PSEUDO-VALUATION RINGS AND $C(X)$

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Abstract. A well-known line of study in the theory of the ring $C(X)$ of continuous real-valued functions on the topological space X is to determine when, for a prime $P \in \operatorname{Spec}(C(X))$, the factor ring $C(X)/P$ is a valuation domain. In the context of commutative rings, the notion of a pseudo-valuation domain plays a fascinating role, and we investigate the question of when $C(X)/P$ is a pseudo-valuation domain for compact Hausdorff space X and, more generally, when A is a pseudo-valuation domain for bounded real closed domain A. In this context, we note that $C(X)/P$ is always a divided domain and show that it may be a pseudo-valuation domain without being a valuation domain.

1. Introduction

We are interested in studying factor domains of rings of continuous functions. Recall that, for a Tychonoff space X (that is, X is completely regular and Hausdorff), $C(X)$ denotes the ring of all real-valued continuous functions on X, while $C^*(X)$ denotes the subring of bounded functions. The ring $C(X)$ is an example of a lattice-ordered ring, where we order $C(X)$ pointwise, that is, $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in X$. An f-ring is a lattice-ordered ring that can be embedded in a product of totally-ordered rings; $C(X)$ is an f-ring as it is a subring of $\mathbb{R}^X$.

For $f \in C(X)$, we denote its *zeroset* by $Z(f) = \{x \in X : f(x) = 0\}$. The set-theoretic complement of $Z(f)$ in X is denoted by $\operatorname{coz}(f)$ and is called the *cozeroset* of f. A subset $V \subseteq X$ is called a zeroset (respectively, cozeroset) if there is some $f \in C(X)$ such that $V = Z(f)$ (respectively, $V = \operatorname{coz}(f)$). We shall use $cl_X V$ and $int_X V$ to denote the closure and interior of V in X, respectively. For a Tychonoff space X, the Stone-Čech compactification of X, denoted βX, is the unique (up to homeomorphism) compact space containing X densely and C^*-embedded. This last property means that every $f \in C^*(X)$ has a continuous extension to all of βX. Our main reference for $C(X)$ is [10].

For $p \in \beta X$ we form two ideals of $C(X)$:

$$M^p = \{f \in C(X) : p \in cl_{\beta X} Z(f)\}$$

and

$$O^p = \{f \in C(X) : cl_{\beta X} Z(f) \text{ is a neighborhood of } p\}$$

It is known that each M^p is a maximal ideal and that every maximal ideal of $C(X)$ is of the form M^p for some (unique) $p \in \beta X$; this is known as the Gelfand-Kolmogoroff Theorem. (When $p \in X$, we note that $M^p = \{f \in C(X) : f(p) = 0\}$ and $O^p = \{f \in C(X) : p \in int_X Z(f)\}$. These are typically denoted by M_p and O_p, respectively.) The ring $C(X)$ is a *pm*-ring, that is, every prime ideal is contained in a unique maximal ideal. (The notion of a *pm*-ring was originally defined by De Marco and Orsatti [7].) In fact, for any prime $P \in \operatorname{Spec}(C(X))$ (where $\operatorname{Spec}(A)$ denotes the set of all prime ideals of the commutative ring

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A), the set of prime ideals containing P forms a chain, so that $C(X)/P$ is local and its prime ideals are totally-ordered by inclusion. (We remark that local rings are not assumed to be Noetherian.)

Some of the first work on studying the prime ideal structure of $C(X)$ was the following.

Theorem 1.1. [10, 14.25] *For a Tychonoff space X, the following are equivalent.*

1. *For each $p \in \beta X$, O^p is a prime ideal.*
2. *X is an F-space, that is, $C(X)$ is a Bézout ring.*
3. *$C(X)$ is an arithmetical ring.*
4. *βX is an F-space.*

In this class of spaces, the central idea is that, for each $p \in \beta X$, the ideal O^p is prime and the unique minimal prime beneath M^p. Moreover, each of the factor domains $C(X)/O^p$ turns out to be a valuation domain. Recall that a domain D is called a *valuation domain* if its lattice of ideals is totally-ordered by inclusion. Examples of F-spaces are all discrete spaces, $\beta\mathbb{N} \smallsetminus \mathbb{N}$, and $\beta\mathbb{R} \smallsetminus \mathbb{R}$. In [5], the authors began the study of conditions equivalent to $C(X)/P$ being a valuation domain, though in a different context. In [13], the authors considered the global condition on the space X that, for every prime $P \in \mathrm{Spec}(C(X))$, the ring $C(X)/P$ be a valuation domain, calling such spaces *SV-spaces*. An F-space is an SV-space, but not conversely.

Between the two extremes of valuation domain, which evidently is rare for integral domains of the form $C(X)/P$, and just the prime ideals totally-ordered by inclusion, which (as noted above) always holds for $C(X)/P$, other conditions on a local integral domain have interested investigators in commutative ring theory. A local integral domain D, with maximal ideal M and field of fractions K, is called a *pseudo-valuation domain* (PVD) if the ring

$$M : M = \{x \in K : xM \subseteq M\}$$

is a valuation domain with maximal ideal M (see [11]). We call D a *divided domain* if $P = PD_P$ for every prime $P \in \mathrm{Spec}(D)$ (see [1] and [8]). Badawi [4] gives a nice characterization of these conditions as follows.

1. D is a valuation domain if and only if for every $a, b \in D$, $a \mid b$ or $b \mid a$.
2. D is a PVD if and only if for every $a, b \in D$ and every $c \in M$, $a \mid b$ or $b \mid ac$.
3. $M : M$ is a valuation domain if and only if for every $a, b \in D$ and every $c \in M$, $a \mid bc$ or $b \mid ac$.
4. D is a divided domain if and only if for every $a, b \in D$, there exists $n \in \mathbb{N}$, such that $a \mid b$ or $b \mid a^n$.
5. The prime ideals of D are totally-ordered by inclusion if and only if for every $a, b \in D$, there exists $n \in \mathbb{N}$, such that $a \mid b^n$ or $b \mid a^n$.

