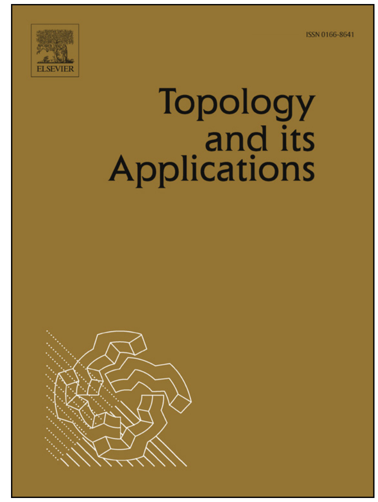

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ON THE LATTICE OF z-IDEALS OF A COMMUTATIVE RING

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This article is dedicated to Professor Aleš Pultr on his 80th birthday.

Abstract. We prove that the lattice of z-ideals of a commutative ring with identity is a coherent frame. We characterize when it is a Yosida frame, and when it satisfies some projectability properties. We also characterize Hilbert rings in terms of ideals that arise naturally in this study. A ring with zero Jacobson radical is shown to be feebly Baer precisely when its frame of z-ideals is feebly projectable. Denote by $\mathrm{ZId}(A)$ the frame of z-ideals of a ring A. We show that the assignment $A \mapsto \mathrm{ZId}(A)$ is the object part of a functor $\mathbf{CRng}_{\mathfrak{z}} \to \mathbf{CohFrm}$, where $\mathbf{CRng}_{\mathfrak{z}}$ designates the category whose objects are commutative rings with identity and whose morphisms are the ring homomorphisms that contract z-ideals to z-ideals.

1. Introduction

Throughout the paper, by "ring" we mean a commutative ring with identity $1 \neq 0$. Almost all our rings are *reduced*, which is to say they have no nonzero nilpotent elements. The history of z-ideals in rings is easy to record. The first usage of the term "z-ideal" was by Kohls [13] in the study of rings of continuous functions. He observed that they could be characterized purely algebraically. Modifying that characterization, it was Mason [16] who initiated the study of z-ideals of commutative rings in earnest.

The interest in lattices of z-ideals started with the paper of Martínez and Zenk [15], in which they prove that the lattice of z-ideals of the ring $C(X)$ is a frame. They actually proved that it is a coherently normal Yosida frame. In her doctoral thesis [9], the first-named author of the present article extended this result of Martínez and Zenk to lattices of z-ideals of the ring $\mathcal{R}L$ of continuous real-valued functions on a completely regular frame. This was further extended by Dube [5] to lattices of z-ideals of an f-ring with bounded inversion.

The present paper (which started as a question raised at the Ordered Algebraic Structures meeting held at Louisiana State University in 2016) significantly improves the earlier results. It is organized as follows.

We recall in Section 2 the necessary background, and we fix notation. In Section 3, we prove that the lattice, $\mathrm{ZId}(A)$, of z-ideals of any ring A is a frame (Theorem 3.1). We obtain this via a device called a "prenucleus" that was invented by Banaschewski [1]. The prenucleus in question is defined on the frame, $\mathrm{RId}(A)$, of radical ideals of A, and it turns out that the fixed elements of the prenucleus are precisely the z-ideals of A. After identifying the compact elements of $\mathrm{ZId}(A)$ (Lemma 3.2), we show that $\mathrm{ZId}(A)$ is a coherent frame (Theorem 3.4).

In [14], Martínez and Zenk introduce what they call a "z-nucleus" on an algebraic frame L. Its fixed elements are called the z-elements of L. Their proof that the lattice of z-ideals of $C(X)$, for X compact, is a frame actually shows that these ideals are the z-elements of the frame of convex ℓ-ideals of $C(X)$, when the latter is viewed as an ℓ-group. In Section 4

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we identify the z-elements of $\mathrm{RId}(A)$ (Theorem 4.4), and this enables us to characterize the rings A for which $\mathrm{ZId}(A)$ is a Yosida frame (Corollary 4.5).

The penultimate section deals with issues surrounding the notion of projectability – a property that is frequently considered in algebraic frames. We prove, among other things, that a ring with zero Jacobson radical is weak Baer if and only if its frame of z-ideals is projectable (Proposition 5.4), and that it is feebly Baer if and only if $\mathrm{RId}(A)$ is feebly Baer.

In the final section, we show that ZId can be made into a functor when we restrict the ring homomorphisms to those that contract z-ideals to z-ideals (Proposition 6.3). We give a comment (Remark 6.2) why this restriction is actually "forced" on us.

2. Background and Notation

2.1. **Algebraic frames.** Our reference for frames and their homomorphisms is [19]. Our notation is, to a large extent, standard. For instance, we denote the pseudocomplement of an element a by a^*, and the right adjoint of a frame homomorphism h by h_*.

Let L be a frame. An element $a \in L$ is *compact* if, for any $X \subseteq L$, $a \leq \bigvee X$ implies that there is a finite $Y \subseteq X$ with $a \leq \bigvee Y$. We denote by $\mathfrak{k}(L)$ the set of all compact elements of L. If every element of L is the join of compact elements below it, then L is said to be *algebraic*. If $a \wedge b \in \mathfrak{k}(L)$ for every $a, b \in \mathfrak{k}(L)$, then L is said to have the *finite intersection property*, throughout abbreviated as FIP. If the top element of L (which we shall denote by 1) is compact and L has FIP, then L is called *coherent*. A frame homomorphism between algebraic frames is called a *coherent map* if it maps compact elements to compact elements.

2.2. **Rings.** We write $\mathrm{Ann}(S)$ for the annihilator of $S \subseteq A$, and abbreviate $\mathrm{Ann}(\{a\})$ as $\mathrm{Ann}(a)$. The ideal generated by a single element a will be written as $\langle a \rangle$. The *radical* of an ideal I of A is the ideal

$$\sqrt{I} = \{x \in A \mid x^n \in I \text{ for some positive integer } n\}.$$

An ideal is called a *radical ideal* if it coincides with its radical. A convenient characterization is that

$$I \text{ is a radical ideal} \qquad \Longleftrightarrow \qquad a^2 \in I \text{ implies } a \in I.$$

The smallest radical ideal containing an element a is denoted by $[a]$. That is, $[a] = \sqrt{\langle a \rangle}$.

The lattice $\mathrm{RId}(A)$ of radical ideals of A, ordered by inclusion, is a coherent frame (see [2]). The meet in $\mathrm{RId}(A)$ is intersection, and the join is the radical of the sum. The principal radical ideal generated by $a \in A$ is denoted by $[a]$. The compact elements of $\mathrm{RId}(A)$ are the finitely generated radical ideals. The top element of $\mathrm{RId}(A)$ is the compact element $[1] = A$, and the bottom element is $[0]$, which is the zero ideal if A is reduced. We let $\mathrm{Max}(A)$ denote the set of maximal ideals of A.

For an element a of a ring A, we write $\mathfrak{Z}(a)$ and $Z(a)$, respectively, for the sets

$$\mathfrak{Z}(a) = \{M \in \mathrm{Max}(A) \mid a \in M\} \qquad \text{and} \qquad Z(a) = \bigcap \mathfrak{Z}(a).$$

An ideal I of A is a *z-ideal* if, for any $a, b \in A$

$$\mathfrak{Z}(a) = \mathfrak{Z}(b) \text{ and } a \in I \quad \Longrightarrow \quad b \in I.$$

Examples of z-ideals are maximal ideals and the minimal prime ideals of Jacobson semi-simple rings. Intersections of z-ideals are z-ideals. A useful characterization (which is known in the literature) is that I is a z-ideal if and only if $Z(a) \subseteq I$ for every $a \in I$. Observe, as well, that $Z(a)$ is the smallest z-ideal containing a.

