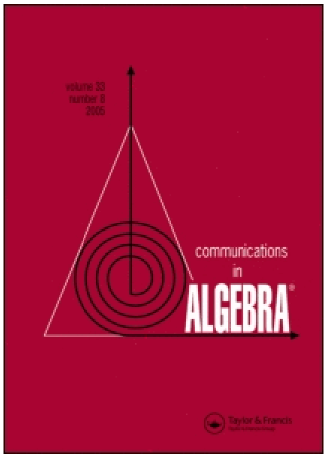

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BÉZOUT SP-DOMAINS

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SP-domains were first introduced by Vaughan and Yeagy (1978) as generalizations of Dedekind domains. In particular, they showed that an SP-domain is an almost Dedekind domain. Later, Olberding (2005) gave a complete characterization of SP-domains within the class of almost Dedekind domains. This article characterizes Bézout SP-domains using the order structure of the group of divisibility of an integral domain. We use this information to construct a Bézout SP-domain with zero Jacobson radical.

Key Words: Bézout domain; Lattice-ordered group of divisibility; Specker group.

Mathematics Subject Classification: Primary 13F05; Secondary 06F20.

1. SP-DOMAINS

Throughout, all rings are commutative and possess an identity.

Recall that an ideal of a ring is said to be *semiprime* if it is the intersection of a collection of prime ideals. (These ideals are also known as *radical* ideals.) Vaughan and Yeagy (1978) defined an *SP-domain* as an integral domain R for which every nontrivial ideal of R is a product of semiprime ideals. This is a natural generalization of a Dedekind domain. In fact, they showed that an SP-domain is an almost Dedekind domain, that is, every localization at a maximal ideal is a Dedekind domain. Later, Yeagy (1979) showed that if R is a union of a tower of Dedekind domains, then R is an SP-domain if and only if R has no critical maximal ideals. Recall that for an integral domain R, a maximal ideal M is said to be *critical* if every finite subset of M is contained in the square of a maximal ideal of R. Finally, this last result was generalized to arbitrary almost Dedekind domains by Olberding (2005). In particular, the following theorem and corollary were demonstrated.

Theorem 1.1 (Olberding, 2005, Theorem 2.1)**.** *The following statements are equivalent for the almost Dedekind domain R.*

(i) *R is an SP-domain*;
(ii) *R has no critical maximal ideals*;

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(iii) *If I is a finitely generated ideal R, then $\sqrt{I}$ is also finitely generated*;
(iv) *Every principal ideal of R is a product of semiprime ideals.*

Corollary 1.2. *Suppose R is an integral domain. R is an SP-domain if and only if R is a Prüfer domain of Krull dimension* 1 *and R has no critical maximal ideals.*

We wish to add another item to Theorem 1.1; one which we shall have cause to use later.

Proposition 1.3. *Suppose R is an almost Dedekind domain. R is an SP-domain if and only if every nonzero maximal ideal contains a nonzero finitely generated semiprime ideal.*

Proof. The necessity follows from (iii). As for the sufficiency we show that R has no critical maximal ideals. Let M be a maximal ideal of R. By hypothesis, let I be a nonzero finitely generated semiprime ideal contained in M. We claim that I is not contained in the square of any maximal ideal. Let N be a maximal ideal of R and suppose that $I \leq N$. Since I is semiprime and R is an almost Dedekind domain it follows that $IR_N = NR_N$, and since R_N is a Dedekind domain it follows that I is not contained in N^2. Therefore, M is not critical, whence R is an SP-domain. □

Throughout the rest of this article we assume that A is a Bézout domain. We let $G(A)$ denote the group of divisibility of A. Since A is Bézout we know that $G(A)$ is an abelian lattice-ordered group (or ℓ-group for short). We assume familiar knowledge of the group of divisibility of a Bézout domain, and elementary knowledge of abelian lattice-ordered groups. For more information on ℓ-groups we urge the readers to consult Gilmer (1992) or Darnel (1995). The article by Heinzer and Ohm (1971), in particular, Section 2 might also be useful. Since the ℓ-groups in question are all abelian, we will use additive notation for the operation.

The *positive cone of* G is the set $G^+ = \{g \in G : g \geq 0\}$. An ℓ-subgroup of G is a subgroup which is also a sublattice. We say the ℓ-subgroup H of G is *convex* if whenever $h_1 \leq g \leq h_2$ with $h_1, h_2 \in H$, then it follows that $g \in G$. A common example of a convex ℓ-subgroup is one of the form

$$u^\perp = \{g \in G : |u| \wedge |g| = 0\}.$$

This set is called the *polar of* u. Another example is given by first observing that given a subset of G there is a least convex ℓ-subgroup containing said set. In particular, we let $G(u)$ denote the convex ℓ-subgroup generated by u and observe that