Clearly (1) implies (2), while (2) implies (3) and (4), and each of (3) and (4) implies (5). In fact, these five conditions actually form a hierarchy, as we now prove.

Theorem 1.2. *If D is a local domain with maximal ideal M such that $M : M$ is a valuation domain, then D is a divided domain.*

Proof. To show that D is a divided domain, let $a, b \in D$; by Badawi's equivalences noted above, it suffices to show that $a \mid b$ or $b \mid a^n$ for some $n \geq 1$. We can suppose that $a, b \in M$ (else at least one of them would be a unit, and we would be done).

Now, set $V = M : M$, a valuation domain by hypothesis. Then, either $Va^2 \subseteq Vb$ or $Vb \subseteq Va^2$. In the former case, we can write $a^2 = bc$ for some $c \in V$, so $a^3 = b(ca)$, where $ca \in M$, so $ca \in D$, and hence $b \mid a^3$ in D. In the latter case, $Vb \subseteq Va^2$, so $b = ca^2 = (ca)a$ for some $c \in V$, that is, $ca \in D$ as above, so $a \mid b$ in D. □

It is known that integral domains of the form $C(X)/P$ are always divided domains; that is, $C(X)/P$ always satisfies condition (4) (as well as condition (5)), which leads naturally to the question of which of conditions (1), (2), or (3) also hold for such factor domains. This question, for compact topological spaces, is the focus of the remainder of this paper. As a reminder, and following notation used in [6], we shall use Ω to denote a compact Hausdorff space.

We will actually work in the class of *real closed rings,* which contains all rings of the form $C(X)$ and their factor domains. In §2 we discuss the definition of real closed rings and summarize properties of real closed domains important for us. In §3 we impose an additional condition on real closed domains, allowing us to characterize when such domains are valuation domains (Theorem 3.1) or PVD's (Theorem 3.3). For compact space Ω and prime $P \in \mathrm{Spec}(C(\Omega))$, the domain $C(\Omega)/P$ satisfies this additional condition, allowing us to specialize these theorems and give conditions equivalent to $C(\Omega)/P$ being a valuation domain (Corollary 3.5) and conditions equivalent to $C(\Omega)/P$ being a PVD (Corollary 3.6). We also show (Corollary 3.7) that conditions (2) and (3) are equivalent for integral domains of the form $C(\Omega)/P$, with Ω compact.

Thus, we are led finally to the question of which of conditions (1) or (2) hold for such domains. It follows from Theorem 1.1 that, if Ω is a compact F-space, then $C(\Omega)/P$ is a valuation domain for every prime $P \in \mathrm{Spec}(C(\Omega))$. In §4, we provide examples to show that the other two possibilities can arise: Example 4.1 gives a compact space Ω and a prime $P \in \mathrm{Spec}(C(\Omega))$ such that $C(\Omega)/P$ is a PVD but not a valuation domain, and Example 4.2 gives a compact space Ω and a prime ideal $P \in \mathrm{Spec}(C(\Omega))$ such that $C(\Omega)/P$ is not a PVD. We also note that Example 4.1 yields a counterexample to the claim in [6, Proposition 4.21 (iii)].

Note that, in general, no two of the above five conditions are equivalent: Example 4.1 shows that (2) does not imply (1); [2, Example 3.2] shows that (3) does not imply (2); Example 4.2 and Corollary 3.7 show that (4) does not imply (3); and [8, Example 2.9] shows that (5) does not imply (4).

2. Real Closed Rings

For our purposes, the appropriate place to work is in the class of real closed rings [18], a generalization to arbitrary commutative rings of the notion of real closed fields, (that is, ordered fields in which each positive element has a square root and each polynomial of odd degree has a root). The study of real closed fields has a long history dating back to the Artin-Schreier Theorem ([3]), which characterizes this class of fields. From the model-theoretic point of view, the following definition appears to be the appropriate generalization (see [15]).

Definition 2.1. A ring A is said to be a *real closed ring* if it satisfies the following conditions.

(1) A is a reduced commutative ring with 1.
(2) The squares of A form a positive cone of a partial order $\leq$ which makes A into an f-ring.
(3) A is 2-convex (see Definition 2.2 below).
(4) For each prime $P \in \mathrm{Spec}(A)$, the quotient field of A/P is a real closed field and the domain A/P is integrally closed.

For example, the field of real numbers and the subfield of real algebraic numbers are real closed. A *lattice-ordered ring* (or ℓ-ring for short) is a ring equipped with a lattice order, denoted $(A, \leq, \vee, \wedge)$, such that, if $a \leq b$ in A and $c \in A$, then $a + c \leq b + c$, and the product of positive elements is again positive. An *f-ring* is an ℓ-ring such that, whenever $a \wedge b = 0$ and

$c \geq 0$, then $ca \wedge b = 0$. Under the assumption of the Axiom of Choice, f-rings are precisely sub-direct products of totally-ordered rings. For any topological space X, the ring $C(X)$ is an f-ring. Condition (2) says that a real closed ring is an f-ring with the strong property that each positive element has a square root. This strong property is often referred to as the f-ring A is *square root closed*.

Some care needs to be taken when perusing the literature on real closed rings. For example, in [5], the authors were interested in determining when a ring of the form $C(X)/P$ for some prime $P \in \mathrm{Spec}(C(X))$ is a real closed ring. However, on further inspection, their definition turns out to be equivalent to 1-convexity and hence to $C(X)/P$ being a valuation domain [5, Definition 20, p. 151, and Proposition 27, p.152], so that their definition is stronger than that of [15], which we use here. Indeed, using an equivalent form of Definition 2.1, Schwartz showed that every $C(X)$ is a real-closed ring, whence each $C(X)/P$ is a real closed domain [18, Theorem 1.2], and hence our results for real closed domains apply to domains of the form $C(X)/P$ as well.