A ring A is *Gelfand* if for any $a, b \in A$ with $a + b = 1$, there exist $r, s \in A$ such that $(1 + ar)(1 + bs) = 0$.

3. Frames of z-ideals

It has been established in a number of papers that the lattices of z-ideals of certain rings are algebraic frames. This was first done for the rings $C(X)$ [15], then for the rings $\mathcal{R}L$ [9], and the most recent case for f-rings with bounded inversion [5]. In the latter two cases, the result is obtained by showing that the lattice in question is a certain quotient of the frame of radical ideals and radical ℓ-ideals, respectively.

For the general case we utilize the notion of a "prenucleus"; a clever device invented by Banaschewski [1]. Let us recall that a *prenucleus* on a frame L is a map $k_0\colon L \to L$ such that, for all $x, y \in L$:

$$x \le k_0(x), \qquad x \le y \implies k_0(x) \le k_0(y), \qquad k_0(x) \wedge y \le k_0(x \wedge y).$$

The set $\mathrm{Fix}(k_0) = \{t \in L \mid k_0(t) = t\}$ is then a frame, and the mapping $k\colon L \to L$ given by

$$k(x) = \bigwedge\{t \in L \mid x \le t = k_0(t)\}$$

is a nucleus on L with $\mathrm{Fix}(k) = \mathrm{Fix}(k_0)$. We denote by $\mathrm{ZId}(A)$ the lattice of z-ideals of a ring A. Observe that for any $a \in A$, $Z(a) \in \mathrm{RId}(A)$.

Theorem 3.1. *For any commutative ring A, the lattice* $\mathrm{ZId}(A)$ *is a frame.*

Proof. Define a map $k_0\colon \mathrm{RId}(A) \to \mathrm{RId}(A)$ by

$$k_0(I) = \bigvee\{Z(a) \mid a \in I\}.$$

We claim that k_0 is a prenucleus on $\mathrm{RId}(A)$. It is clear that, for any $I, J \in \mathrm{RId}(A)$, $I \subseteq k_0(I)$, and $I \subseteq J$ implies $k_0(I) \subseteq k_0(J)$. Note that if $a \in I$ and $u \in J$, then $[u] \cap Z(a) \subseteq Z(au)$. To see this, let $x \in [u] \cap Z(a)$. Pick a positive integer n and $r \in A$ such that $x^n = ru$. Consider any maximal ideal N of A with $au \in N$. We aim to show that $x \in N$. Since N is a prime ideal (as it is maximal in a ring with identity), $a \in N$ or $u \in N$. In the former case, we have that $x \in N$ because $x \in Z(a)$, which is the intersection of all maximal ideals containing a. In the latter case, $x^n \in N$, and hence $x \in N$, by primeness. We have thus shown that x belongs to every maximal ideal containing au, so $x \in Z(au)$, establishing the stated containment. Now,

$$J \cap k_0(I) = J \cap \bigvee\{Z(a) \mid a \in I\} = \bigvee\{J \cap Z(a) \mid a \in I\},$$

and since $J = \bigvee\{[u] \mid u \in J\}$, we have, for any $a \in I$,

$$\begin{aligned}
Z(a) \cap J &= Z(a) \cap \bigvee\{[u] \mid u \in J\} \\
&= \bigvee\{Z(a) \cap [u] \mid u \in J\} \\
&\subseteq \bigvee\{Z(au) \mid u \in J\} && \text{by what we noted above} \\
&\subseteq \bigvee\{Z(t) \mid t \in I \cap J\} \\
&= k_0(I \cap J),
\end{aligned}$$

which leads to $J \cap k_0(I) \subseteq k_0(I \cap J)$, thus showing that k_0 is a prenucleus. Consequently, $\mathrm{Fix}(k_0)$ is a frame. Next, we show that $\mathrm{Fix}(k_0) = \mathrm{ZId}(A)$. Let I be a z-ideal in A. Then $Z(a) \subseteq I$ for each $a \in I$, which then shows that I is an upper bound for the set $\{Z(a) \mid a \in I\}$. Let J be the supremum of this set. For any $a \in I$, $Z(a) \subseteq J$, and so $a \in J$, showing that $I \subseteq J$. Therefore $I = \bigvee\{Z(a) \mid a \in I\} = k_0(I)$. Thus, $\mathrm{ZId}(A) \subseteq \mathrm{Fix}(k_0)$. For the other inclusion, if $I = k_0(I)$, then $Z(a) \subseteq I$ for every $a \in I$, which says I is a z-ideal. So $\mathrm{Fix}(k_0) \subseteq \mathrm{ZId}(A)$, and hence equality follows. $\square$

We point out that the join in $\mathrm{ZId}(A)$ is the smallest z-ideal containing the sum.

With an aim to show that $\mathrm{ZId}(A)$ is a coherent frame, we now describe the compact elements of this frame.

Lemma 3.2. *For any commutative ring A, the compact elements of $\mathrm{ZId}(A)$ are precisely the ideals of the form $Z(a_1) \vee \cdots \vee Z(a_n)$, for some finitely many elements $a_i \in A$. That is,*

$$\mathfrak{k}(\mathrm{ZId}(A)) = \{Z(a_1) \vee \cdots \vee Z(a_k) \mid a_1, \ldots, a_k \in A\}.$$

Proof. To show that each $Z(a_1) \vee \cdots \vee Z(a_k)$ is compact, we first show that each $Z(a)$ is compact. Consider a directed collection $\{I_\alpha \mid \alpha \in \Gamma\} \subseteq \mathrm{ZId}(A)$ with $Z(a) \leq \bigvee\{I_\alpha \mid \alpha \in \Gamma\}$. Then $Z(a) \subseteq \bigcup\{I_\alpha \mid \alpha \in \Gamma\}$. Since $a \in Z(a)$, we have that $a \in I_\beta$ for some $\beta \in \Gamma$, which then implies $Z(a) \subseteq I_\beta$ since I_β is a z-ideal. Therefore $Z(a)$ is compact, and hence $Z(a_1) \vee \cdots \vee Z(a_k)$ is compact. On the other hand, let $K \in \mathfrak{k}(\mathrm{ZId}(A))$. Since $K = \bigvee\{Z(a) \mid a \in K\}$, and since K is compact, we can find finitely many $a_1, \ldots, a_n \in A$ such that $K = Z(a_1) \vee \cdots \vee Z(a_n)$. This proves the lemma. □

The goal is to show that $\mathrm{ZId}(A)$ is a coherent frame. We thus need to be able to describe the meet of two compact elements. For that we need the following lemma.

Lemma 3.3. *For any $a, b \in A$, $Z(a) \cap Z(b) = Z(ab)$.*

Proof. Since $\mathfrak{Z}(a) \subseteq \mathfrak{Z}(ab)$, it follows that $Z(ab) \subseteq Z(a)$, and similarly, $Z(ab) \subseteq Z(b)$. Therefore, $Z(ab) \subseteq Z(a) \cap Z(b)$. For the other inclusion, observe that since maximal ideals are prime,

$$\mathfrak{Z}(ab) = \mathfrak{Z}(a) \cup \mathfrak{Z}(b),$$

which then shows that $Z(a) \cap Z(b) \subseteq Z(ab)$. □

Now, let us recall from [1] that if $k\colon L \to L$ is a nucleus such that $\mathrm{Fix}(k)$ is closed under directed joins calculated in L, then $\mathrm{Fix}(k)$ is compact if L is compact. To apply this to $\mathrm{ZId}(A)$, let $k\colon \mathrm{RId}(A) \to \mathrm{RId}(A)$ be the nucleus induced by the prenucleus k_0 defined in Theorem 3.1. For later use, we remark that $k(I)$ is the smallest z-ideal containing I because

$$\begin{aligned} k(I) &= \bigwedge\{J \in \mathrm{ZId}(A) \mid I \leq J = k_0(J)\} \\ &= \bigcap\{J \in \mathrm{ZId}(A) \mid J \supseteq I\}. \end{aligned}$$

Theorem 3.4. $\mathrm{ZId}(A)$ *is a coherent frame.*

Proof. We show first that $\mathrm{ZId}(A)$ is compact. Since $\mathrm{RId}(A)$ is compact, we may apply the criterion cited above from [1]. So let $\{I_\alpha \mid \alpha \in \Gamma\}$ be a directed collection of z-ideals. The join of this collection considered in $\mathrm{RId}(A)$ is just the union $\bigcup_\alpha I_\alpha$. But clearly any directed union of z-ideals is a z-ideal. Thus, $\mathrm{Fix}(k)$ is closed under directed joins taken in $\mathrm{RId}(A)$, which then proves that $\mathrm{ZId}(A)$ is compact.

