId}(A)$, of $z$-ideals of any ring $A$ is a frame (Theorem 3.1). We obtain this via a device called a “prenucleus” that was invented by Banaschewski [1]. The prenucleus in question is defined on the frame, $\text{RId}(A)$, of radical ideals of $A$, and it turns out that the fixed elements of the prenucleus are precisely the $z$-ideals of $A$. After identifying the compact elements of $\text{ZId}(A)$ (Lemma 3.2), we show that $\text{ZId}(A)$ is a coherent frame (Theorem 3.4).

In [14], Martínez and Zenk introduce what they call a “$z$-nucleus” on an algebraic frame $L$. Its fixed elements are called the $z$-elements of $L$. Their proof that the lattice of $z$-ideals of $C(X)$, for $X$ compact, is a frame actually shows that these ideals are the $z$-elements of the frame of convex $\ell$-ideals of $C(X)$, when the latter is viewed as an $\ell$-group. In Section 4
we identify the \( z \)-elements of \( \text{RId}(A) \) (Theorem 4.4), and this enables us to characterize the rings \( A \) for which \( \text{ZId}(A) \) is a Yosida frame (Corollary 4.5).

The penultimate section deals with issues surrounding the notion of projectability – a property that is frequently considered in algebraic frames. We prove, among other things, that a ring with zero Jacobson radical is weak Baer if and only if its frame of \( z \)-ideals is projectable (Proposition 5.4), and that it is feebly Baer if and only if \( \text{RId}(A) \) is feebly Baer.

In the final section, we show that \( \text{ZId} \) can be made into a functor when we restrict the ring homomorphisms to those that contract \( z \)-ideals to \( z \)-ideals (Proposition 6.3). We give a comment (Remark 6.2) why this restriction is actually “forced” on us.

2. Background and Notation

2.1. Algebraic frames. Our reference for frames and their homomorphisms is [19]. Our notation is, to a large extent, standard. For instance, we denote the pseudocomplement of an element \( a \) by \( a^* \), and the right adjoint of a frame homomorphism \( h \) by \( h^* \).

Let \( L \) be a frame. An element \( a \in L \) is compact if, for any \( X \subseteq L \), \( a \leq \bigvee X \) implies that there is a finite \( Y \subseteq X \) with \( a \leq \bigvee Y \). We denote by \( \mathfrak{t}(L) \) the set of all compact elements of \( L \). If every element of \( L \) is the join of compact elements below it, then \( L \) is said to be algebraic. If \( a \wedge b \in \mathfrak{t}(L) \) for every \( a, b \in \mathfrak{t}(L) \), then \( L \) is said to have the finite intersection property, throughout abbreviated as FIP. If the top element of \( L \) (which we shall denote by 1) is compact and \( L \) has FIP, then \( L \) is called coherent. A frame homomorphism between algebraic frames is called a coherent map if it maps compact elements to compact elements.

2.2. Rings. We write \( \text{Ann}(S) \) for the annihilator of \( S \subseteq A \), and abbreviate \( \text{Ann}(\{a\}) \) as \( \text{Ann}(a) \). The ideal generated by a single element \( a \) will be written as \( \langle a \rangle \). The radical of an ideal \( I \) of \( A \) is the ideal

\[ \sqrt{I} = \{ x \in A \mid x^n \in I \text{ for some positive integer } n \}. \]

An ideal is called a radical ideal if it coincides with its radical. A convenient characterization is that

\[ I \text{ is a radical ideal } \iff a^2 \in I \text{ implies } a \in I. \]

The smallest radical ideal containing an element \( a \) is denoted by \([a]\). That is, \([a] = \sqrt{\langle a \rangle}\).

The lattice \( \text{RId}(A) \) of radical ideals of \( A \), ordered by inclusion, is a coherent frame (see [2]). The meet in \( \text{RId}(A) \) is intersection, and the join is the radical of the sum. The principal radical ideal generated by \( a \in A \) is denoted by \([a]\). The compact elements of \( \text{RId}(A) \) are the finitely generated radical ideals. The top element of \( \text{RId}(A) \) is the compact element \([1] = A \), and the bottom element is \([0]\), which is the zero ideal if \( A \) is reduced. We let \( \text{Max}(A) \) denote the set of maximal ideals of \( A \).

For an element \( a \) of a ring \( A \), we write \( \mathfrak{z}(a) \) and \( \mathfrak{Z}(a) \), respectively, for the sets

\[ \mathfrak{z}(a) = \{ M \in \text{Max}(A) \mid a \in M \} \quad \text{and} \quad \mathfrak{Z}(a) = \bigcap \mathfrak{z}(a). \]

An ideal \( I \) of \( A \) is a \( z \)-ideal if, for any \( a, b \in A \)

\[ \mathfrak{z}(a) = \mathfrak{z}(b) \quad \text{and} \quad a \in I \implies b \in I. \]

Examples of \( z \)-ideals are maximal ideals and the minimal prime ideals of Jacobson semi-simple rings. Intersections of \( z \)-ideals are \( z \)-ideals. A useful characterization (which is known in the literature) is that \( I \) is a \( z \)-ideal if and only if \( \mathfrak{Z}(a) \subseteq I \) for every \( a \in I \). Observe, as well, that \( \mathfrak{Z}(a) \) is the smallest \( z \)-ideal containing \( a \).

A ring \( A \) is Gelfand if for any \( a, b \in A \) with \( a + b = 1 \), there exist \( r, s \in A \) such that

\[ (1 + ar)(1 + bs) = 0. \]
3. Frames of $z$-ideals

It has been established in a number of papers that the lattices of $z$-ideals of certain rings are algebraic frames. This was first done for the rings $C(X)$ [15], then for the rings $\mathcal{RL}$ [9], and the most recent case for $f$-rings with bounded inversion [5]. In the latter two cases, the result is obtained by showing that the lattice in question is a certain quotient of the frame of radical ideals and radical $f$-ideals, respectively.

For the general case we utilize the notion of a “prenucleus”; a clever device invented by Banaschewski [1]. Let us recall that a prenucleus on a frame $L$ is a map $k_0: L \to L$ such that, for all $x, y \in L$:

$$x \leq k_0(x), \quad x \leq y \implies k_0(x) \leq k_0(y), \quad k_0(x) \wedge y \leq k_0(x \wedge y).$$

The set $\text{Fix}(k_0) = \{t \in L \mid k_0(t) = t\}$ is then a frame, and the mapping $k: L \to L$ given by

$$k(x) = \bigwedge \{t \in L \mid x \leq t = k_0(t)\}$$

is a nucleus on $L$ with $\text{Fix}(k) = \text{Fix}(k_0)$. We denote by $\text{ZId}(A)$ the lattice of $z$-ideals of a ring $A$. Observe that for any $a \in A$, $Z(a) \in \text{RId}(A)$.