The following definition is due to Larson [14].

Definition 2.2. Let $(A, \leq)$ be an f-ring and $m \in \mathbb{N}$. Then A is said to be *m-convex* if, whenever $0 \leq a \leq b^m$ in A, then there is some $c \in A$ such that $a = cb$, that is, $b \mid a$.

Schwartz showed that every real closed domain is a divided domain [16, Proposition 1.5]. His proof uses the fact that a real closed domain is 2-convex; in fact m-convex for any $m \in \mathbb{N}$ suffices.

Theorem 2.3. *If D is a local, totally-ordered, m-convex integral domain for some $m \in \mathbb{N}$, then D is a divided domain.*

Proof. To show that D is a divided domain, let $a, b \in D$; again by Badawi's equivalences noted above, it suffices to show that $a \mid b$ or $b \mid a^n$ for some $n \geq 1$. If $|b| \leq |a|^m$, then m-convexity yields that $a \mid b$, while if $|a|^m \leq |b|$, then $|a|^{m^2} \leq |b|^m$ implies that $b \mid a^{m^2}$. □

Real closed rings have some nice properties; we summarize those which we shall find most useful. If A is a real closed ring and $P \in \mathrm{Spec}(A)$ is a prime ideal, then the domain A/P is also a real closed ring and hence square root closed (as noted above). In fact, the prime ideal P is a convex ℓ-subgroup, so that A/P is a totally-ordered integral domain. Also, 2-convexity (condition (3)) implies that A has *bounded inversion*, that is, if $1 \leq a$, then a is invertible. (This was originally called "closed under inversion" by Henriksen and Johnson [12]; an interesting property of f-rings with bounded inversion is that they are *pm*-rings. For any f-ring, bounded inversion is equivalent to the statement that every maximal ideal is a convex ℓ-subgroup.) In general, an m-convex f-ring has bounded inversion, since $1 \leq a$ implies $1 \leq a^m$, so that m-convexity yields that a divides 1, that is, a is invertible. A totally-ordered domain with bounded inversion is easily seen to be local, so that Theorem 2.3 applies to A/P in case A is a real closed ring and $P \in \mathrm{Spec}(A)$ is a prime ideal.

If A is a totally-ordered m-convex domain, then A has bounded inversion, and one can use this to show that A is also $(m+1)$-convex. We now give an example which shows that the converse fails, and an example which shows that bounded inversion does not imply m-convexity for any $m \in \mathbb{N}$.

Example 2.4. ($(m+1)$-convexity does not imply m-convexity.) The power series ring in one variable $\mathbb{R}[[X]]$ over $\mathbb{R}$ is a totally-ordered ring with respect to the lexicographic ordering $1 \gg X \gg X^2 \gg X^3 \gg \ldots$. Let A be the subring consisting of those power series whose linear and cubic terms are 0. Then A is not 2-convex, because $X^5 \leq X^4 = (X^2)^2$, but X^2

does not divide X^5 in A. One easily checks that this ring is indeed m-convex for every $m \geq 3$ (hence A does have bounded inversion).

Example 2.5. (Bounded inversion does not imply m-convexity.) Consider the power series ring in two variables $\mathbb{R}[[X,Y]]$ over $\mathbb{R}$, ordered so that

$$1 \gg X \gg X^2 \gg X^3 \gg \ldots \gg Y \gg Y^2 \gg \ldots$$

Since an element of $\mathbb{R}[[X,Y]]$ is invertible if and only if its constant term is non-zero, it follows that $\mathbb{R}[[X,Y]]$ has bounded inversion. Moreover, the primes of $\mathbb{R}[[X,Y]]$ are not linearly ordered, since the prime ideals generated by X and Y, respectively, are not comparable. It follows that $\mathbb{R}[[X,Y]]$ is not divided, and therefore, by Theorem 2.3, not m-convex for any $m \in \mathbb{N}$.

3. Bounded Real Closed Domains

We are interested in a certain type of f-ring which generalizes the idea of $C(\Omega)$ with Ω a compact Hausdorff space. For a general f-ring A, the set

$$A^* = \{a \in A : |a| \leq n \text{ for some } n \in \mathbb{N}\}$$

is a subring of A known as the set of *bounded* elements. Notice that A^* is the convex ℓ-subgroup of A generated by the identity element. When $A = A^*$, A is said to be a *bounded* f-ring. If $(A, \leq)$ is a bounded real closed ring, then so is A/P for every prime $P \in \operatorname{Spec}(A)$. In particular, $C(\Omega)/P$ is a bounded real closed domain for every prime $P \in \operatorname{Spec}(C(\Omega))$.

For the remainder of this section, we assume that $(A, \leq)$ is a bounded real closed domain. As noted in the previous section, A is local, totally-ordered, 2-convex, square root closed, and has bounded inversion. Let M be the maximal ideal of A, and K the quotient field of A. It is well known that K can be totally-ordered in a way that extends the order on A, and so that $(K, \leq)$ is a totally-ordered field; indeed, K is a real closed field. The set K^* of bounded elements of K, also called the set of finite elements of K, is a valuation domain with maximal ideal

$$K^0 = \{a \in K : n|a| \leq 1 \text{ for all } n \in \mathbb{N}\}$$

called the set of infinitesimal elements of K. Since A is assumed to be bounded, $A \subseteq K^*$, and since A is a totally-ordered domain with bounded inversion, $M \subseteq K^0$.

Our characterization of when a bounded real closed domain is a valuation domain generalizes [6, Proposition 4.32]; we add in a new equivalence. Furthermore, it is a good time to suggest the reader compare our results to those in section 1 of [17].