$$G(u) = \{g \in G : |g| \leq nu \text{ for some } n \in \mathbb{N}\}.$$

When $G = G(u)$, the element u is called a *strong order unit*. A weak order unit is an element $u \in G$ such that $u^\perp = 0$. Obviously, a strong order unit is a weak order unit. The convex ℓ-subgroup P is called a *prime subgroup* whenever $a, b \in G^+$ and $a \wedge b = 0$, then either $a \in P$ or $b \in P$. (This is equivalent to the usual property that algebraists associate with prime, namely, if $a, b \in G^+$ and $a \wedge b \in P$, then $a \in P$

or $b \in P$.) It is a straightforward Zorn's Lemma argument to show that every prime subgroup contains in a minimal prime subgroup. We denote the collection of minimal prime subgroups of the ℓ-group G by $Min(G)$. For $g \in G$, we define $U(g) = \{P \in Min(G) | g \notin P\}$.

There is an order reversing bijection between the nonzero prime ideals of the Bézout domain A and the prime subgroups of $G(A)$. Since we already know that if A is an SP-domain, then its Krull dimension is 1 which translates to $G(A)$ having the property that every prime subgroup is maximal (and hence minimal). Such ℓ-groups are well-studied in the literature.

Definition 1.4. Let G be an ℓ-group. G is called *archimedean* if whenever $a, b \in G^+$ and $a \leq nb$ for all $n \in \mathbb{N}$, then $a = 0$. Not every ℓ-homomorphic image of an *archimedean* ℓ-group is archimedean. The ℓ-group G is called *hyperarchimedean* when every homomorphic image of G is archimedean. This definition is due to Conrad (1974). (They were orignally called *epi-archimedean groups* but the use of this name has faded over the years.) The next theorem can be found in Conrad (1974) or in Theorem 55.1 of Darnel (1995).

Theorem 1.5. *The following are equivalent*:

(i) *G is hyperarchimedean*;
(ii) *Every prime subgroup of G is maximal* (*and hence minimal*);
(iii) *For every $g \in G$, $G = G(g) \oplus g^{\perp}$*;
(iv) *G is ℓ-isomorphic to an ℓ-subgroup of $\mathbb{R}^I$, say G^*, such that for each $0 < g, h \in G^*$ there exists an $n \in N$ for which $h(i) < ng(i)$ for all $i \in \{i \in I : g(i) \neq 0\}$.*

Moreover, each representation of a hyperarchimedean ℓ-group as a group of real-valued functions must satisfy (iv).

Example 1.6. It is known that the class of hyperarchimedean ℓ-groups is closed under homomorphic images and ℓ-subgroups. These properties will be useful later. Also, note that for any Tychonoff space X, the set of bounded integer-valued continuous functions on X, denoted $C^*(X, \mathbb{Z})$, is a hyperarchimedean ℓ-group.

Returning to Bézout domains, we point out that for a prime ideal M of A, the group of divisibility of the localization A_M is ℓ-isomorphic to $G(A)/P$ where P is the prime subgroup associated to M. Thus, whenever A_M is a Dedekind domain it follows that $G(A)/P$ is a copy of $\mathbb{Z}$. Therefore, if A is a Bézout almost Dedekind domain, then every totally-orderd homomorphic image is a copy of $\mathbb{Z}$. Such ℓ-groups are called *hyper-$\mathbb{Z}$*.

Definition 1.7. Let G be an ℓ-group. We call the element $s \in G^+$ *singular* if whenever $0 \leq t \leq s$, then $t \wedge (s - t) = 0$. If s is singular it follows that for any prime subgroup P, either $s \in P$ or $s + P$ is the least positive element of G/P. We denote the collection of singular elements by S, and let $[S]$ denote the subgroup generated by S. Theorem 4.3 of Conrad and McAlister (1969) shows that $[S]$ is an ℓ-subgroup of G. Observe that for any Tychonoff space X, the singular elements of $C(X, \mathbb{Z})$ are precisely the characteristic functions on clopen subsets of X. Furthermore, $[S]$

is the collection of bounded integer-valued continuous functions on X. Whenever it happens that $G = [S]$, then G is called an *S-group* (or a *Specker group*). This definition is due to Conrad (1974) where this class of groups was investigated. Some other interesting results on this class of ℓ-groups can be found in Martinez (1975). One item of note is that every S-group is a hyper-$\mathbb{Z}$ group.

Lemma 1.8. *Suppose G is a hyperarchimedean ℓ-group. G is an S-group if and only if for each $P \in Min(G)$ there is a singular element $s \in S \setminus P$.*

Proof. Suppose G is an S-group and let $P \in Min(G)$. Choose $g \in G^+ \setminus P$. By hypothesis we can write $g = n_1 s_1 + \cdots + n_k s_k$, where $s_1, \ldots, s_k \in S$. If each s_i belongs to P, then so does g, whence there is a singular element of G not belonging to P.