**Theorem 3.1.** *For any commutative ring* $A$, *the lattice* $\text{ZId}(A)$ *is a frame.*

**Proof.** Define a map $k_0: \text{RId}(A) \to \text{RId}(A)$ by

$$k_0(I) = \bigvee \{Z(a) \mid a \in I\}.$$ 

We claim that $k_0$ is a prenucleus on $\text{RId}(A)$. It is clear that, for any $I, J \in \text{RId}(A)$, $I \subseteq k_0(I)$, and $I \subseteq J$ implies $k_0(I) \subseteq k_0(J)$. Note that if $a \in I$ and $u \in J$, then $[u] \cap Z(a) \subseteq Z(au)$. To see this, let $x \in [u] \cap Z(a)$. Pick a positive integer $n$ and $r \in A$ such that $x^n = ru$. Consider any maximal ideal $N$ of $A$ with $au \in N$. We aim to show that $x \in N$. Since $N$ is a prime ideal (as it is maximal in a ring with identity), $a \in N$ or $u \in N$. In the former case, we have that $x \in N$ because $x \in Z(a)$, which is the intersection of all maximal ideals containing $a$. In the latter case, $x^n \in N$, and hence $x \in N$, by primeness. We have thus shown that $x$ belongs to every maximal ideal containing $au$, so $x \in Z(au)$, establishing the stated containment. Now,

$$J \cap k_0(I) = J \cap \bigvee \{Z(a) \mid a \in I\} = \bigvee \{J \cap Z(a) \mid a \in I\},$$

and since $J = \bigvee \{[u] \mid u \in J\}$, we have, for any $a \in I$,

$$Z(a) \cap J = Z(a) \cap \bigvee \{[u] \mid u \in J\}$$

$$= \bigvee \{Z(a) \cap [u] \mid u \in J\}$$

$$\subseteq \bigvee \{Z(au) \mid u \in J\}$$

$$\subseteq \bigvee \{Z(t) \mid t \in I \cap J\}$$

$$= k_0(I \cap J),$$

which leads to $J \cap k_0(I) \subseteq k_0(I \cap J)$, thus showing that $k_0$ is a prenucleus. Consequently, $\text{Fix}(k_0)$ is a frame. Next, we show that $\text{Fix}(k_0) = \text{ZId}(A)$. Let $I$ be a $z$-ideal in $A$. Then $Z(a) \subseteq I$ for each $a \in I$, which then shows that $I$ is an upper bound for the set $\{Z(a) \mid a \in I\}$. Let $J$ be the supremum of this set. For any $a \in I$, $Z(a) \subseteq J$, and so $a \in J$, showing that $I \subseteq J$. Therefore $I = \bigvee \{Z(a) \mid a \in I\} = k_0(I)$. Thus, $\text{ZId}(A) \subseteq \text{Fix}(k_0)$. For the other inclusion, if $I = k_0(I)$, then $Z(a) \subseteq I$ for every $a \in I$, which says $I$ is a $z$-ideal. So $\text{Fix}(k_0) \subseteq \text{ZId}(A)$, and hence equality follows. \qed
We point out that the join in $\text{ZId}(A)$ is the smallest $z$-ideal containing the sum.

With an aim to show that $\text{ZId}(A)$ is a coherent frame, we now describe the compact elements of this frame.

**Lemma 3.2.** For any commutative ring $A$, the compact elements of $\text{ZId}(A)$ are precisely the ideals of the form $\text{Z}(a_1) \lor \cdots \lor \text{Z}(a_n)$, for some finitely many elements $a_i \in A$. That is,

$$\mathcal{t}(\text{ZId}(A)) = \{ \text{Z}(a_1) \lor \cdots \lor \text{Z}(a_k) \mid a_1, \ldots, a_k \in A \}.$$  

**Proof.** To show that each $\text{Z}(a_1) \lor \cdots \lor \text{Z}(a_k)$ is compact, we first show that each $\text{Z}(a)$ is compact. Consider a directed collection $\{I_a \mid a \in \Gamma\} \subseteq \text{ZId}(A)$ with $\text{Z}(a) \leq \bigvee \{I_a \mid a \in \Gamma\}$. Then $\text{Z}(a) \subseteq \bigcup \{I_a \mid a \in \Gamma\}$. Since $a \in \text{Z}(a)$, we have that $a \in I_\beta$ for some $\beta \in \Gamma$, which then implies $\text{Z}(a) \subseteq I_\beta$ since $I_\beta$ is a $z$-ideal. Therefore $\text{Z}(a)$ is compact, and hence $\text{Z}(a_1) \lor \cdots \lor \text{Z}(a_k)$ is compact. On the other hand, let $K \in \mathcal{t}(\text{ZId}(A))$. Since $K = \bigvee \{\text{Z}(a) \mid a \in K\}$, and since $K$ is compact, we can find finitely many $a_1, \ldots, a_n \in A$ such that $K = \text{Z}(a_1) \lor \cdots \lor \text{Z}(a_n)$. This proves the lemma. \hfill $\square$

The goal is to show that $\text{ZId}(A)$ is a coherent frame. We thus need to be able to describe the meet of two compact elements. For that we need the following lemma.

**Lemma 3.3.** For any $a, b \in A$, $\text{Z}(a) \land \text{Z}(b) = \text{Z}(ab)$.

**Proof.** Since $\mathfrak{z}(a) \subseteq \mathfrak{z}(ab)$, it follows that $\text{Z}(ab) \subseteq \text{Z}(a)$, and similarly, $\text{Z}(ab) \subseteq \text{Z}(b)$. Therefore, $\text{Z}(ab) \subseteq \text{Z}(a) \land \text{Z}(b)$. For the other inclusion, observe that since maximal ideals are prime,

$$\mathfrak{z}(ab) = \mathfrak{z}(a) \lor \mathfrak{z}(b),$$

which then shows that $\text{Z}(a) \land \text{Z}(b) \subseteq \text{Z}(ab)$. \hfill $\square$

Now, let us recall from [1] that if $k : L \to L$ is a nucleus such that Fix$(k)$ is closed under directed joins calculated in $L$, then Fix$(k)$ is compact if $L$ is compact. To apply this to $\text{ZId}(A)$, let $k : \text{RId}(A) \to \text{RId}(A)$ be the nucleus induced by the prenucleus $k_0$ defined in Theorem 3.1. For later use, we remark that $k(I)$ is the smallest $z$-ideal containing $I$ because

$$k(I) = \bigwedge \{J \in \text{ZId}(A) \mid I \leq J = k_0(J)\} = \bigcap \{J \in \text{ZId}(A) \mid J \supseteq I\}.$$  

**Theorem 3.4.** $\text{ZId}(A)$ is a coherent frame.

**Proof.** We show first that $\text{ZId}(A)$ is compact. Since $\text{RId}(A)$ is compact, we may apply the criterion cited above from [1]. So let $\{I_a \mid a \in \Gamma\}$ be a directed collection of $z$-ideals. The join of this collection considered in $\text{RId}(A)$ is just the union $\bigcup_a I_a$. But clearly any directed union of $z$-ideals is a $z$-ideal. Thus, Fix$(k)$ is closed under directed joins taken in $\text{RId}(A)$, which then proves that $\text{ZId}(A)$ is compact.