Theorem 3.1. *If $(A, \leq)$ is a bounded real closed domain with maximal ideal M and quotient field K, then the following are equivalent.*

(1) *A is a valuation domain.*
(2) *$K^* = A$.*
(3) *$K^0 = M$ and the natural map $A/M \to K^*/K^0$ is surjective.*
(4) *A is 1-convex.*

Proof. (1) implies (2). Since A is assumed to be bounded, $A \subseteq K^*$. Conversely, let $0 < q \in K^*$, so that $q \leq n$, for some $n \in \mathbb{N}$. Using the hypothesis, $q \in A$ or $q^{-1} \in A$. If $q \in A$, then we are done. If $q^{-1} \in A$, then $q \leq n$ implies that $1 \leq nq^{-1}$, so bounded inversion makes nq^{-1} a unit in A. But n is a unit in A, so q^{-1} is a unit in A, and hence $q \in A$, as required.

(2) implies (3). Since A is a totally-ordered domain with bounded inversion, $M \subseteq K^0$. Conversely, if $q \in K^0$, then $q \in K^* = A$ by (2). Furthermore, q^{-1} is infinitely larger than 1,

and so q is not an invertible element of the bounded domain A, that is, $q \in M$. Then clearly $A/M = K^*/K^0$.

(3) implies (4). Suppose $0 \leq a \leq b$ for $a, b \in A$. Then the element $q = \frac{a}{b} \leq 1$. Since $q \in K^*$, by hypothesis there is some $t \in A$ such that $t + K^0 = q + K^0$. This means that $q - t \in K^0 = M$. But then $q = t + m \in A$ for some $m \in M$. Consequently, $b \mid a$ in A.

(4) implies (1). Let $a, b \in A$. Since A is totally-ordered, we can assume that $0 \leq a \leq b$, and so 1-convexity implies that $b \mid a$ in A. □

Remark 3.2. The equivalence of (1) and (2) is [6, Proposition 4.32]. The equivalence of (1) and (3) is stated without proof in [6]. Both of these are for domains of the form $C(\Omega)/P$.

We are now in position to state and prove our second main theorem; a characterization of when a bounded real closed domain is a PVD.

Theorem 3.3. *If $(A, \leq)$ is a bounded real closed domain with maximal ideal M and quotient field K, then the following are equivalent.*

(1) *A is a PVD.*
(2) *M is an interval of K.*
(3) *M is an interval of K^*.*
(4) *M is an ideal of K^*.*
(5) *$K^* \subseteq M : M$.*

Proof. (1) implies (2). Suppose that A is a PVD and $0 < q < b$ for some $q \in K$ and $b \in M$. Let $V = M : M$, by hypothesis a valuation domain with maximal ideal M. If $q^{-1} \in V$, then $1 < q^{-1}b$, where $q^{-1}b \in M$ (because M is an ideal of V), contradicting the fact that $M \subseteq K^0$. Therefore, $q^{-1} \notin V$, and hence $q \in M$, because V is a valuation domain with maximal ideal M. Thus, M is an interval in K.

(2) implies (3) is patent.

(3) implies (4). Suppose M is an interval of K^* and let $0 < x \in K^*$. Then $x \leq n$ for some $n \in \mathbb{N}$. For any $0 < m \in M$, $0 < xm \leq nm$ so that (3) implies that $xm \in M$. Therefore, M is an ideal of K^*.

(4) and (5) are obviously equivalent.

(5) implies (2). Let $0 < k < m$ for $k \in K$ and $m \in M$. Observe that $\frac{k}{m} < 1$, whence $\frac{k}{m} \in K^*$. By hypothesis, $k = \frac{k}{m}m \in M$. Thus, M is an interval of K.

(2) implies (1). Suppose that M is an interval of K. Let $V = M : M$, and suppose that $a \in K^*$, so that $|a| \leq n$ for some $n \in \mathbb{N}$. Then for all $m \in M$, $|am| \leq n|m|$ with $n|m| \in M$ implies $am \in M$, and hence $a \in V$. That is, $K^* \subseteq V$, so that V is an overring of the valuation domain K^*, and therefore V is also a valuation domain. Finally, suppose that $a \in V \setminus M$ and $m \in M$. Then since A is closed under square roots of positive elements, $\sqrt{|m|} \in M$, so that $\sqrt{|m|} < |a|$ (since M is an interval of K). It follows that $|a^{-1}m| \leq \sqrt{|m|}$, and hence $a^{-1}m \in M$ (again because M is an interval of K). Thus, $a^{-1} \in M : M = V$, so that a is a unit in V, and therefore M is the maximal ideal of V. □

Our third and final main theorem shows that, under the standing hypotheses of this section, if $M : M$ is a valuation domain, then M must be its unique maximal ideal. (Compare [2, Example 3.2], which shows that this is not the case for local integral domains in general.) Consequently, in the hierarchy of the introduction, as applied to the class of bounded real closed domains, (2) and (3) are equivalent.

Theorem 3.4. *If $(A, \leq)$ is a bounded real closed domain with maximal ideal M and quotient field K, then A is a PVD if and only if $M : M$ is a valuation domain.*

Proof. For the nontrivial direction, suppose that $V = M : M$ is a valuation domain and that $0 < a < m$ for some $a \in K$ and $m \in M$. Since V is a valuation domain, $a \in V$ or $a^{-1} \in V$. But $1 < ma^{-1}$ would imply that $ma^{-1} \notin M$ (since $M \subseteq K^0$), so $a^{-1} \notin V$, and hence $a \in V$. Now $0 < am < m^2$ with $am \in M$ implies that $m \mid am$ in A (since A is 2-convex), whence $a \in A$. Then $A \subseteq V$ and $a^{-1} \notin V$ implies $a \in M$, the maximal ideal of A. Therefore, M is an interval in K, so A is a PVD by Theorem 3.3. □

If Ω is a compact Hausdorff space and $P \in \operatorname{Spec}(C(\Omega))$ is prime, we consider the bounded real closed domain $A = C(\Omega)/P$. It is known that both A/M and K^*/K^0 are isomorphic to the real numbers via the inclusion $\mathbb{R} \subseteq A \subseteq K^*$. Applying Theorem 3.1 produces the following corollary.