To see coherence, let $K_1, K_2 \in \mathfrak{k}(\mathrm{ZId}(A))$ with, say,

$$K_1 = Z(a_1) \vee \cdots \vee Z(a_k) \quad \text{and} \quad K_2 = Z(b_1) \vee \cdots \vee Z(b_n).$$

Then

$$\begin{aligned} K_1 \wedge K_2 = \Big(Z(a_1) \cap Z(b_1)\Big) \vee \cdots \vee \Big(Z(a_1) \cap Z(b_n)\Big) \vee \cdots \vee \\ \Big(Z(a_k) \cap Z(b_1)\Big) \vee \cdots \vee \Big(Z(a_k) \cap Z(b_n)\Big) \\ = Z(a_1 b_1) \vee \cdots \vee Z(a_k b_n), \end{aligned}$$

which, in view of the previous lemma, implies $K_1 \wedge K_2$ is compact. Therefore, $\mathrm{ZId}(A)$ is a coherent frame. $\square$

Since $\mathrm{ZId}(A)$ is the fix-set of some nucleus on $\mathrm{RId}(A)$, we have that $\mathrm{ZId}(A)$ is a quotient of $\mathrm{RId}(A)$. We can actually say more. Recall that a frame homomorphism $h\colon L \to M$ is *dense* if, for any $x \in L$, $h(x) = 0$ implies $x = 0$. This is equivalent to saying $h_*(0) = 0$, where h_* denotes the right adjoint of h. On the other hand, h is *codense* if, for any $x \in L$, $h(x) = 1$ implies $x = 1$. We have remarked that the nucleus $k\colon \mathrm{RId}(A) \to \mathrm{RId}(A)$ for which $\mathrm{ZId}(A) = \mathrm{Fix}(k)$ sends a radical ideal I to the intersection of the z-ideals containing I. Let us denote by

$$\kappa\colon \mathrm{RId}(A) \to \mathrm{ZId}(A)$$

the frame homomorphism induced by k.

Proposition 3.5. *Let A be a ring and $\kappa\colon \mathrm{RId}(A) \to \mathrm{ZId}(A)$ be the homomorphism above.*

(a) *κ is a coherent map.*
(b) *κ is codense.*
(c) *If A has zero Jacobson radical, then κ is dense.*

Proof. (a) To prove that κ is a coherent map, it suffices to show that $\kappa([a]) \in \mathfrak{k}(\mathrm{ZId}(A))$, for each $a \in A$. Since a z-ideal contains a if and only if it contains $[a]$, it is easy to see that $\kappa([a]) = Z(a)$. Therefore, κ is a coherent map.

(b) Let $I \in \mathrm{RId}(A)$ be such that $\kappa(I) = 1_{\mathrm{ZId}(A)}$. Since maximal ideals are z-ideals, this implies I is contained in no proper ideal of A, and hence I is the whole ring A. This shows that κ is codense.

(c) If A has zero Jacobson radical, then in both $\mathrm{RId}(A)$ and $\mathrm{ZId}(A)$ the bottom element is the zero ideal of A. Since the right adjoint of κ is the inclusion map $\mathrm{ZId}(A) \hookrightarrow \mathrm{RId}(A)$, it follows that κ is dense. $\square$

An upshot of items (b) and (c) in this proposition is the following result. Recall that a frame L is *normal* if whenever $a \vee b = 1$, there exist u and v in L such that

$$u \wedge v = 0 \qquad \text{and} \qquad u \vee a = 1 = v \vee b.$$

It is routine to verify that if $h\colon L \to M$ is a surjective, dense and codense frame homomorphism, then L is normal if and only if M is normal. Now, in [2, Proposition 1], Banaschewski proves that $\mathrm{RId}(A)$ is normal if and only if A is a Gelfand ring. We therefore have the following corollary.

Corollary 3.6. *For any ring A with zero Jacobson radical, $\mathrm{ZId}(A)$ is normal iff A is a Gelfand ring.*

Other "piggyback" results based on Banaschewski's theorems in [2] concern von Neumann regular rings and clean rings. The reader will recall that A is a *clean ring* if for each $a \in A$, there is an idempotent $e \in A$ such that $a + e$ is invertible. (In the commutative case these rings also go by the name of exchange rings. For a nice history of clean rings see [18]). In [2], Banaschewski calls a frame L *weakly zero-dimensional* if whenever $a \vee b = 1$ in L, there exists a complemented $c \in L$ such that $c \leq a$ and $c^* \leq b$. He then proves in [2, Proposition 2] that A is a clean ring if and only if $\mathrm{RId}(A)$ is weakly zero-dimensional. Another corollary to Proposition 3.5 is the following result.

Corollary 3.7. *Let A be a ring with zero Jacobson radical.*

(a) *A is a clean ring if and only if $\mathrm{ZId}(A)$ is weakly zero-dimensional.*
(b) *A is von Neumann regular if and only if $\mathrm{ZId}(A)$ is a regular frame.*

It is rather interesting that we can characterize local rings in terms of their frames of z-ideals, as we show below and more generally. Note that the bottom element of $\mathrm{ZId}(A)$ is of course the Jacobson radical of A.

Proposition 3.8. *A ring A is local iff $\mathrm{ZId}(A)$ is a two-element Boolean algebra.*

We prove this is a more general setting.

Proposition 3.9. *A ring A has exactly n maximal ideals iff $\mathrm{ZId}(A)$ is isomorphic to 2^n as a Boolean algebra.*

Proof. Clearly, if $\mathrm{ZId}(A)$ is isomorphic to 2^n, then there are exactly n many maximal elements of $\mathrm{ZId}(A)$ and hence n many maximal ideals of A.

Conversely, suppose A has exactly n many maximal ideals. By the Chinese Remainder Theorem, for any $M_1, M_2, \cdots, M_t \in \mathrm{Max}(A)$,

$$M_1 \cap \cdots \cap M_t = M_1 M_2 \cdots M_t$$

and so by primality, each subset of $\mathrm{Max}(A)$ produces a unique z-ideal. Next, let J be a (proper) z-ideal and let $M_1, M_2, \cdots, M_t \in \mathrm{Max}(A)$ be the collection of maximal ideals containing J. Clearly, $J \subseteq M_1 \cap \cdots \cap M_t$. Also, for any element $a \in M_1 \cap \cdots \cap M_t$, $Z(a) \subseteq J$ and so $a \in J$ since J is a z-ideal..

□

We close this section by describing the prime elements of $\mathrm{ZId}(A)$, denoted by $\mathrm{Spec}(\mathrm{ZId}(A))$. As is well known (and not difficult to show), the prime elements of $\mathrm{RId}(A)$ are precisely the prime ideals of A. Also, the primes of any sublocale of a frame L are exactly the primes of L that belong to the sublocale. We consequently have the following.