To see coherence, let $K_1, K_2 \in \mathcal{t}(\text{ZId}(A))$ with, say,

$$K_1 = \text{Z}(a_1) \lor \cdots \lor \text{Z}(a_k) \quad \text{and} \quad K_2 = \text{Z}(b_1) \lor \cdots \lor \text{Z}(b_n).$$

Then

$$K_1 \land K_2 = \left(\text{Z}(a_1) \land \text{Z}(b_1)\right) \lor \cdots \lor \left(\text{Z}(a_k) \land \text{Z}(b_n)\right) \lor \cdots \lor \left(\text{Z}(a_k) \land \text{Z}(b_n)\right) = \text{Z}(a_1b_1) \lor \cdots \lor \text{Z}(a_kb_n),$$
which, in view of the previous lemma, implies \( K_1 \wedge K_2 \) is compact. Therefore, \( Z\text{Id}(A) \) is a coherent frame.

Since \( Z\text{Id}(A) \) is the fix-set of some nucleus on \( R\text{Id}(A) \), we have that \( Z\text{Id}(A) \) is a quotient of \( R\text{Id}(A) \). We can actually say more. Recall that a frame homomorphism \( h: L \rightarrow M \) is dense if, for any \( x \in L \), \( h(x) = 0 \) implies \( x = 0 \). This is equivalent to saying \( h_+(0) = 0 \), where \( h_+ \) denotes the right adjoint of \( h \). On the other hand, \( h \) is codense if, for any \( x \in L \), \( h(x) = 1 \) implies \( x = 1 \). We have remarked that the nucleus \( k: R\text{Id}(A) \rightarrow R\text{Id}(A) \) for which \( Z\text{Id}(A) = \text{Fix}(k) \) sends a radical ideal \( I \) to the intersection of the \( z \)-ideals containing \( I \). Let us denote by

\[
\kappa: R\text{Id}(A) \rightarrow Z\text{Id}(A)
\]

the frame homomorphism induced by \( k \).

**Proposition 3.5.** Let \( A \) be a ring and \( \kappa: R\text{Id}(A) \rightarrow Z\text{Id}(A) \) be the homomorphism above.

(a) \( \kappa \) is a coherent map.
(b) \( \kappa \) is codense.
(c) If \( A \) has zero Jacobson radical, then \( \kappa \) is dense.

**Proof.** (a) To prove that \( \kappa \) is a coherent map, it suffices to show that \( \kappa([a]) \in \mathfrak{I}(Z\text{Id}(A)) \), for each \( a \in A \). Since a \( z \)-ideal contains \( a \) if and only if it contains \([a]\), it is easy to see that \( \kappa([a]) = Z(a) \). Therefore, \( \kappa \) is a coherent map.

(b) Let \( I \in R\text{Id}(A) \) be such that \( \kappa(I) = 1_{Z\text{Id}(A)} \). Since maximal ideals are \( z \)-ideals, this implies \( I \) is contained in no proper ideal of \( A \), and hence \( I \) is the whole ring \( A \). This shows that \( \kappa \) is codense.

(c) If \( A \) has zero Jacobson radical, then in both \( R\text{Id}(A) \) and \( Z\text{Id}(A) \) the bottom element is the zero ideal of \( A \). Since the right adjoint of \( \kappa \) is the inclusion map \( Z\text{Id}(A) \rightarrow R\text{Id}(A) \), it follows that \( \kappa \) is dense. \( \square \)

An upshot of items (b) and (c) in this proposition is the following result. Recall that a frame \( L \) is normal if whenever \( a \lor b = 1 \), there exist \( u \) and \( v \) in \( L \) such that

\[
u \land v = 0 \quad \text{and} \quad u \lor a = 1 = v \lor b.
\]

It is routine to verify that if \( h: L \rightarrow M \) is a surjective, dense and codense frame homomorphism, then \( L \) is normal if and only if \( M \) is normal. Now, in [2, Proposition 1], Banaschewski proves that \( R\text{Id}(A) \) is normal if and only if \( A \) is a Gelfand ring. We therefore have the following corollary.

**Corollary 3.6.** For any ring \( A \) with zero Jacobson radical, \( Z\text{Id}(A) \) is normal iff \( A \) is a Gelfand ring.

Other “piggyback” results based on Banaschewski’s theorems in [2] concern von Neumann regular rings and clean rings. The reader will recall that \( A \) is a clean ring if for each \( a \in A \), there is an idempotent \( e \in A \) such that \( a + e \) is invertible. (In the commutative case these rings also go by the name of exchange rings. For a nice history of clean rings see [18]). In [2], Banaschewski calls a frame \( L \) weakly zero-dimensional if whenever \( a \lor b = 1 \) in \( L \), there exists a complemented \( c \in L \) such that \( c \leq a \) and \( c' \leq b \). He then proves in [2, Proposition 2] that \( A \) is a clean ring if and only if \( R\text{Id}(A) \) is weakly zero-dimensional. Another corollary to Proposition 3.5 is the following result.

**Corollary 3.7.** Let \( A \) be a ring with zero Jacobson radical.

(a) \( A \) is a clean ring if and only if \( Z\text{Id}(A) \) is weakly zero-dimensional.
(b) \( A \) is von Neumann regular if and only if \( Z\text{Id}(A) \) is a regular frame.
It is rather interesting that we can characterize local rings in terms of their frames of \( z \)-ideals, as we show below and more generally. Note that the bottom element of \( \text{ZId}(A) \) is of course the Jacobson radical of \( A \).

**Proposition 3.8.** A ring \( A \) is local iff \( \text{ZId}(A) \) is a two-element Boolean algebra.

We prove this is a more general setting.

**Proposition 3.9.** A ring \( A \) has exactly \( n \) maximal ideals iff \( \text{ZId}(A) \) is isomorphic to \( 2^n \) as a Boolean algebra.

**Proof.** Clearly, if \( \text{ZId}(A) \) is isomorphic to \( 2^n \), then there are exactly \( n \) many maximal elements of \( \text{ZId}(A) \) and hence \( n \) many maximal ideals of \( A \).

Conversely, suppose \( A \) has exactly \( n \) many maximal ideals. By the Chinese Remainder Theorem, for any \( M_1, M_2, \ldots, M_t \in \text{Max}(A) \),

\[
M_1 \cap \cdots \cap M_t = M_1 M_2 \cdots M_t
\]

and so by primality, each subset of \( \text{Max}(A) \) produces a unique \( z \)-ideal. Next, let \( J \) be a (proper) \( z \)-ideal and let \( M_1, M_2, \ldots, M_t \in \text{Max}(A) \) be the collection of maximal ideals containing \( J \). Clearly, \( J \subseteq M_1 \cap \cdots \cap M_t \). Also, for any element \( a \in M_1 \cap \cdots \cap M_t \), \( Z(a) \subseteq J \) and so \( a \in J \) since \( J \) is a \( z \)-ideal.

\[\square\]

We close this section by describing the prime elements of \( \text{ZId}(A) \), denoted by \( \text{Spec}(\text{ZId}(A)) \). As is well known (and not difficult to show), the prime elements of \( \text{RId}(A) \) are precisely the prime ideals of \( A \). Also, the primes of any sublocale of a frame \( L \) are exactly the primes of \( L \) that belong to the sublocale. We consequently have the following.