Corollary 3.5. *For a compact space Ω and prime $P \in \operatorname{Spec}(C(\Omega))$, the following are equivalent for the bounded real closed domain $A = C(\Omega)/P$.*

(1) *A is a valuation domain.*
(2) *$K^* = A$.*
(3) *$K^0 = M$*
(4) *A is 1-convex.*

Any prime ideal satisfying any of the equivalent conditions of the corollary is known as a *valuation prime*. Using the same notation, the prime $P \in \operatorname{Spec}(C(\Omega))$ is called a *strongly convex prime* if M is an interval of K (see [6]). The following is a corollary to Theorem 3.3.

Corollary 3.6. *For a compact space Ω and prime $P \in \operatorname{Spec}(C(\Omega))$, the following are equivalent for the bounded real closed domain $A = C(\Omega)/P$.*

(1) *A is a PVD.*
(2) *P is strongly convex.*
(3) *M is an ideal of K^*.*
(4) *$K^* \subseteq M : M$.*

Our third and final result about $C(\Omega)$ is the application of Theorem 3.4.

Corollary 3.7. *For a compact space Ω and prime $P \in \operatorname{Spec}(C(\Omega))$, the bounded real closed domain $A = C(\Omega)/P$ is a PVD if and only if $M : M$ is a valuation domain.*

Remark 3.8. It should be noted that Theorem 3.1 holds for arbitrary (totally-ordered) bounded domains with bounded inversion. What is of main importance is that M be an interval of A. Of course, statement (4) implies that bounded inversion is necessary. For example, if $A = \mathbb{Z}_{(2)}$, the localization of the integers at the prime 2, then A is a totally-ordered valuation domain which does not have bounded inversion. In this case, $K^* = \mathbb{Q}$ and $K^0 = \{0\}$, so none of conditions (2), (3), and (4) is satisfied.

Theorem 3.3, however, appears to require more than taking a bounded domain with bounded inversion. Notice that in the proof of (2) implies (1), we do use that A is a real closed ring, inasmuch as A is square root closed. Likewise, the proof of Theorem 3.4, which relies on Theorem 3.3, also uses the 2-convexity condition.

We conclude this section with an example of a totally-ordered bounded domain with bounded inversion for which the first part of Theorem 3.1 (3) holds but the second part fails, illustrating the importance of the residue fields in that theorem.

Example 3.9. Let $\mathbb{A}$ denote the field of real algebraic numbers and let A be the subring of the power series ring in one variable $\mathbb{R}[[X]]$ over $\mathbb{R}$ consisting of those series whose constant term belongs to $\mathbb{A}$. Order $\mathbb{R}[[X]]$ as in Example 2.4, so using the lexicographic ordering

$1 \gg X \gg X^2 \gg X^3 \gg \ldots$. Then one easily checks that A is bounded and has bounded inversion, and $K^* = \mathbb{R}[[X]]$ with $M = X\mathbb{R}[[X]]$ the maximal ideal of both A and K^*, and hence $K^0 = M$ but $A \neq K^*$. Note that K^* is a valuation domain with maximal ideal M, so that $M : M = K^*$, and hence A is a PVD but not a valuation domain.

4. Examples

In this section, we continue the practice that Ω will always be a compact Hausdorff topological space and $P \in \mathrm{Spec}(C(\Omega))$ a prime ideal. Following the notation used in [5], we use A_P to denote the factor domain $C(\Omega)/P$, M_P its maximal ideal, and K_P its quotient field. We supply two examples. The aim of the first is to show that A_P can be a PVD without being a valuation domain (Example 4.1), while the aim of the second is to show that A_P need not be a PVD (Example 4.2). In the process, we show that our first example provides a counterexample to [6, Proposition 4.21 (iii)].

Our first example comes from [10, Exercise 14.G], as well as various results in [5] and [6].

Example 4.1. Let $\mathbb{N}^*$ be the one-point compactification of the countably infinite discrete space $\mathbb{N}$, so that $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, where the open sets consist of all subsets of $\mathbb{N}$ as well as all cofinite subsets of $\mathbb{N}^*$ containing ∞. (Equivalently, we can view $\mathbb{N}^* = \{0\} \cup \{\frac{1}{k} : k \in \mathbb{N}\}$, with the relative topology from $\mathbb{R}$.) The ring $C(\mathbb{N}^*)$ is isomorphic to the ring of convergent sequences of real numbers, under pointwise operations.

Clearly, $\mathbb{N}^*$ is compact, so the (distinct) maximal ideals of $C(\mathbb{N}^*)$ consist of the ideals $M_p = \{f \in C(\mathbb{N}^*) : f(p) = 0\}$ as p ranges over all elements of $\mathbb{N}^*$. One easily checks that, for $p \in \mathbb{N}$, the maximal ideal M_p is also a minimal prime. The maximal ideal M_∞ is not a minimal prime, however; the minimal primes contained in M_∞ can be constructed as follows. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and set

$$P_{\mathcal{U}} = \{f \in C(\mathbb{N}^*) : Z(f) = S \cup \{\infty\} \text{ for some } S \in \mathcal{U}\}$$

Then $P_{\mathcal{U}}$ is a prime ideal, and the (distinct) minimal prime ideals of $C(\mathbb{N}^*)$ contained in M_∞ consist of the ideals $P_{\mathcal{U}}$ as $\mathcal{U}$ ranges over all free ultrafilters on $\mathbb{N}$.

We are interested in determining whether A_P is a valuation domain, PVD, or neither, for prime ideals $P \subseteq C(\mathbb{N}^*)$. Certainly we can restrict our attention to non-maximal primes P. Moreover, if $P \subseteq Q$ are prime ideals of $C(\mathbb{N}^*)$, with Q non-maximal, then one easily checks that A_Q is a valuation domain (respectively, PVD) if A_P is, so we can further restrict our attention to the factor domains $A_{P_{\mathcal{U}}}$ as $\mathcal{U}$ ranges over all free ultrafilters on $\mathbb{N}$.