Proposition 3.10. *The primes of $\mathrm{ZId}(A)$ are precisely the prime z-ideals of A.*

4. When $\mathrm{ZId}(A)$ is a Yosida frame

Recall from [15] that a *Yosida frame* is an algebraic frame in which every compact element is a meet of maximal elements. If the frame in question has FIP, then it is Yosida if and only if for each pair of compact elements $a < b$, there is a z (not necessarily compact) such that $a \vee z < 1 = b \vee z$. Our goal in this section is to characterize the rings whose frames of z-ideals are Yosida frames. Towards that end, we need to identify the z-elements (we will recall the definition shortly) of $\mathrm{ZId}(A)$. We will see that they coincide with the z-elements of $\mathrm{RId}(A)$. First, some notation.

For a ring A and any $F \subseteq A$, we set

$$\mathfrak{Z}(F) = \{M \in \mathrm{Max}(A) \mid F \subseteq M\}.$$

For a finite set $\{a_1, \ldots, a_n\}$, we abbreviate $\mathfrak{Z}(\{a_1, \ldots, a_n\})$ as $\mathfrak{Z}(a_1, \ldots, a_n)$.

Definition 4.1. An ideal I of A is a *Martínez-Zenk ideal* (abbreviated mz-ideal) if for any finite $F \subseteq A$ and $a \in A$, $\mathfrak{Z}(F) = \mathfrak{Z}(a)$ and $F \subseteq I$ imply $a \in I$.

Example 4.2. We see immediately that every mz-ideal is a z-ideal, and hence a radical ideal. Also, annihilator ideals are mz-ideals, as well as minimal prime ideals in rings with zero Jacobson radicals. Given a finite set $F = \{a_1, a_2, \cdots, a_n\} \subseteq A$, the mz-ideal generated by F is the strong z-ideal generated by them, namely $Z(F)$. However, in general it is possible that the z-ideal generated by F is smaller than $Z(F)$. Unfortunately, we do not have an example of a z-ideal which is not an mz-ideal.

The following reformulations of the definition, which we record as a lemma, will be useful. They parallel the analogous characterizations of z-ideals (see [16]).

Lemma 4.3. *The following are equivalent for an ideal I of a ring A.*

(1) *I is an mz-ideal.*
(2) *For any finite $F \subseteq I$ and any $a \in A$, $\mathfrak{Z}(F) \subseteq \mathfrak{Z}(a)$ implies $a \in I$.*
(3) *For any finite $F \subseteq I$, $\bigcap \mathfrak{Z}(F) \subseteq I$.*

Proof. (1) $\Rightarrow$ (2): Assume that I is an mz-ideal. Let $\{a_1, \dots, a_n\} \subseteq I$ and $a \in A$ be such that $\mathfrak{Z}(a_1, \dots, a_n) \subseteq \mathfrak{Z}(a)$. We claim that $\mathfrak{Z}(aa_1, \dots, aa_n) = \mathfrak{Z}(a)$. The containment $\supseteq$ is immediate. For the other, let $M \in \mathfrak{Z}(aa_1, \dots, aa_n)$. If each $a_i \in M$, then $M \in \mathfrak{Z}(a_1, \dots, a_n) \subseteq \mathfrak{Z}(a)$. If some $a_k \notin M$, then the fact that $aa_k \in M$ implies $a \in M$, by primeness, so that $M \in \mathfrak{Z}(a)$. Now, since $\{aa_1, \dots, aa_n\} \subseteq I$ and I is an mz-ideal, by hypothesis, it follows that $a \in I$.

(2) $\Rightarrow$ (3): Let F be a finite set with $F \subseteq I$, and let $a \in \bigcap \mathfrak{Z}(F)$. If $M \in \mathfrak{Z}(F)$, then $a \in M$; and so $\mathfrak{Z}(F) \subseteq \mathfrak{Z}(a)$. It, therefore, follows from (2) that $a \in I$, which then proves that $\bigcap \mathfrak{Z}(F) \subseteq I$.

(3) $\Rightarrow$ (1): This follows from the fact that, for any $a \in A$, $a \in \bigcap \mathfrak{Z}(a)$. □

Condition (3) in this lemma makes it particularly apparent that every maximal ideal is an mz-ideal. Regarding z-elements, we shall use the description in [14, Definition & Remarks 6.3] since we are dealing with compact algebraic frames. For a compact algebraic frame L, the nucleus $\mathrm{ar}\colon L \to L$ is given by

$$\mathrm{ar}(x) = \bigwedge \{m \in \mathrm{Max}(L) \mid x \leq m\}.$$

In particular, for any finite set $\{a_1, \dots, a_n\} \subseteq A$, if we let K and H be the compact elements

$$K = [a_1] \vee \cdots \vee [a_n] \qquad \text{and} \qquad H = Z(a_1) \vee \cdots \vee Z(a_n)$$

of $\mathrm{RId}(A)$ and $\mathrm{ZId}(A)$, respectively, then

$$\mathrm{ar}(K) = \bigcap \mathfrak{Z}(a_1, \dots, a_n) = \mathrm{ar}(H).$$

The *z-nucleus* on L is defined by

$$z(x) = \bigvee \{\mathrm{ar}(c) \mid c \in \mathfrak{k}(L), c \leq x\},$$

and an $x \in L$ is called a *z-element* if $z(x) = x$.

Theorem 4.4. *For any ring A,*

$$z(\mathrm{RId}(A)) = z(\mathrm{ZId}(A)) = \{I \subseteq A \mid I \textit{ is an } mz\textit{-ideal in } A\}.$$

Proof. We prove the equality of the first and last sets displayed above, and indicate how the equality of the second and last sets follow similarly. For brevity, let us write $\mathrm{MId}(A)$ for the lattice of mz-ideals of A. Suppose that $I \in z(\mathrm{RId}(A))$. Then,

$$I = \bigvee_{\mathrm{ZId}(A)} \{\mathrm{ar}(K) \mid K \in \mathfrak{k}(\mathrm{RId}(A)), K \subseteq I\}.$$

Let $F = \{a_1, \dots, a_n\} \subseteq I$. To show that I is an mz-ideal, it suffices, by Lemma 4.3, to prove that $\bigcap \mathfrak{Z}(F) \subseteq I$. Put $K = [a_1] \vee \cdots \vee [a_n]$, and note that $K \subseteq I$. Now, as observed above, $\mathrm{ar}(K) = \bigcap \mathfrak{Z}(F)$, which then implies $\bigcap \mathfrak{Z}(F) \subseteq I$ since $\mathrm{ar}(K) \subseteq I$. Therefore, I is an mz-ideal. Thus, $z(\mathrm{RId}(A)) \subseteq \mathrm{MId}(A)$.

On the other hand, let $J \in \mathrm{MId}(A)$. For any $a \in J$, $a \in \mathrm{ar}([a])$, which implies

$$J \subseteq \bigvee \{\mathrm{ar}([a]) \mid a \in J\} \subseteq \bigvee \{\mathrm{ar}(K) \mid K \in \mathfrak{k}(\mathrm{RId}(A)), K \subseteq J\}.$$

But now if K is a compact element in $\mathrm{RId}(A)$ and $K \subseteq J$, then $\mathrm{ar}(K) \subseteq J$ since J is an mz-ideal. We therefore have

$$\bigvee\{\mathrm{ar}(K) \mid K \in \mathfrak{k}(\mathrm{RId}(A)), K \subseteq J\} \subseteq J,$$

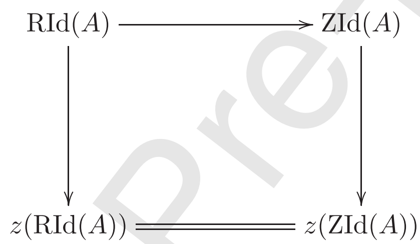
and consequently

$$J = \bigvee\{\mathrm{ar}(K) \mid K \in \mathfrak{k}(\mathrm{RId}(A)), K \subseteq J\},$$

which says $J \in z(\mathrm{RId}(A))$. This proves that $\mathrm{MId}(A) \subseteq z(\mathrm{RId}(A))$, and hence we have the claimed equality.