**Proposition 3.10.** The primes of \( \text{ZId}(A) \) are precisely the prime \( z \)-ideals of \( A \).

### 4. When \( \text{ZId}(A) \) is a Yosida Frame

Recall from [15] that a *Yosida frame* is an algebraic frame in which every compact element is a meet of maximal elements. If the frame in question has FIP, then it is Yosida if and only if for each pair of compact elements \( a < b \), there is a \( z \) (not necessarily compact) such that \( a \vee z < 1 = b \vee z \). Our goal in this section is to characterize the rings whose frames of \( z \)-ideals are Yosida frames. Towards that end, we need to identify the \( z \)-elements (we will recall the definition shortly) of \( \text{ZId}(A) \). We will see that they coincide with the \( z \)-elements of \( \text{RId}(A) \). First, some notation.

For a ring \( A \) and any \( F \subseteq A \), we set

\[
3(F) = \{ M \in \text{Max}(A) \mid F \subseteq M \}.
\]

For a finite set \( \{a_1, \ldots, a_n\} \), we abbreviate \( 3(\{a_1, \ldots, a_n\}) \) as \( 3(a_1, \ldots, a_n) \).

**Definition 4.1.** An ideal \( I \) of \( A \) is a Martínez-Zenk ideal (abbreviated \( mz \)-ideal) if for any finite \( F \subseteq A \) and \( a \in A \), \( 3(F) = 3(a) \) and \( F \subseteq I \) imply \( a \in I \).

**Example 4.2.** We see immediately that every \( mz \)-ideal is a \( z \)-ideal, and hence a radical ideal. Also, annihilator ideals are \( mz \)-ideals, as well as minimal prime ideals in rings with zero Jacobson radicals. Given a finite set \( F = \{a_1, a_2, \ldots, a_n\} \subseteq A \), the \( mz \)-ideal generated by \( F \) is the strong \( z \)-ideal generated by them, namely \( Z(F) \). However, in general it is possible that the \( z \)-ideal generated by \( F \) is smaller than \( Z(F) \). Unfortunately, we do not have an example of a \( z \)-ideal which is not an \( mz \)-ideal.
The following reformulations of the definition, which we record as a lemma, will be useful. They parallel the analogous characterizations of $z$-ideals (see [16]).

**Lemma 4.3.** The following are equivalent for an ideal $I$ of a ring $A$.

1. $I$ is an $mz$-ideal.
2. For any finite $F \subseteq I$ and any $a \in A$, $3(F) \subseteq 3(a)$ implies $a \in I$.
3. For any finite $F \subseteq I$, $\cap 3(F) \subseteq I$.

**Proof.** (1) $\Rightarrow$ (2): Assume that $I$ is an $mz$-ideal. Let $\{a_1, \ldots, a_n\} \subseteq I$ and $a \in A$ be such that $3(a_1, \ldots, a_n) \subseteq 3(a)$. We claim that $3(aa_1, \ldots, aa_n) = 3(a)$. The containment is immediate. For the other, let $M \in 3(aa_1, \ldots, aa_n)$. If each $a_i \in M$, then $M \in 3(a_1, \ldots, a_n) \subseteq 3(a)$. If some $a_k \notin M$, then the fact that $aa_k \in M$ implies $a \in M$, by primeness, so that $M \in 3(a)$. Now, since $\{aa_1, \ldots, aa_n\} \subseteq I$ and $I$ is an $mz$-ideal, by hypothesis, it follows that $a \in I$.

(2) $\Rightarrow$ (3): Let $F$ be a finite set with $F \subseteq I$, and let $a \in \cap 3(F)$. If $M \in 3(F)$, then $a \in M$; and so $3(F) \subseteq 3(a)$. It, therefore, follows from (2) that $a \in I$, which then proves that $\cap 3(F) \subseteq I$.

(3) $\Rightarrow$ (1): This follows from the fact that, for any $a \in A$, $a \in \cap 3(a)$. \hfill \Box

Condition (3) in this lemma makes it particularly apparent that every maximal ideal is an $mz$-ideal. Regarding $z$-elements, we shall use the description in [14, Definition & Remarks 6.3] since we are dealing with compact algebraic frames. For a compact algebraic frame $L$, the nucleus $ar : L \rightarrow L$ is given by

$$ar(x) = \bigwedge \{m \in \text{Max}(L) | x \leq m\}.$$

In particular, for any finite set $\{a_1, \ldots, a_n\} \subseteq A$, if we let $K$ and $H$ be the compact elements

$$K = [a_1] \lor \cdots \lor [a_n] \quad \text{and} \quad H = Z(a_1) \lor \cdots \lor Z(a_n)$$

de RId$(A)$ and $ZId(A)$, respectively, then

$$ar(K) = \bigcap Z(a_1, \ldots, a_n) = ar(H).$$

The $z$-nucleus on $L$ is defined by

$$z(x) = \bigvee \{ar(c) | c \in t(L), c \leq x\},$$

and an $x \in L$ is called a $z$-element if $z(x) = x$.

**Theorem 4.4.** For any ring $A$,

$$z(\text{RId}(A)) = z(\text{ZId}(A)) = \{I \subseteq A | I \text{ is an } mz\text{-ideal in } A\}.$$

**Proof.** We prove the equality of the first and last sets displayed above, and indicate how the equality of the second and last sets follow similarly. For brevity, let us write $MId(A)$ for the lattice of $mz$-ideals of $A$. Suppose that $I \in z(\text{RId}(A))$. Then,

$$I = \bigvee_{\text{ZId}(A)} \{ar(K) | K \in t(\text{RId}(A)), K \subseteq I\}.$$

Let $F = \{a_1, \ldots, a_n\} \subseteq I$. To show that $I$ is an $mz$-ideal, it suffices, by Lemma 4.3, to prove that $\cap 3(F) \subseteq I$. Put $K = [a_1] \lor \cdots \lor [a_n]$, and note that $K \subseteq I$. Now, as observed above, $ar(K) = \bigcap 3(F)$, which then implies $\cap 3(F) \subseteq I$ since $ar(K) \subseteq I$. Therefore, $I$ is an $mz$-ideal. Thus, $z(\text{RId}(A)) \subseteq MId(A)$.

On the other hand, let $J \in MId(A)$. For any $a \in J$, $a \in ar([a])$, which implies

$$J \subseteq \bigvee \{ar([a]) | a \in J\} \subseteq \bigvee \{ar(K) | K \in t(\text{RId}(A)), K \subseteq J\}.$$
But now if $K$ is a compact element in $\text{RId}(A)$ and $K \subseteq J$, then $\text{ar}(K) \subseteq J$ since $J$ is an $mz$-ideal. We therefore have

$$\bigvee \{ \text{ar}(K) \mid K \in \mathfrak{t}(\text{RId}(A)), K \subseteq J \} \subseteq J,$$

and consequently

$$J = \bigvee \{ \text{ar}(K) \mid K \in \mathfrak{t}(\text{RId}(A)), K \subseteq J \},$$

which says $J \in z(\text{RId}(A))$. This proves that $\text{MId}(A) \subseteq z(\text{RId}(A))$, and hence we have the claimed equality.