Using the characterization given in [5, Definition 29 (31), p.153], we shall call the free ultrafilter $\mathcal{U}$ a *P-point* if for every partition $\{P_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $P_k \notin \mathcal{U}$ for each $k \in \mathbb{N}$, there exists a set $U \in \mathcal{U}$ such that $U \cap P_k$ is finite for each $k \in \mathbb{N}$. We remark that the existence of P-points is independent of ZFC, while the existence of non-P-points is provable in ZFC.

We claim that, if $\mathcal{U}$ is not a P-point, then $A_{P_{\mathcal{U}}}$ is a PVD but not a valuation domain. From [5, Proposition 27, p.152, and Theorem 1, p.172], we see that $A_{P_{\mathcal{U}}}$ is a valuation domain if and only if $\mathcal{U}$ is a P-point, so it suffices to show that, for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the factor domain $A_{P_{\mathcal{U}}}$ is a PVD. Equivalently, by Corollary 3.6 it suffices to show that every prime $P_{\mathcal{U}}$ of $C(\mathbb{N}^*)$ is strongly convex. This is claimed as [6, Proposition 4.36], but, as the proof given there is sketchy at best, for completeness we include a proof here.

We must show that $M_{P_{\mathcal{U}}}$ is an interval of $K_{P_{\mathcal{U}}}$. To be efficient in our notation, we shall write the coset $f + P_{\mathcal{U}}$ in $A_{P_{\mathcal{U}}} = C(\mathbb{N}^*)/P_{\mathcal{U}}$ as simply $\bar{f}$. Thus, suppose that $0 < \frac{\bar{f}}{\bar{g}} < \bar{h}$ for some $\bar{f}, \bar{g} \in A_{P_{\mathcal{U}}}$ and $\bar{h} \in M_{P_{\mathcal{U}}}$. If $\bar{g} \notin M_{P_{\mathcal{U}}}$, then $\bar{g}$ is a unit in $A_{P_{\mathcal{U}}}$, so $\frac{\bar{f}}{\bar{g}} \in A_{P_{\mathcal{U}}}$ implies

$\frac{\bar{f}}{\bar{g}} \in A_{P_{\mathcal{U}}} \cap K_P^0 = M_{P_{\mathcal{U}}}$, using the fact that $\bar{h}$ is infinitesimal and $0 < \frac{\bar{f}}{\bar{g}} < \bar{h}$. Therefore, we can suppose that $\bar{g} \in M_{P_{\mathcal{U}}}$, which forces $\bar{f} \in M_{P_{\mathcal{U}}}$ as well, since $\bar{h}$ finite and $0 < \frac{\bar{f}}{\bar{g}} < \bar{h}$ implies $\frac{\bar{f}}{\bar{g}}$ is finite also. Note that the canonical map from $C(\mathbb{N}^*)$ to $A_{P_{\mathcal{U}}}$ is a lattice homomorphism ([10, Theorem 5.3]), so that $h + P_{\mathcal{U}} = |h + P_{\mathcal{U}}| = |h| + P_{\mathcal{U}}$, and hence we can replace h by $|h|$, so that $h \geq 0$ in $C(\mathbb{N}^*)$. Similarly, since $A_{P_{\mathcal{U}}}$ is totally-ordered and $\frac{\bar{f}}{\bar{g}} > 0$, we can replace f and g by $|f|$ and $|g|$, respectively, so that $f \geq 0$ and $g \geq 0$ in $C(\mathbb{N}^*)$ as well. In fact, since $0 < \bar{f} < \bar{g}\bar{h}$, we can replace f by $f \wedge gh$, so that $0 \leq f \leq gh$ in $C(\mathbb{N}^*)$.

Now define a function $k : \mathbb{N}^* \to \mathbb{R}$ by $k(n) = \frac{f(n)}{g(n)}$ if $n \in \mathbb{N} \smallsetminus Z(g)$, and $k(n) = 0$ otherwise (including $k(\infty) = 0$). Then k is continuous on the discrete subspace $\mathbb{N}$ of $\mathbb{N}^*$, and $0 \leq k(n) \leq h(n)$ for all $n \in \mathbb{N}$, while $\lim_{n\to\infty} h(n) = h(\infty) = 0$ (because $\bar{h} \in M_{P_{\mathcal{U}}} = M_\infty/P_{\mathcal{U}}$ implies $h \in M_\infty$) forces $\lim_{n\to\infty} k(n) = 0 = k(\infty)$. Therefore, k is continuous on $\mathbb{N}^*$, and $f(n) = k(n)g(n)$ for all $n \in \mathbb{N}^*$ (because $0 \leq f \leq gh$ implies that $Z(g) \subseteq Z(f)$), so that $\frac{\bar{f}}{\bar{g}} = \bar{k} \in M_{P_{\mathcal{U}}}$, completing the proof that $P_{\mathcal{U}}$ is strongly convex.

It now appears that we have reached a contradiction in the results of [5] and [6]. By [6, Proposition 4.21 (iii)], $M_{P_{\mathcal{U}}}$ is cofinal in $K^0_{P_{\mathcal{U}}}$. As we have just shown, $M_{P_{\mathcal{U}}}$ is an interval in $K_{P_{\mathcal{U}}}$, as is $K^0_{P_{\mathcal{U}}}$ (by definition), from which it follows that $K^0_{P_{\mathcal{U}}} = M_{P_{\mathcal{U}}}$. But then by Corollary 3.5, $P_{\mathcal{U}}$ is a valuation prime, and hence [5, Proposition 27, p.152, and Theorem 1, p.172] imply that $\mathcal{U}$ is a P-point. That is, we have shown that every free ultrafilter on $\mathbb{N}$ is a P-point, which is definitely not the case. The culprit is [6, Proposition 4.21 (iii)], which is false, as we now show by example.