The equality $z(\mathrm{ZId}(A)) = \mathrm{MId}(A)$ is proved similarly, by replacing $K = [a_1] \vee \cdots \vee [a_n]$ with $K = Z(a_1) \vee \cdots \vee Z(a_n)$, and $[a]$ with $Z(a)$. □

We consequently have the following commutative diagram, where the horizontal arrow is the homomorphism $\kappa\colon \mathrm{RId}(A) \to \mathrm{ZId}(A)$, considered earlier, and the vertical arrows are the homomorphisms induced by the respective z-nuclei.



Now let us recall [15, Proposition 2.5(a)], which says a compact algebraic frame L is Yosida if and only if $L = zL$. Applying it to $\mathrm{ZId}(A)$, and taking into account the foregoing theorem, we have the following result.

Corollary 4.5. *The following statements are equivalent.*

(1) $\mathrm{ZId}(A)$ *is Yosida.*
(2) $\mathrm{ZId}(A) = z(\mathrm{RId}(A))$.
(3) *Every z-ideal of A is an mz-ideal.*
(4) *Every prime z-ideal is an mz-ideal.*
(5) *For every finite set $F \subseteq A$, $\bigvee\{Z(a) \mid a \in F\} = \bigcap \mathfrak{Z}(F)$.*

Proof. The equivalences (1) ⇔ (2) ⇔ (3) follow from Theorem 4.4. The implication (3) ⇒ (4) is trivial.

(4) ⇒ (3): Suppose every prime z-ideal is an mz-ideal. Let I be a z-ideal. Since $\mathrm{ZId}(A)$ is spatial, then, by Proposition 3.10,

$$I = \bigwedge\{P \in \mathrm{Spec}(\mathrm{ZId}(A)) \mid I \leq P\} = \bigcap\{P \in \mathrm{Spec}(\mathrm{ZId}(A)) \mid I \subseteq P\}.$$

Since each prime z-ideal is an mz-ideal, by hypothesis, and intersections of mz-ideals are mz-ideals, we have that I is an intersection of mz-ideals. Therefore I is an mz-ideal.

(1) ⇒ (5): Assume that $\mathrm{ZId}(A)$ is Yosida. Let $F = \{a_1, \dots, a_n\} \subseteq A$. Observe that $\mathfrak{Z}(F) \subseteq \mathfrak{Z}(a_i)$ for each i, so that $Z(a_i) \subseteq \bigcap \mathfrak{Z}(F)$, and hence $Z(a_1) \vee \cdots \vee Z(a_n) \subseteq \bigcap \mathfrak{Z}(F)$. Suppose, by way of contradiction, that $\bigcap \mathfrak{Z}(F) \nsubseteq \bigvee_i Z(a_i)$. Then take $x \in \bigcap \mathfrak{Z}(F) \smallsetminus \bigvee_i Z(a_i)$, and put

$$J = Z(a_1) \vee \cdots \vee Z(a_n) \vee Z(x).$$

Then J is a compact element of $\mathrm{ZId}(A)$ with $Z(a_1) \vee \cdots \vee Z(a_n) < J$. Since $\mathrm{ZId}(A)$ is Yosida, there is an $H \in \mathrm{ZId}(A)$ such that

$$Z(a_1) \vee \cdots \vee Z(a_n) \vee H < Z(a_1) \vee \cdots \vee Z(a_n) \vee Z(x) \vee H = A.$$

Thus, $Z(a_1) \vee \cdots \vee Z(a_n) \vee H$ is a proper ideal in A, and so there is a maximal ideal M in A with $Z(a_1) \vee \cdots \vee Z(a_n) \vee H \subseteq M$. Consequently, $F \subseteq M$, which then implies $x \in M$ since $\bigcap \mathfrak{Z}(F) \subseteq M$. But now $x \in M$ implies $J \subseteq M$, and hence $M = A$ since $M \supseteq H$ as well. This contradiction proves that $\bigcap \mathfrak{Z}(F) \subseteq Z(a_1) \vee \cdots \vee Z(a_n)$, and so we have the claimed equality.

(5) $\Rightarrow$ (2): Clearly, condition (5) implies $K = \mathrm{ar}(K)$, for each $K \in \mathfrak{k}(\mathrm{ZId}(A))$. Thus, for any $I \in \mathrm{ZId}(A)$,

$$z(I) = \bigvee\{\mathrm{ar}(K) \mid K \in \mathfrak{k}(\mathrm{ZId})(A), K \leq I\} = \bigvee\{K \in \mathfrak{k}(\mathrm{ZId}(A)), K \leq I\} = I,$$

which says $z(\mathrm{ZId}(A)) = \mathrm{ZId}(A)$. □

Remark 4.6. Examples of rings whose frames of z-ideals are Yosida frames abound. By [5, Theorem 3.5], they include all reduced f-rings with bounded inversion. Recall that a *Bézout ring* is a ring in which every finitely generated ideal is principal. It is easy to see that in a Bézout ring every z-ideal is an mz-ideal. Consequently, Bézout rings are also of this type.

Theorem 4.4 tells us which ideals of A are fixed by the z-nucleus on $\mathrm{RId}(A)$. We now describe the ideals fixed by the ar-nucleus on $\mathrm{RId}(A)$. In a general algebraic frame L, $\mathrm{Fix}(\mathrm{ar})$ is denoted by $\mathbf{a}^{\uparrow}(L)$. As in [16], we say an ideal of a ring A is a *strong z-ideal* if it is an intersection of maximal ideals.

Proposition 4.7. *For any ring A,*

$$\mathbf{a}^{\uparrow}(\mathrm{RId}(A)) = \mathbf{a}^{\uparrow}(\mathrm{ZId}(A)) = \{I \subseteq A \mid I \textit{ is a strong } z\textit{-ideal}\}.$$

Recall that a *Hilbert* ring is a ring in which every prime ideal is an intersection of maximal ideals. Every $I \in \mathrm{RId}(A)$ is a meet of primes; that is, is an intersection of prime ideals of A. Thus, if A is a Hilbert ring, then every $I \in \mathrm{RId}(A)$ is an intersection of maximal ideals. The converse holds as well, as one sees immediately. Now, based on the fact that

$$\mathbf{a}^{\uparrow}(\mathrm{ZId}(A)) \subseteq z(\mathrm{ZId}(A)) \subseteq \mathrm{ZId}(A) \subseteq \mathrm{RId}(A).$$

We therefore, have the following corollary, which we state in ring-theoretic terms.

Corollary 4.8. *The following are equivalent for a ring A.*

1. *A is a Hilbert ring.*
2. *Every radical ideal of A is a strong z-ideal.*
3. *Every z-ideal of A is a strong z-ideal.*
4. *Every mz-ideal of A is a strong z-ideal.*

We end this section with an example (within Hilbert rings, no less) that shows that the "operator" $\mathfrak{Z}(\cdot)$ does not treat singletons with regard to intersections as it does with regard to unions. More precisely, we observed in the course of the proof of Lemma 3.3 that $\mathfrak{Z}(a) \cup \mathfrak{Z}(b) = \mathfrak{Z}(ab)$, for all elements a and b. The result fails if we replace $\cup$ with $\cap$, as the example below shows.