Conversely, if $[S] < G$, then choose $g \in G^+ \setminus [S]$. As mentioned before, a Zorn's Lemma argument assures us that there is a prime subgroup P (which is minimal since G is hyperarchimedean) for which $[S] \leq P$. By hypothesis, there is an $s \in S \setminus P$. This contradicts that $s \in [S] \leq P$. □

Theorem 1.9. *Let A be a Bézout almost Dedekind domain and set $G = G(A)$. The following are equivalent*:

(i) *A is an SP-domain*;
(ii) *For every $P \in Min(G)$, there exists a $g \in G^+ \setminus P$ such that whenever $h \in G^+ \setminus P$ and $U(h) = U(g)$, then $g \leq h$*;
(iii) *G is an S-group.*

Proof. Suppose that A is an SP-domain and let $P \in Min(G)$. Let $M \in Max(A)$ be its corresponding maximal ideal. Since A is an SP-domain, there is a nonzero finitely generated semiprime ideal contained in M. Since A is Bézout it follows that we can assume said semiprime ideal is principal, say $aA = \sqrt{aA} \leq M$. Let g be the element of G^+ corresponding to a. Then $g \in G^+ \setminus P$. We claim g has the desired property from (ii). Suppose $h \in G^+$ with $U(g) = U(h)$. Let $b \in A$ correspond to h. It follows that a and b belong to exactly the same maximal ideals, and hence prime ideals since A is of Krull dimension 1. Therefore, $\sqrt{bA} = \sqrt{aA} = aA$, whence $b \in aA$. This translates to the inequality $g \leq h$. Hence (i) $\Rightarrow$ (ii).

Next, suppose (ii). We show that every nonzero maximal ideal contains a nonzero principal semiprime ideal, and thus A is an SP-domain by Proposition 1.3. To that end, let M be a (nonzero) maximal of A and let P be its corresponding minimal prime subgroup of G. Let $g \in G^+ \setminus P$ with the property given by (ii). Let $a \in A$ be any element corresponding to $g \in G^+$. First, it is clear that $a \in M$. Next, we show that $\sqrt{aA} = aA$. Let $b \in \sqrt{aA}$. Since A is a Bézout domain, we assume without any loss of generality that $aA \leq bA \leq \sqrt{aA}$. This means that b belongs to exactly the same prime ideals as a. Let $h \in G^+$ be the corresponding element to $b \in A$. Then $U(g) = U(h)$ and so $g \leq h$, i.e., $bA \leq aA$. Therefore, $aA = \sqrt{aA}$, whence (ii) $\Rightarrow$ (i).

By the lemma (iii) $\Rightarrow$ (ii). So suppose A is an SP-domain. We show that G is an S-group. Let $P \in Min(G)$ and let $g \in G^+ \setminus P$ have the property that if $h \in G^+$ and $U(g) = U(h)$, then $g \leq h$. We claim that $g \in S$, whence G is an S-group by Lemma 1.8. To that end, suppose that g is not a singular element of G. In particular, this means that there is an $0 \leq h \leq g$ such that $0 \neq h \wedge (g - h)$. Let $t = h \wedge (g - h) \neq 0$. We prove that $U(g - t) = U(g)$.

First of all by properties of ℓ-groups we gather that

$$g - t = (g - h) \vee h.$$

Thus, if $g - t \notin Q$, then either $g - h \notin Q$ or $h \notin Q$. Since $h \leq g$ and $g - h \leq g$ it follows that in either case $g \notin Q$, and so $U(g - t) \subseteq U(g)$. Conversely, suppose $g \notin Q$. We consider two cases: $h \in Q$ or $h \notin Q$. In the first case, $g - h \notin Q$, and thus $g - t = (g - h) \vee h \notin Q$, by convexity of Q. Similarly, if $h \notin Q$, then also $g - t = (g - h) \vee h \notin Q$. Therefore, we obtain the reverse containment $U(g) \subseteq U(g - t)$. The property assumed on g yields that $g \leq g - t$, whence $t = 0$, a contradiction. We conclude that g is singular. □

As we noted before a Bézout domain A is an almost Dedekind domain precisely when $G(A)$ is a hyper-$\mathbb{Z}$ group. Since there are hyper-$\mathbb{Z}$ groups which are not S-groups it follows that there are Bézout almost Dedekind domains which are not SP-domains. Also, we point out that since a Specker group G is ℓ-isomorphic to a $C(X, \mathbb{Z})$ if and only if G possesses a weak-order unit, we conclude that a Bézout domain A is an SP-domain with nonzero Jacobson radical if and only if $G(A) \cong C(X, \mathbb{Z})$ for some compact zero-dimensional Hausdorff space. Thus, Olberding's construction (2005) is the best possible such construction. Finally, we observe that since there are Specker groups without weak-order units (e.g., the direct sum of infinitely many copies of $\mathbb{Z}$ ordered pointwise) it follows that there are SP-domains with zero Jacobson radical. In the literature all of the examples of SP-domains have nonzero Jacobson radicals.

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