Let $\mathcal{V}$ be a non-P-point, so that there is a partition $\{P_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $P_k \notin \mathcal{V}$ for each $k \in \mathbb{N}$, and such that for every set $U \in \mathcal{V}$, there is some $k \in \mathbb{N}$ such that $U \cap P_k$ is infinite. We show that $M_{P_{\mathcal{V}}}$ is not cofinal in $K^0_{P_{\mathcal{V}}}$. To the end, define real-valued functions f and g by $f(n) = \frac{1}{kn}$ and $g(n) = \frac{1}{n}$ for each $n \in P_k$ and each $k \in \mathbb{N}$, and let $f(\infty) = g(\infty) = 0$. Clearly $f, g \in C(\mathbb{N}^*)$, and we begin by showing that $\frac{\bar{f}}{\bar{g}} \in K^0_{P_{\mathcal{V}}}$. Given $k \in \mathbb{N}$, define the real-valued function h_k by $h_k(n) = |\frac{1}{k}g(n) - f(n)|$ for $n \in P_1 \cup \ldots \cup P_k$, and $h_k(n) = 0$ otherwise (including $n = \infty$). If $1 \leq i \leq k$, then for $n \in P_i$, one computes that $h_k(n) = \frac{k-i}{ikn} \leq \frac{1}{n}$, so it follows that $h_k \in C(\mathbb{N}^*)$. In fact, $\mathbb{N} \smallsetminus (P_1 \cup \ldots \cup P_k) \subseteq Z(h_k) \smallsetminus \{\infty\}$, where $P_1, \ldots, P_k \notin \mathcal{V}$, forces $Z(h_k) \smallsetminus \{\infty\} \in \mathcal{V}$, so that $h_k \in P_{\mathcal{V}}$. Then $f(n) \leq \frac{1}{k}g(n) + h_k(n)$ for all $n \in \mathbb{N}^*$ implies $\bar{f} \leq \frac{1}{k}\bar{g}$ in $A_{P_{\mathcal{V}}}$, so that, since $k \in \mathbb{N}$ was arbitrary, $\frac{\bar{f}}{\bar{g}} \in K^0_{P_{\mathcal{V}}}$.

Suppose that $\frac{\bar{f}}{\bar{g}} \leq \bar{h}$ for some $h \in C(\mathbb{N}^*)$, so that $0 \leq \bar{f} \leq \bar{g}\bar{h}$ in $A_{P_{\mathcal{V}}}$. Thus, $f \leq gh + l$ for some $l \in P_{\mathcal{V}}$. Let $U = Z(l) \smallsetminus \{\infty\}$, so that $U \in \mathcal{V}$. By assumption, there is some $k \in \mathbb{N}$ such that $U \cap P_k$ is infinite. Now, for $n \in U \cap P_k$, we have

$$\frac{1}{kn} \leq \frac{1}{n}h(n) + 0$$

so that $\frac{1}{k} \leq h(n)$. Since the set $U \cap P_k$ is infinite, it follows that

$$h(\infty) = \lim_{n\to\infty} h(n) \neq 0$$

and hence $\bar{h} \notin M_{P_{\mathcal{V}}}$. Therefore, $M_{P_{\mathcal{V}}}$ is not cofinal in $K^0_{P_{\mathcal{V}}}$.

Our second example is essentially [6, Theorem 4.40].

Example 4.2. Here we provide an example of a compact space Ω and a minimal prime Q in $C(\Omega)$ such that $A_Q = C(\Omega)/Q$ is not a PVD. Let X be a countable disjoint union of copies of

$\mathbb{N}^*$, so $X = \mathbb{N} \times \mathbb{N}^*$ with the product topology. Set $\Omega = \beta X$, the Stone-Čech compactification of X. We shall work in $C^*(X)$, which is isomorphic to $C(\Omega)$.

Let $\mathcal{U}$ and $\mathcal{V}$ be free ultrafilters on $\mathbb{N}$. For each $F \in C^*(X)$ and each $m \in \mathbb{N}$, define

$$\sigma_{F,m} = \{n \in \mathbb{N} : F(m,n) = 0\}$$

and

$$\sigma_F^{\mathcal{V}} = \{m \in \mathbb{N} : \sigma_{F,m} \in \mathcal{V}\}$$

and

$$Q_{\mathcal{U},\mathcal{V}} = \{F \in C^*(X) : \sigma_F^{\mathcal{V}} \in \mathcal{U}\}$$

We claim that $Q_{\mathcal{U},\mathcal{V}}$ is a minimal prime ideal of $C^*(X)$. Observe that since $\sigma_{F,m} \cup \sigma_{G,m} = \sigma_{FG,m}$ for all $F, G \in C^*(X)$ and $m \in \mathbb{N}$, then $\sigma_{FG}^{\mathcal{V}} = \sigma_F^{\mathcal{V}} \cup \sigma_G^{\mathcal{V}}$. Therefore, since $\mathcal{U}$ is an ultrafilter, and hence a prime filter, $Q_{\mathcal{U},\mathcal{V}}$ is a prime ideal. Furthermore, if $F \in Q_{\mathcal{U},\mathcal{V}}$, then one can define a $G \in C^*(X)$ such that $G \notin Q_{\mathcal{U},\mathcal{V}}$ and $FG = 0$, whence $Q_{\mathcal{U},\mathcal{V}}$ is a minimal prime ideal of $C^*(X)$. Observe that this holds for any free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$.