Example 4.9. It is known that if A is a Hilbert ring then so is the polynomial ring $A[x]$. It follows that the ring $A = \mathbb{C}[X, Y]$ is a Hilbert ring. Interestingly, this ring has the property that there are $f, g \in A$ such that $\mathfrak{Z}(f) \cap \mathfrak{Z}(g)$ is not of the form $\mathfrak{Z}(h)$ for any $h \in A$.

The reader is invited to also contrast this with the result proved in Lemma 3.3, regarding the "operator" $Z(\cdot)$, stating that for any a and b, there is a c (in fact, $c = ab$) such that $Z(a) \cap Z(b) = Z(c)$.

5. On projectabilty properties

In this section we seek conditions on A that make $\mathrm{ZId}(A)$ satisfy the various projectability properties, such as those described in [12]. We shall assume that all rings in this section have zero Jacobson radical, so that the zero ideal is a z-ideal, and hence for any ideal I, $\mathrm{Ann}(I)$ is a z-ideal by [16, Proposition 1.3]. Note, in particular, that for any $I \in \mathrm{ZId}(A)$, the pseudocomplement of I is $\mathrm{Ann}(I)$.

Projectability conditions involve pseudocomplements and double pseudocomplements of compact elements. So we start by describing them. First, we deal with the simplest compact elements, namely, the ideals $Z(a)$.

Lemma 5.1. *If A has zero Jacobson radical, then for any $a \in A$, $\mathrm{Ann}(Z(a)) = \mathrm{Ann}(a)$, and hence $\mathrm{Ann}^2(Z(a)) = \mathrm{Ann}^2(a)$.*

Proof. Since $a \in Z(a)$, it is clear that $\mathrm{Ann}(Z(a)) \subseteq \mathrm{Ann}(a)$. To show the reverse inclusion, let $x \in \mathrm{Ann}(a)$, so that $xa = 0$, and suppose, by way of contradiction, that x does not annihilate $Z(a)$. Then pick $r \in Z(a)$ such that $xr \neq 0$. Since A has zero Jacobson radical, this implies there is a maximal ideal, say N, such that $xr \notin N$. Consequently, $x \notin N$. Since $xa = 0 \in N$ and N is prime, we must have $a \in N$. Since $r \in Z(a)$, which says r belongs to every maximal ideal that contains a, we have $r \in N$. But now this implies $xr \in N$; and we have a contradiction. □

Corollary 5.2. *For any $a_1, \ldots, a_n \in A$,*

$$\Big(Z(a_1) \vee \cdots \vee Z(a_n)\Big)^* = \mathrm{Ann}(a_1, \ldots, a_n),$$

and consequently,

$$\Big(Z(a_1) \vee \cdots \vee Z(a_n)\Big)^{**} = \mathrm{Ann}^2(a_1, \ldots, a_n).$$

Proof. We need only prove the first part. Applying Lemma 5.1, we have

$$\begin{aligned}\Big(Z(a_1) \vee \cdots \vee Z(a_n)\Big)^* &= Z(a_1)^* \wedge \cdots \wedge Z(a_n)^* \\ &= \mathrm{Ann}(a_1) \cap \cdots \cap \mathrm{Ann}(a_n) \\ &= \mathrm{Ann}(a_1, \ldots, a_n),\end{aligned}$$

which proves the result. □

We recall that an algebraic frame L is called *projectable* if $c^{**} \vee c^* = 1$ for every $c \in \mathfrak{k}(L)$. Recall that a ring A is *weak Baer* if for every $a \in A$, $\mathrm{Ann}(a)$ is generated by an idempotent. This is equivalent to saying the annihilator of any finitely generated ideal is a principal ideal generated by an idempotent. As observed in [8, 4.2], $\mathrm{RId}(A)$ is projectable if and only if A is weak Baer. The same holds for $\mathrm{ZId}(A)$, as we show below. In proving one of the equivalences, we shall use [6, Lemma 5.2], which ensures that if $h\colon L \to M$ is a surjective dense and codense coherent map, then L is projectable if and only if M is projectable.

Let us observe the following about the frame of z-elements.

Lemma 5.3. *Let L be an algebraic frame, and write $z\colon L \to zL$ for the coherent map induced by the z-nucleus on L.*

(a) *The homomorphism $z\colon L \to zL$ is dense.*
(b) *If L is compact and each $x < 1$ in L is below some $m \in \mathrm{Max}(L)$, then z is codense.*

Proof. (a) If $z(x) = 0$, then $\mathrm{ar}(c) = 0$ for each compact $c \leq x$. Since each $c \leq \mathrm{ar}(c)$, this says $x = 0$ in light of coherence of L.

(b) Suppose $z(x) = 1$. By compactness, there are finitely many $c_1, \dots, c_n \in \mathfrak{k}(L)$, each below x, such that $\mathrm{ar}(c_1) \vee \cdots \vee \mathrm{ar}(c_n) = 1$. Hence $c = c_1 \vee \cdots \vee c_n$ is a compact element below x such that $\mathrm{ar}(c) = 1$. Thus, by the hypothesis on L, there is no maximal element above c, which then implies $c = 1$, and hence $x = 1$. □

The result that follows is an improvement of [6, Proposition 5.3].

Proposition 5.4. *The following are equivalent for a ring A with zero Jacobson radical.*

(1) $\mathrm{ZId}(A)$ *is projectable.*
(2) $z(\mathrm{RId}(A))$ *is projectable.*
(3) A *is weak Baer.*

Proof. (1) ⇔ (2): This follows from Lemma 5.3, since $z\colon \mathrm{ZId}(A) \to z(\mathrm{ZId}(A))$ is dense and codense.

(1) ⇔ (3): Suppose that $\mathrm{ZId}(A)$ is projectable, and let $a \in A$. Then $Z(a)^* \vee Z(a)^{**} = A$, from which, by Lemma 5.1, we deduce that $\mathrm{Ann}(a) + \mathrm{Ann}^2(a)$ is contained in no maximal ideal of A. Thus, $\mathrm{Ann}(a) + \mathrm{Ann}^2(a) = A$, and hence $\mathrm{Ann}(a)$ is a direct summand since $\mathrm{Ann}(a) \cap \mathrm{Ann}^2(a) = 0$, as A is reduced. Therefore, A is weak Baer.

Conversely, suppose that A is weak Baer, and let $K = Z(a_1) \vee \cdots \vee Z(a_n)$ be a compact element in $\mathrm{ZId}(A)$. By Corollary 5.2, $K^* = \mathrm{Ann}(a_1, \dots, a_n)$, and since A is weak Baer, there is an idempotent $e \in A$ such that $K^* = \langle e \rangle$. We deduce from this (by an argument as above) that $K^* \vee K^{**} = A$, showing that $\mathrm{ZId}(A)$ is projectable. □

In [12], the authors call an algebraic frame L *feebly projectable* if whenever $a \wedge b = 0$ in $\mathfrak{k}(L)$, there exists $c \in \mathfrak{k}(L)$ such that $c^* \vee c^{**} = 1$, $a \leq c^*$ and $b \leq c^{**}$. They show that for an algebraic frame that has a dense compact element (such as a coherent frame), this is equivalent to requiring that whenever $a \wedge b = 0 \in \mathfrak{k}(L)$, there exists a complemented $c \in L$ such that $a \leq c$ and $b \leq c^*$. In [11], the authors call a ring A *feebly Baer* if whenever $ab = 0$ in A, there is an idempotent $e \in A$ such that $a \in \langle e \rangle$ and $b \in \langle 1 - e \rangle$.

The proof of the lemma that follows is omitted since it is fairly routine if one takes into account the fact that a surjective dense and codense frame homomorphism preserves and reflects complementedness.