The equality $z(\text{ZId}(A)) = \text{MId}(A)$ is proved similarly, by replacing $K = [a_1] \cup \cdots \cup [a_n]$ with $K = Z(a_1) \cup \cdots \cup Z(a_n)$, and $[a]$ with $Z(a)$. \hfill \Box

We consequently have the following commutative diagram, where the horizontal arrow is the homomorphism $\kappa: \text{RId}(A) \to \text{ZId}(A)$, considered earlier, and the vertical arrows are the homomorphisms induced by the respective $z$-nuclei.

$$\begin{array}{ccc}
\text{RId}(A) & \longrightarrow & \text{ZId}(A) \\
\downarrow & & \downarrow \\
\text{z(\text{RId}(A))} & \longrightarrow & \text{z(\text{ZId}(A))}
\end{array}$$

Now let us recall [15, Proposition 2.5(a)], which says a compact algebraic frame $L$ is Yosida if and only if $L = zL$. Applying it to $\text{ZId}(A)$, and taking into account the foregoing theorem, we have the following result.

**Corollary 4.5.** The following statements are equivalent.

1. $\text{ZId}(A)$ is Yosida.
2. $\text{ZId}(A) = z(\text{RId}(A))$.
3. Every $z$-ideal of $A$ is an $mz$-ideal.
4. Every prime $z$-ideal is an $mz$-ideal.
5. For every finite set $F \subseteq A$, $\{Z(a) \mid a \in F\} = \bigcap \mathfrak{z}(F)$.

**Proof.** The equivalences (1) $\iff$ (2) $\iff$ (3) follow from Theorem 4.4. The implication (3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (3): Suppose every prime $z$-ideal is an $mz$-ideal. Let $I$ be a $z$-ideal. Since $\text{ZId}(A)$ is spatial, then, by Proposition 3.10,

$$I = \bigwedge \{ P \in \text{Spec}(\text{ZId}(A)) \mid I \subseteq P \} = \bigcap \{ P \in \text{Spec}(\text{ZId}(A)) \mid I \subseteq P \}.$$

Since each prime $z$-ideal is an $mz$-ideal, by hypothesis, and intersections of $mz$-ideals are $mz$-ideals, we have that $I$ is an intersection of $mz$-ideals. Therefore $I$ is an $mz$-ideal.

(1) $\Rightarrow$ (5): Assume that $\text{ZId}(A)$ is Yosida. Let $F = \{a_1, \ldots, a_n\} \subseteq A$. Observe that $\mathfrak{z}(F) \subseteq \mathfrak{z}(a_i)$ for each $i$, so that $Z(a_i) \subseteq \bigcap \mathfrak{z}(F)$, and hence $Z(a_1) \cup \cdots \cup Z(a_n) \subseteq \bigcap \mathfrak{z}(F)$. Suppose, by way of contradiction, that $\bigcap \mathfrak{z}(F) \not\subseteq \bigvee \mathcal{Z}(a_i)$. Then take $x \in \bigcap \mathfrak{z}(F) \setminus \bigvee \mathcal{Z}(a_i)$, and put

$$J = Z(a_1) \cup \cdots \cup Z(a_n) \cup Z(x).$$

Then $J$ is a compact element of $\text{ZId}(A)$ with $Z(a_1) \cup \cdots \cup Z(a_n) < J$. Since $\text{ZId}(A)$ is Yosida, there is an $H \in \text{ZId}(A)$ such that

$$Z(a_1) \cup \cdots \cup Z(a_n) \cup H < Z(a_1) \cup \cdots \cup Z(a_n) \cup Z(x) \cup H = A.$$
Thus, $Z(a_1) \lor \cdots \lor Z(a_n) \lor H$ is a proper ideal in $A$, and so there is a maximal ideal $M$ in $A$ with $Z(a_1) \lor \cdots \lor Z(a_n) \lor H \subseteq M$. Consequently, $F \subseteq M$, which then implies $x \in M$ since $\bigcap \mathfrak{r}(F) \subseteq M$. But now $x \in M$ implies $J \subseteq M$, and hence $M = A$ since $M \supseteq H$ as well. This contradiction proves that $\bigcap \mathfrak{r}(F) \subseteq Z(a_1) \lor \cdots \lor Z(a_n)$, and so we have the claimed equality.

(5) $\Rightarrow$ (2): Clearly, condition (5) implies $K = \text{ar}(K)$, for each $K \in \mathfrak{t}(Z\text{Id}(A))$. Thus, for any $I \in Z\text{Id}(A)$,

$$z(I) = \bigvee \{\text{ar}(K) \mid K \in \mathfrak{t}(Z\text{Id}(A)), K \subseteq I\} = \bigvee \{K \in \mathfrak{t}(Z\text{Id}(A)), K \subseteq I\} = I,$$

which says $z(Z\text{Id}(A)) = Z\text{Id}(A)$.

\[ \square \]

**Remark 4.6.** Examples of rings whose frames of $z$-ideals are Yosida frames abound. By [5, Theorem 3.5], they include all reduced $f$-rings with bounded inversion. Recall that a Bézout ring is a ring in which every finitely generated ideal is principal. It is easy to see that in a Bézout ring every $z$-ideal is an $mz$-ideal. Consequently, Bézout rings are also of this type.

Theorem 4.4 tells us which ideals of $A$ are fixed by the $z$-nucleus on $R\text{Id}(A)$. We now describe the ideals fixed by the $\text{ar}$-nucleus on $R\text{Id}(A)$. In a general algebraic frame $L$, $\text{Fix}(\text{ar})$ is denoted by $\mathfrak{a}^\uparrow(L)$. As in [16], we say an ideal of a ring $A$ is a strong $z$-ideal if it is an intersection of maximal ideals.

**Proposition 4.7.** For any ring $A$,

$$\mathfrak{a}^\uparrow(R\text{Id}(A)) = \mathfrak{a}^\uparrow(Z\text{Id}(A)) = \{I \subseteq A \mid I$ is a strong $z$-ideal\}.

Recall that a Hilbert ring is a ring in which every prime ideal is an intersection of maximal ideals. Every $I \in R\text{Id}(A)$ is a meet of primes; that is, is an intersection of prime ideals of $A$. Thus, if $A$ is a Hilbert ring, then every $I \in R\text{Id}(A)$ is an intersection of maximal ideals. The converse holds as well, as one sees immediately. Now, based on the fact that $\mathfrak{a}^\uparrow(Z\text{Id}(A)) \subseteq z(Z\text{Id}(A)) \subseteq Z\text{Id}(A) \subseteq R\text{Id}(A)$.

We therefore, have the following corollary, which we state in ring-theoretic terms.

**Corollary 4.8.** The following are equivalent for a ring $A$.

1. $A$ is a Hilbert ring.
2. Every radical ideal of $A$ is a strong $z$-ideal.
3. Every $z$-ideal of $A$ is a strong $z$-ideal.
4. Every $mz$-ideal of $A$ is a strong $z$-ideal.

We end this section with an example (within Hilbert rings, no less) that shows that the “operator” $3(\cdot)$ does not treat singletons with regard to intersections as it does with regard to unions. More precisely, we observed in the course of the proof of Lemma 3.3 that $3(a) \cup 3(b) = 3(ab)$, for all elements $a$ and $b$. The result fails if we replace $\cup$ with $\cap$, as the example below shows.