Lemma 5.5. *If $h\colon L \to M$ is a surjective dense and codense coherent map, then L is feebly projectable iff M is feebly projectable.*

Recall that the join and the meet of finitely many complemented elements are complemented, and, in fact, if $c_1, \dots, c_n$ are complemented, then

$$(c_1 \wedge \cdots \wedge c_n)^* = c_1^* \vee \cdots \vee c_n^*.$$

Recall also that if A is a reduced ring (and, hence, if A has zero Jacobson radical), then the bottom of $\mathrm{RId}(A)$ is the zero ideal.

Theorem 5.6. *The following conditions regarding a ring A with zero Jacobson radical are equivalent.*

(1) A *is feebly Baer.*
(2) $\mathrm{RId}(A)$ *is feebly projectable.*
(3) $\mathrm{ZId}(A)$ *is feebly projectable.*
(4) $z(\mathrm{RId}(A))$ *is feebly projectable.*

Proof. (1) $\Rightarrow$ (2): Assume that A is feebly Baer. Suppose K and H are two compact elements of $\mathrm{RId}(A)$ with $K \wedge H = 0$. Then, by repeating elements if necessary, we may assume that there is a positive integer n and elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ with

$$K = [a_1] \vee \cdots \vee [a_n] \qquad \text{and} \qquad [b_1] \vee \cdots \vee [b_n].$$

The equality $K \wedge H = 0$ implies that, for any pair (i, j) of indices, $a_i b_j = 0$, and so there are idempotents e_{ij} in A with $a_i \in \langle e_{ij} \rangle$ and $b_j \in \langle 1 - e_{ij} \rangle$. We therefore have complemented elements γ_{ij} in $\mathrm{RId}(A)$ with the property exhibited in the matrix below, where α_i and β_j abbreviate, respectively, $[a_i]$ and $[b_j]$.

$$\begin{pmatrix} \alpha_1 \leq \gamma_{11} \text{ and } \beta_1 \leq \gamma_{11}^* & \alpha_1 \leq \gamma_{12} \text{ and } \beta_2 \leq \gamma_{12}^* & \cdots & \alpha_1 \leq \gamma_{1n} \text{ and } \beta_n \leq \gamma_{1n}^* \\ \alpha_2 \leq \gamma_{21} \text{ and } \beta_1 \leq \gamma_{21}^* & \alpha_2 \leq \gamma_{22} \text{ and } \beta_2 \leq \gamma_{22}^* & \cdots & \alpha_2 \leq \gamma_{2n} \text{ and } \beta_n \leq \gamma_{2n}^* \\ \vdots & \vdots & & \vdots \\ \alpha_n \leq \gamma_{n1} \text{ and } \beta_1 \leq \gamma_{n1}^* & \alpha_n \leq \gamma_{n2} \text{ and } \beta_2 \leq \gamma_{n2}^* & \cdots & \alpha_n \leq \gamma_{nn} \text{ and } \beta_n \leq \gamma_{nn}^* \end{pmatrix}$$

Now note that, for each index i and each index j,

$$\alpha_i \leq \gamma_{i1} \wedge \gamma_{i2} \wedge \cdots \wedge \gamma_{in} \qquad \text{and} \qquad \beta_j \leq \gamma_{1j}^* \wedge \gamma_{2j}^* \cdots \wedge \gamma_{nj}^*,$$

so that

$$\begin{aligned} H = \beta_1 \vee \cdots \vee \beta_n &\leq \Big(\bigwedge_{i=1}^{n} \gamma_{i1}^*\Big) \vee \cdots \vee \Big(\bigwedge_{i=1}^{n} \gamma_{in}^*\Big) \\ &= \Big(\bigvee_{i=1}^{n} \gamma_{i1}\Big)^* \vee \cdots \vee \Big(\bigvee_{i=1}^{n} \gamma_{in}\Big)^* \\ &= \Bigg(\Big(\bigvee_{i=1}^{n} \gamma_{i1}\Big) \wedge \cdots \wedge \Big(\bigvee_{i=1}^{n} \gamma_{in}\Big)\Bigg)^* \end{aligned}$$

and

$$K = \alpha_1 \vee \cdots \vee \alpha_n \leq \Big(\bigwedge_{j=1}^{n} \gamma_{1j}\Big) \vee \cdots \vee \Big(\bigwedge_{j=1}^{n} \gamma_{nj}\Big) \leq \Big(\bigvee_{i=1}^{n} \gamma_{i1}\Big) \vee \cdots \vee \Big(\bigvee_{i=1}^{n} \gamma_{in}\Big).$$

Thus, the element $C = \Big(\bigvee_{i=1}^{n} \gamma_{i1}\Big) \vee \cdots \vee \Big(\bigvee_{i=1}^{n} \gamma_{in}\Big)$ is complemented in $\mathrm{RId}(A)$ and satisfies $K \leq C$ and $H \leq C^*$. Therefore, $\mathrm{RId}(A)$ is feebly projectable.

(2) $\Rightarrow$ (1): Assume that $\mathrm{RId}(A)$ is feebly projectable. Let $ab = 0$ in A. Then $[a] \wedge [b] = [ab] = 0_{\mathrm{RId}(A)}$ in $\mathfrak{k}(\mathrm{RId}(A))$. Thus, there is a complemented $C \in \mathrm{RId}(A)$ such that $[a] \leq C$ and $[b] \leq C^*$. By [2, Lemma 1], there is an idempotent $e \in A$ such that $C = [e]$, and hence $C^* = [1 - e]$. From this we deduce easily that $a \in \langle e \rangle$ and $b \in \langle 1 - e \rangle$. Therefore, A is feebly Baer.

(2) $\Leftrightarrow$ (3): This follows from Proposition 3.5 and Lemma 5.5, since the homomorphism $\kappa\colon \mathrm{RId}(A) \to \mathrm{ZId}(A)$, defined just prior to Proposition 3.5 is dense and codense if A has zero Jacobson radical.

(2) $\Leftrightarrow$ (4): This follows from Lemmas 5.3 and 5.5, since $z\colon \mathrm{RId}(A) \to z(\mathrm{RId}(A))$ is dense and codense. □

Remark 5.7. That A is feebly Baer if and only if $\mathrm{RId}(A)$ is feebly projectable does not require A to have zero Jacobson radical. It holds for any reduced ring – which is all that is needed for the zero of $\mathrm{RId}(A)$ to be the zero ideal of A.

6. ZId is a functor

Following the idea (but not the notation) in [7], we let $\mathbf{CRng}_{\mathfrak{z}}$ denote the category whose objects are rings, and whose morphisms are ring homomorphisms $\phi\colon A \to B$ that contract z-ideals to z-ideals. We will show that the assignment $A \mapsto \mathrm{ZId}(A)$ is the object part of a functor $\mathbf{CRng}_{\mathfrak{z}} \to \mathbf{CohFrm}$, and that the association $A \mapsto \mathfrak{z}_A\colon \mathrm{RId} \to \mathrm{ZId}$ is a natural transformation. We shall need the following lemma.

Lemma 6.1. *Let $\phi\colon A \to B$ be a ring homomorphism that contracts z-ideals to z-ideals. The mapping $\overline{\phi}\colon \mathfrak{k}(\mathrm{ZId}(A)) \to \mathfrak{k}(\mathrm{ZId}(B))$, defined by*

$$\overline{\phi}\big(Z(a_1) \vee \cdots \vee Z(a_n)\big) = Z(\phi(a_1)) \vee \cdots \vee Z(\phi(a_n)),$$

is well defined, and is a lattice homomorphism.