**Example 4.9.** It is known that if $A$ is a Hilbert ring then so is the polynomial ring $A[x]$. It follows that the ring $A = \mathbb{C}[X,Y]$ is a Hilbert ring. Interestingly, this ring has the property that there are $f, g \in A$ such that $3(f) \cap 3(g)$ is not of the form $3(h)$ for any $h \in A$.

The reader is invited to also contrast this with the result proved in Lemma 3.3, regarding the “operator” $Z(\cdot)$, stating that for any $a$ and $b$, there is a $c$ (in fact, $c = ab$) such that $Z(a) \cap Z(b) = Z(c)$. 


5. ON PROJECTABILITY PROPERTIES

In this section we seek conditions on \( A \) that make \( Z\text{Id}(A) \) satisfy the various projectability properties, such as those described in [12]. We shall assume that all rings in this section have zero Jacobson radical, so that the zero ideal is a \( z \)-ideal, and hence for any ideal \( I \), \( \text{Ann}(I) \) is a \( z \)-ideal by [16, Proposition 1.3]. Note, in particular, that for any \( I \in Z\text{Id}(A) \), the pseudocomplement of \( I \) is \( \text{Ann}(I) \).

Projectability conditions involve pseudocomplements and double pseudocomplements of compact elements. So we start by describing them. First, we deal with the simplest compact elements, namely, the ideals \( Z(a) \).

**Lemma 5.1.** If \( A \) has zero Jacobson radical, then for any \( a \in A \), \( \text{Ann}(Z(a)) = \text{Ann}(a) \), and hence \( \text{Ann}^2(Z(a)) = \text{Ann}^2(a) \).

**Proof.** Since \( a \in Z(a) \), it is clear that \( \text{Ann}(Z(a)) \subseteq \text{Ann}(a) \). To show the reverse inclusion, let \( x \in \text{Ann}(a) \), so that \( xa = 0 \), and suppose, by way of contradiction, that \( x \) does not annihilate \( Z(a) \). Then pick \( r \in Z(a) \) such that \( xr \neq 0 \). Since \( A \) has zero Jacobson radical, this implies there is a maximal ideal, say \( N \), such that \( xr \notin N \). Consequently, \( x \notin N \). Since \( xa = 0 \in N \) and \( N \) is prime, we must have \( a \notin N \). Since \( r \in Z(a) \), which says \( r \) belongs to every maximal ideal that contains \( a \), we have \( r \in N \). But now this implies \( xr \in N \); and we have a contradiction. \( \square \)

**Corollary 5.2.** For any \( a_1, \ldots, a_n \in A \),
\[
\left( Z(a_1) \vee \cdots \vee Z(a_n) \right)^* = \text{Ann}(a_1, \ldots, a_n),
\]
and consequently,
\[
\left( Z(a_1) \vee \cdots \vee Z(a_n) \right)^{**} = \text{Ann}^2(a_1, \ldots, a_n).
\]

**Proof.** We need only prove the first part. Applying Lemma 5.1, we have
\[
\left( Z(a_1) \vee \cdots \vee Z(a_n) \right)^* = Z(a_1)^* \wedge \cdots \wedge Z(a_n)^* = \text{Ann}(a_1) \cap \cdots \cap \text{Ann}(a_n) = \text{Ann}(a_1, \ldots, a_n),
\]
which proves the result. \( \square \)

We recall that an algebraic frame \( L \) is called **projectable** if \( c^{**} \vee c^* = 1 \) for every \( c \in \mathfrak{t}(L) \). Recall that a ring \( A \) is **weak Baer** if for every \( a \in A \), \( \text{Ann}(a) \) is generated by an idempotent. This is equivalent to saying the annihilator of any finitely generated ideal is a principal ideal generated by an idempotent. As observed in [8, 4.2], \( \text{RId}(A) \) is projectable if and only if \( A \) is weak Baer. The same holds for \( Z\text{Id}(A) \), as we show below. In proving one of the equivalences, we shall use [6, Lemma 5.2], which ensures that if \( h: L \to M \) is a surjective dense and codense coherent map, then \( L \) is projectable if and only if \( M \) is projectable.

Let us observe the following about the frame of \( z \)-elements.

**Lemma 5.3.** Let \( L \) be an algebraic frame, and write \( z: L \to zL \) for the coherent map induced by the \( z \)-nucleus on \( L \).

(a) The homomorphism \( z: L \to zL \) is dense.

(b) If \( L \) is compact and each \( x < 1 \) in \( L \) is below some \( m \in \text{Max}(L) \), then \( z \) is codense.
Proof. (a) If \( z(x) = 0 \), then \( ar(c) = 0 \) for each compact \( c \leq x \). Since each \( c \leq ar(c) \), this says \( x = 0 \) in light of coherence of \( L \).

(b) Suppose \( z(x) = 1 \). By compactness, there are finitely many \( c_1, \ldots, c_n \in \mathfrak{t}(L) \), each below \( x \), such that \( ar(c_1) \lor \cdots \lor ar(c_n) = 1 \). Hence \( c = c_1 \lor \cdots \lor c_n \) is a compact element below \( x \) such that \( ar(c) = 1 \). Thus, by the hypothesis on \( L \), there is no maximal element above \( c \), which then implies \( c = 1 \), and hence \( x = 1 \). \]

The result that follows is an improvement of [6, Proposition 5.3].

**Proposition 5.4.** The following are equivalent for a ring \( A \) with zero Jacobson radical.

1. \( \text{ZId}(A) \) is projectable.
2. \( z(\text{RId}(A)) \) is projectable.
3. \( A \) is weak Baer.

**Proof.** (1) \( \iff \) (2): This follows from Lemma 5.3, since \( z : \text{ZId}(A) \to z(\text{ZId}(A)) \) is dense and codense.

(1) \( \iff \) (3): Suppose that \( \text{ZId}(A) \) is projectable, and let \( a \in A \). Then \( Z(a)^* \lor Z(a)^{**} = A \), from which, by Lemma 5.1, we deduce that \( \text{Ann}(a) + \text{Ann}^2(a) \) is contained in no maximal ideal of \( A \). Thus, \( \text{Ann}(a) + \text{Ann}^2(a) = A \), and hence \( \text{Ann}(a) \) is a direct summand since \( \text{Ann}(a) \cap \text{Ann}^2(a) = 0 \), as \( A \) is reduced. Therefore, \( A \) is weak Baer.

Conversely, suppose that \( A \) is weak Baer, and let \( K = Z(a_1) \lor \cdots \lor Z(a_n) \) be a compact element in \( \text{ZId}(A) \). By Corollary 5.2, \( K^* = \text{Ann}(a_1, \ldots, a_n) \), and since \( A \) is weak Baer, there is an idempotent \( e \in A \) such that \( K^* = \langle e \rangle \). We deduce from this (by an argument as above) that \( K^* \lor K^{**} = A \), showing that \( \text{ZId}(A) \) is projectable.

In [12], the authors call an algebraic frame \( L \) *feebly projectable* if whenever \( a \land b = 0 \) in \( \mathfrak{t}(L) \), there exists \( c \in \mathfrak{t}(L) \) such that \( c^* \lor c^{**} = 1 \), \( a \leq c^* \) and \( b \leq c^{**} \). They show that for an algebraic frame that has a dense compact element (such as a coherent frame), this is equivalent to requiring that whenever \( a \land b = 0 \in \mathfrak{t}(L) \), there exists a complemented \( c \in L \) such that \( a \leq c \) and \( b \leq c^* \). In [11], the authors call a ring \( A \) *feebly Baer* if whenever \( ab = 0 \) in \( A \), there is an idempotent \( e \in A \) such that \( a \in \langle e \rangle \) and \( b \in (1 - e) \). The proof of the lemma that follows is omitted since it is fairly routine if one takes into account the fact that a surjective dense and codense frame homomorphism preserves and reflects complementedness.

**Lemma 5.5.** If \( h : L \to M \) is a surjective dense and codense coherent map, then \( L \) is feebly projectable iff \( M \) is feebly projectable.

Recall that the join and the meet of finitely many complemented elements are complemented, and, in fact, if \( c_1, \ldots, c_n \) are complemented, then

\[
(c_1 \land \cdots \land c_n)^* = c_1^* \lor \cdots \lor c_n^*.
\]

Recall also that if \( A \) is a reduced ring (and, hence, if \( A \) has zero Jacobson radical), then the bottom of \( \text{RId}(A) \) is the zero ideal.

**Theorem 5.6.** The following conditions regarding a ring \( A \) with zero Jacobson radical are equivalent.

1. \( A \) is feebly Baer.
2. \( \text{RId}(A) \) is feebly projectable.
3. \( \text{ZId}(A) \) is feebly projectable.
4. \( z(\text{RId}(A)) \) is feebly projectable.
Proof. (1) ⇒ (2): Assume that $A$ is feebly Baer. Suppose $K$ and $H$ are two compact elements of $RId(A)$ with $K \land H = 0$. Then, by repeating elements if necessary, we may assume that there is a positive integer $n$ and elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ with

$$K = [a_1] \lor \ldots \lor [a_n] \quad \text{and} \quad [b_1] \lor \ldots \lor [b_n].$$

The equality $K \land H = 0$ implies that, for any pair $(i, j)$ of indices, $a_i b_j = 0$, and so there are idempotents $e_{ij}$ in $A$ with $a_i \in \langle e_{ij} \rangle$ and $b_j \in \langle 1 - e_{ij} \rangle$. We therefore have complemented elements $\gamma_{ij}$ in $RId(A)$ with the property exhibited in the matrix below, where $\alpha_i$ and $\beta_j$ abbreviate, respectively, $[a_i]$ and $[b_j]$.

$$
\begin{pmatrix}
\alpha_1 \leq \gamma_{11} & \leq & \gamma_{12} & \leq & \gamma_{1n} \\
\alpha_2 \leq \gamma_{21} & \leq & \gamma_{22} & \leq & \gamma_{2n} \\
\vdots & & \vdots & & \vdots \\
\alpha_n \leq \gamma_{n1} & \leq & \gamma_{n2} & \leq & \gamma_{nn}
\end{pmatrix}
$$

Now note that, for each index $i$ and each index $j$,

$$\alpha_i \leq \gamma_{i1} \land \gamma_{i2} \land \ldots \land \gamma_{in} \quad \text{and} \quad \beta_j \leq \gamma_{1j} \land \gamma_{2j} \ldots \land \gamma_{nj},$$

so that

$$
H = \beta_1 \lor \ldots \lor \beta_n \leq \left( \bigwedge_{i=1}^n \gamma_{1i} \right) \lor \ldots \lor \left( \bigwedge_{i=1}^n \gamma_{in} \right)
$$

and

$$K = \alpha_1 \lor \ldots \lor \alpha_n \leq \left( \bigvee_{i=1}^n \gamma_{i1} \right) \lor \ldots \lor \left( \bigvee_{i=1}^n \gamma_{in} \right) \quad \text{and} \quad \left( \bigvee_{i=1}^n \gamma_{i1} \right) \land \ldots \land \left( \bigvee_{i=1}^n \gamma_{in} \right) \leq \left( \bigvee_{i=1}^n \gamma_{i1} \right) \lor \ldots \lor \left( \bigvee_{i=1}^n \gamma_{in} \right).$$

Thus, the element $C = \left( \bigvee_{i=1}^n \gamma_{i1} \right) \lor \ldots \lor \left( \bigvee_{i=1}^n \gamma_{in} \right)$ is complemented in $RId(A)$ and satisfies $K \leq C$ and $H \leq C^*$. Therefore, $RId(A)$ is feebly projectable.

(2) ⇒ (1): Assume that $RId(A)$ is feebly projectable. Let $ab = 0$ in $A$. Then $[a] \land [b] = [ab] = 0_{RId(A)}$ in $t(RId(A))$. Thus, there is a complemented $C \in RId(A)$ such that $[a] \leq C$ and $[b] \leq C^*$. By [2, Lemma 1], there is an idempotent $e \in A$ such that $C = [e]$, and hence $C^* = [1 - e]$. From this we deduce easily that $a \in \langle e \rangle$ and $b \in \langle 1 - e \rangle$. Therefore, $A$ is feebly Baer.

(2) ⇔ (3): This follows from Proposition 3.5 and Lemma 5.5, since the homomorphism $\kappa : RId(A) \to ZId(A)$, defined just prior to Proposition 3.5 is dense and codense if $A$ has zero Jacobson radical.

(2) ⇔ (4): This follows from Lemmas 5.3 and 5.5, since $z : RId(A) \to z(RId(A))$ is dense and codense.

Remark 5.7. That $A$ is feebly Baer if and only if $RId(A)$ is feebly projectable does not require $A$ to have zero Jacobson radical. It holds for any reduced ring – which is all that is needed for the zero of $RId(A)$ to be the zero ideal of $A$. □
6. $Z\text{Id}$ is a Functor

Following the idea (but not the notation) in [7], we let $\text{CRng}_i$ denote the category whose objects are rings, and whose morphisms are ring homomorphisms $\phi : A \to B$ that contract $z$-ideals to $z$-ideals. We will show that the assignment $A \mapsto Z\text{Id}(A)$ is the object part of a functor $\text{CRng}_i \to \text{CohFrm}$, and that the association $A \mapsto 3_A : \text{RId} \to Z\text{Id}$ is a natural transformation. We shall need the following lemma.

**Lemma 6.1.** Let $\phi : A \to B$ be a ring homomorphism that contracts $z$-ideals to $z$-ideals. The mapping $\bar{\phi} : \mathfrak{t}(Z\text{Id}(A)) \to \mathfrak{t}(Z\text{Id}(B))$, defined by

$$\bar{\phi}(Z(a_1) \lor \cdots \lor Z(a_n)) = Z(\phi(a_1)) \lor \cdots \lor Z(\phi(a_n)),$$

is well defined, and is a lattice homomorphism.