Proof. (i) Let us show first that $\overline{\phi}$ is well defined. Consider elements $a_1, \dots, a_n$ and $b_1, \dots, b_k$ in A such that $Z(a_1) \vee \cdots \vee Z(a_n) = Z(b_1) \vee \cdots \vee Z(b_k)$. We must show that

$$Z(\phi(a_1)) \vee \cdots \vee Z(\phi(a_n)) = Z(\phi(b_1)) \vee \cdots \vee Z(\phi(b_k)).$$

For brevity, put $J = Z(\phi(b_1)) \vee \cdots \vee Z(\phi(b_k))$. Now, for any index $i = 1, \dots, k$ we have $\phi(b_i) \in J$, which implies $b_i \in \phi^{-1}[J]$. Since $\phi^{-1}[J]$ is a z-ideal, this implies

$$Z(b_1) \vee \cdots \vee Z(b_k) \subseteq \phi^{-1}[J],$$

and hence

$$Z(a_1) \vee \cdots \vee Z(a_n) \subseteq \phi^{-1}[J].$$

Consequently, for each index $j = 1, \dots, n$, $a_i \in \phi^{-1}[J]$, so that $\phi(a_j) \in J$, whence we deduce that $Z(\phi(a_1)) \vee \cdots \vee Z(\phi(a_n)) \subseteq J$, that is,

$$Z(\phi(a_1)) \vee \cdots \vee Z(\phi(a_n)) \subseteq Z(\phi(b_1)) \vee \cdots \vee Z(\phi(b_k)).$$

A similar argument establishes the reverse inclusion; so we have the desired equality.

(ii) Now we show that $\overline{\phi}$ is a lattice homomorphism. Since for any ring R the bottom and the top elements of $\mathrm{ZId}(R)$ are $Z(0)$ and $Z(1)$, it is immediate that $\overline{\phi}$ preserves the bottom and the top. If

$$K = Z(a_1) \vee \cdots \vee Z(a_n) \qquad \text{and} \qquad H = Z(b_1) \vee \cdots \vee Z(b_k)$$

are two elements of $\mathfrak{k}(\mathrm{ZId}(A))$, then, as shown in the proof of Theorem 3.4,

$$K \wedge H = Z(a_1 b_1) \vee \cdots \vee Z(a_i b_j) \vee \cdots \vee Z(a_n b_k).$$

Since $\phi(a_i b_j) = \phi(a_i)\phi(b_j)$ for any a_i and b_j, it follows routinely that $\overline{\phi}(K \wedge H) = \overline{\phi}(K) \wedge \overline{\phi}(H)$. We also have

$$\begin{aligned}\overline{\phi}(K \vee H) &= \overline{\phi}\Big(Z(a_1) \vee \cdots \vee Z(a_n) \vee Z(b_1) \vee \cdots \vee Z(b_k)\Big)\\ &= \Big(Z(\phi(a_1)) \vee \cdots \vee Z(\phi(a_n))\Big) \vee \Big(Z(\phi(b_1)) \vee \cdots \vee Z(\phi(b_k))\Big)\\ &= \overline{\phi}(K) \vee \overline{\phi}(H),\end{aligned}$$

and so $\overline{\phi}$ is a lattice homomorphism. $\square$

Thus, by [10, p. 64], for any $\phi\colon A \to B$ in $\mathbf{CRng}_{\mathfrak{z}}$, there is a unique coherent map $\tilde{\phi}\colon \mathrm{ZId}(A) \to \mathrm{ZId}(B)$ that extends $\overline{\phi}$. Explicitly, it is given by

$$\tilde{\phi}(J) = \bigvee_{\mathrm{ZId}(B)} \{Z(\phi(a)) \mid a \in J\}.$$

Remark 6.2. A comment regarding the need to restrict to homomorphisms that contract z-ideals to z-ideals is in order. Suppose $\phi\colon A \to B$ is a ring homomorphism that fails to contract all z-ideals to z-ideals. Then, by [16, Lemma 1.7], there is a maximal ideal M in B for which $\phi^{-1}(M)$ not a z-ideal in A. Then there exists $x, y \in A$ such that $\mathfrak{Z}(x) = \mathfrak{Z}(y)$, $x \in \phi^{-1}(M)$, and $y \notin \phi^{-1}(M)$. Then $Z(x) = Z(y)$, but $Z(\phi(x)) \neq Z(\phi(y))$ since M is a maximal ideal of B containing $\phi(x)$ but not $\phi(y)$. Thus, for such a homomorphism, $\overline{\phi}$ as in Lemma 6.1 would not be well defined.

As in Proposition 3.5, for any $A \in \mathbf{CRng}_{\mathfrak{z}}$, let $\kappa_A\colon \mathrm{RId}(A) \to \mathrm{ZId}(A)$ be the coherent map that sends a radical ideal to the smallest z-ideal containing it. Recall that we showed in Proposition 3.5 that $\kappa_A([a]) = Z(a)$, for any $a \in A$. Recall also that, for any ring homomorphism $\phi\colon A \to B$, $\mathrm{RId}(\phi)([a]) = [\phi(a)]$.

Proposition 6.3. *The mappings $A \mapsto \mathrm{ZId}(A)$ and $\phi \mapsto \tilde{\phi}$ define a functor*

$$\mathrm{ZId}\colon \mathbf{CRng}_{\mathfrak{z}} \to \mathbf{CohFrm}$$

such that $A \mapsto \kappa_A\colon \mathrm{RId} \to \mathrm{ZId}$ is a natural transformation.

Proof. That ZId preserves identity follows from the fact that $J = \bigvee\{Z(a) \mid a \in J\}$, for any $J \in \mathrm{ZId}(A)$. To see that ZId preserves composites, let $\phi\colon A \to B$ and $\tau\colon B \to C$ be morphisms in $\mathbf{CRng}_{\mathfrak{z}}$. To check that $\mathrm{ZId}(\tau \cdot \phi) = \mathrm{ZId}(\tau) \cdot \mathrm{ZId}(\phi)$, we need only check that these two coherent maps agree on basic compact elements. Consider then any basic compact element $Z(a) \in \mathfrak{k}(\mathrm{ZId}(A))$, and observe that

$$\Big(\mathrm{ZId}(\tau) \cdot \mathrm{ZId}(\phi)\Big)\big(Z(a)\big) = \mathrm{ZId}(\tau)\big(Z(\phi(a))\big) = Z(\tau(\phi(a))) = \mathrm{ZId}(\tau \cdot \phi)\big(Z(a)\big).$$

Therefore, $\mathrm{ZId}\colon \mathbf{CRng}_{\mathfrak{z}} \to \mathbf{CohFrm}$ is a functor.

To prove the claimed naturality, given $\phi\colon A \to B \in \mathbf{CRng}_{\mathfrak{z}}$, we must show that the square

$$\begin{array}{ccc} \mathrm{RId}(A) & \xrightarrow{\kappa_A} & \mathrm{ZId}(A) \\ {\scriptstyle \mathrm{RId}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathrm{ZId}(\phi)} \\ \mathrm{RId}(B) & \xrightarrow{\kappa_B} & \mathrm{ZId}(B) \end{array}$$

commutes. For any basic compact element $[a] \in \mathfrak{k}(\mathrm{RId}(A))$, we have

$$\big(\mathrm{ZId}(\phi) \cdot \kappa_A\big)([a]) = \mathrm{ZId}(\phi)(Z(a)) = Z(\phi(a)) = \kappa_B([\phi(a)]) = \big(\kappa_B \cdot \mathrm{RId}(\phi)\big)([a]),$$

which then shows that $\mathrm{ZId}(\phi) \cdot \kappa_A = \kappa_B \cdot \mathrm{RId}(\phi)$, by coherence. □

The functorial properties of ZId will be a subject of another occasion.

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