**Proof.** (i) Let us show first that $\bar{\phi}$ is well defined. Consider elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ in $A$ such that $Z(a_1) \lor \cdots \lor Z(a_n) = Z(b_1) \lor \cdots \lor Z(b_k)$. We must show that

$$Z(\phi(a_1)) \lor \cdots \lor Z(\phi(a_n)) = Z(\phi(b_1)) \lor \cdots \lor Z(\phi(b_k)).$$

For brevity, put $J = Z(\phi(b_1)) \lor \cdots \lor Z(\phi(b_k))$. Now, for any index $i = 1, \ldots, k$ we have $\phi(b_i) \in J$, which implies $b_i \in \phi^{-1}[J]$. Since $\phi^{-1}[J]$ is a $z$-ideal, this implies

$$Z(b_1) \lor \cdots \lor Z(b_k) \subseteq \phi^{-1}[J],$$

and hence

$$Z(a_1) \lor \cdots \lor Z(a_n) \subseteq \phi^{-1}[J].$$

Consequently, for each index $j = 1, \ldots, n$, $a_i \in \phi^{-1}[J]$, so that $\phi(a_i) \in J$, whence we deduce that $Z(\phi(a_1)) \lor \cdots \lor Z(\phi(a_n)) \subseteq J$, that is,

$$Z(\phi(a_1)) \lor \cdots \lor Z(\phi(a_n)) \subseteq Z(\phi(b_1)) \lor \cdots \lor Z(\phi(b_k)).$$

A similar argument establishes the reverse inclusion; so we have the desired equality.

(ii) Now we show that $\bar{\phi}$ is a lattice homomorphism. Since for any ring $R$ the bottom and the top elements of $Z\text{Id}(R)$ are $Z(0)$ and $Z(1)$, it is immediate that $\bar{\phi}$ preserves the bottom and the top. If

$$K = Z(a_1) \lor \cdots \lor Z(a_n) \quad \text{and} \quad H = Z(b_1) \lor \cdots \lor Z(b_k)$$

are two elements of $\mathfrak{t}(Z\text{Id}(A))$, then, as shown in the proof of Theorem 3.4,

$$K \land H = Z(a_1 b_1) \lor \cdots \lor Z(a_n b_k) \lor \cdots \lor Z(a_n b_k).$$

Since $\phi(a_i b_j) = \phi(a_i) \phi(b_j)$ for any $a_i$ and $b_j$, it follows routinely that $\bar{\phi}(K \land H) = \bar{\phi}(K) \land \bar{\phi}(H)$. We also have

$$\bar{\phi}(K \lor H) = \bar{\phi}\left(Z(a_1) \lor \cdots \lor Z(a_n) \lor Z(b_1) \lor \cdots \lor Z(b_k)\right)$$

$$= \left(Z(\phi(a_1)) \lor \cdots \lor Z(\phi(a_n)) \lor Z(\phi(b_1)) \lor \cdots \lor Z(\phi(b_k))\right)$$

$$= \bar{\phi}(K) \lor \bar{\phi}(H),$$

and so $\bar{\phi}$ is a lattice homomorphism. \qed

Thus, by [10, p. 64], for any $\phi : A \to B$ in $\text{CRng}_i$, there is a unique coherent map $\phi : Z\text{Id}(A) \to Z\text{Id}(B)$ that extends $\bar{\phi}$. Explicitly, it is given by

$$\hat{\phi}(J) = \bigvee_{Z\text{Id}(B)} \{ Z(\phi(a)) \mid a \in J \}.$$
Remark 6.2. A comment regarding the need to restrict to homomorphisms that contract $z$-ideals to $z$-ideals is in order. Suppose $\phi: A \to B$ is a ring homomorphism that fails to contract all $z$-ideals to $z$-ideals. Then, by [16, Lemma 1.7], there is a maximal ideal $M$ in $B$ for which $\phi^{-1}(M)$ not a $z$-ideal in $A$. Then there exists $x, y \in A$ such that $3(x) = 3(y)$, $x \in \phi^{-1}(M)$, and $y \notin \phi^{-1}(M)$. Then $Z(x) = Z(y)$, but $Z(\phi(x)) \neq Z(\phi(y))$ since $M$ is a maximal ideal of $B$ containing $\phi(x)$ but not $\phi(y)$. Thus, for such a homomorphism, $\overline{\phi}$ as in Lemma 6.1 would not be well defined.

As in Proposition 3.5, for any $A \in \text{CRng}_3$, let $\kappa_A: \text{RId}(A) \to Z\text{Id}(A)$ be the coherent map that sends a radical ideal to the smallest $z$-ideal containing it. Recall that we showed in Proposition 3.5 that $\kappa_A([a]) = Z(a)$, for any $a \in A$. Recall also that, for any ring homomorphism $\phi: A \to B$, $\text{RId}(\phi)([a]) = [\phi(a)]$.

Proposition 6.3. The mappings $A \mapsto Z\text{Id}(A)$ and $\phi \mapsto \overline{\phi}$ define a functor $Z\text{Id}: \text{CRng}_3 \to \text{CohFrm}$ such that $A \mapsto \kappa_A: \text{RId} \to Z\text{Id}$ is a natural transformation.

Proof. That $Z\text{Id}$ preserves identity follows from the fact that $J = \bigvee \{Z(a) \mid a \in J\}$, for any $J \in Z\text{Id}(A)$. To see that $Z\text{Id}$ preserves composites, let $\phi: A \to B$ and $\tau: B \to C$ be morphisms in $\text{CRng}_3$. To check that $Z\text{Id}(\tau \cdot \phi) = Z\text{Id}(\tau) \cdot Z\text{Id}(\phi)$, we need only check that these two coherent maps agree on basic compact elements. Consider then any basic compact element $Z(a) \in \tau(Z\text{Id}(A))$, and observe that

$$\left( Z\text{Id}(\tau) \cdot Z\text{Id}(\phi) \right)(Z(a)) = Z\text{Id}(\tau)(Z(\phi(a))) = Z(\tau(\phi(a))) = Z\text{Id}(\tau \cdot \phi)(Z(a)).$$

Therefore, $Z\text{Id}: \text{CRng}_3 \to \text{CohFrm}$ is a functor.

To prove the claimed naturality, given $\phi: A \to B \in \text{CRng}_3$, we must show that the square

$$\begin{array}{ccc}
\text{RId}(A) & \xrightarrow{\kappa_A} & Z\text{Id}(A) \\
\text{RId}(\phi) \downarrow & & \downarrow Z\text{Id}(\phi) \\
\text{RId}(B) & \xrightarrow{\kappa_B} & Z\text{Id}(B)
\end{array}$$

commutes. For any basic compact element $[a] \in \tau(\text{RId}(A))$, we have

$$\left( Z\text{Id}(\phi) \cdot \kappa_A \right)([a]) = Z\text{Id}(\phi)(Z(a)) = Z(\phi(a)) = \kappa_B([\phi(a)]) = (\kappa_B \cdot \text{RId}(\phi))(\phi),$$

which then shows that $Z\text{Id}(\phi) \cdot \kappa_A = \kappa_B \cdot \text{RId}(\phi)$, by coherence. \hfill $\Box$

The functorial properties of $Z\text{Id}$ will be a subject of another occasion.

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REFERENCES