We claim that

$$M^{\mathcal{U}} = \{F \in C^*(X) : \forall \epsilon > 0, \{m \in \mathbb{N} : |F(m,\infty)| < \epsilon\} \in \mathcal{U}\}$$

is the unique maximal ideal containing $Q_{\mathcal{U},\mathcal{V}}$. Since $\mathcal{U}$ is closed under finite intersections and elements of $C^*(X)$ are bounded functions, it follows easily that $M^{\mathcal{U}}$ is an ideal of $C^*(X)$. To see that $M^{\mathcal{U}}$ is a maximal ideal, suppose $F \in C^*(X) \setminus M^{\mathcal{U}}$; we show that there is some $G \in M^{\mathcal{U}}$ such that $F + G$ is a unit in $C^*(X)$. Note that $C^*(X)$ consists of the bounded continuous functions on X, so a function in $C^*(X)$ is a unit if and only if it is bounded away from 0. Since $F \notin M^{\mathcal{U}}$, there is a positive ϵ such that $S = \{m \in \mathbb{N} : |F(m,\infty)| < \epsilon\} \notin \mathcal{U}$, and hence $(\mathbb{N} \setminus S) \in \mathcal{U}$ (because $\mathcal{U}$ is an ultrafilter). For each $m \in S$, define $G(m,n) = \epsilon - F(m,n)$ for all $n \in \mathbb{N}^*$. For each $m \in (\mathbb{N} \setminus S)$, since $|F(m,\infty)| \geq \epsilon$, there exists some $k_m \in \mathbb{N}$ such that $|F(m,n)| > \epsilon/2$ for all $n \geq k_m$; let $G(m,n) = \epsilon - F(m,n)$ for $n < k_m$, and $G(m,n) = 0$ for $n \geq k_m$, including $G(m,\infty) = 0$. Then one easily checks that G is continuous on X (being continuous on each fixed first coordinate), and $\{m \in \mathbb{N} : G(m,\infty) = 0\} = (\mathbb{N} \setminus S) \in \mathcal{U}$, so that $G \in M^{\mathcal{U}}$. Moreover, $|F(m,n) + G(m,n)| \geq \epsilon/2$ for every $(m,n) \in X$, so that $F + G$ is a unit in $C^*(X)$, as required.

To see that $Q_{\mathcal{U},\mathcal{V}} \subseteq M^{\mathcal{U}}$, let $F \in Q_{\mathcal{U},\mathcal{V}}$, so that $\sigma_F^{\mathcal{V}} \in \mathcal{U}$. Then if $m \in \sigma_F^{\mathcal{V}}$, we get $\sigma_{F,m} = \{n \in \mathbb{N} : F(m,n) = 0\} \in \mathcal{V}$, where $\mathcal{V}$ is a free ultrafilter, so that $\{n \in \mathbb{N} : F(m,n) = 0\}$ must be an infinite subset of $\mathbb{N}$, and therefore $F(m,\infty) = 0$. That is, $F(m,\infty) = 0$ for m in the set $\sigma_F^{\mathcal{V}} \in \mathcal{U}$, from which it follows that $F \in M^{\mathcal{U}}$, and hence $Q_{\mathcal{U},\mathcal{V}} \subseteq M^{\mathcal{U}}$. Since (as noted earlier) $Q_{\mathcal{U},\mathcal{V}}$ is contained in a unique maximal ideal, this completes the proof of the claim.

Now, suppose further that $\mathcal{V}$ is not a P-point, and for the sake of convenience, set $Q = Q_{\mathcal{U},\mathcal{V}}$. With $A_Q = C^*(X)/Q$ and $M_Q = M^{\mathcal{U}}/Q$, we show that $M_Q : M_Q$ is not a valuation domain, which will complete the proof that A_Q is not a PVD.

Since $\mathcal{V}$ is not a P-point, there is a partition $\{P_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $P_k \notin \mathcal{V}$ for each $k \in \mathbb{N}$, and such that for every set $U \in \mathcal{V}$, there is some $k \in \mathbb{N}$ such that $U \cap P_k$ is infinite. Let $f, g \in C(\mathbb{N}^*)$ be defined as in Example 4.1, so that $f(n) = \frac{1}{kn}$ and $g(n) = \frac{1}{n}$ for $n \in P_k$, while $f(\infty) = 0 = g(\infty)$. Next, for each $m \in \mathbb{N}$, let $S_m = \mathbb{N} \setminus \bigcup_{j=1}^{m} P_j$, and note that $S_m \in \mathcal{V}$. Define $F, G \in C^*(X)$ by

$$F(m,n) = \chi_{S_m}(n) f(n)$$

and

$$G(m,n) = \chi_{S_m}(n) g(n)$$

where χ_{S_m} is the characteristic function of S_m, and $F(m,\infty) = 0 = G(m,\infty)$. Let $H \in C^*(X)$ be defined by $H(m,n) = \frac{1}{m}$ for all $(m,n) \in X$, and observe that $H \in M^{\mathcal{U}}$.

For ease of notation, let $\bar{F}$, $\bar{G}$, and $\bar{H}$ denote the cosets $F+Q$, $G+Q$, and $H+Q$, respectively. Then $\bar{H} \in M_Q$, and, to show that $M_Q : M_Q$ is not a valuation domain, it suffices to show that $\bar{G}/\bar{F}, \bar{F}/\bar{G} \notin M_Q : M_Q$, which we do by showing that $(\bar{G}/\bar{F})\bar{H}, (\bar{F}/\bar{G})\bar{H} \notin M_Q$.

If $n \in P_k$, then $\chi_{S_m}(n) \neq 0$ if and only if $n \notin \bigcup_{j=1}^m P_j$ if and only if $k > m$, so that, by construction $F \leq GH$ in $C^*(X)$. Hence, $1 \leq (\bar{G}/\bar{F})\bar{H}$ in $A_Q = C^*(X)/Q$, from which it follows that $(\bar{G}/\bar{F})\bar{H} \notin M_Q$ (since $M_Q \subseteq K_Q^0$).

Suppose, by way of contradiction, that $(\bar{F}/\bar{G})\bar{H} \in M_Q$, so that $FH - GE \in Q$ for some $E \in M^{\mathcal{U}}$. Then for $(m,n) \in \mathbb{N} \times \mathbb{N}$, we compute that

$$\begin{aligned}(FH - GE)(m,n) &= \chi_{S_m}(n)f(n) \cdot \frac{1}{m} - \chi_{S_m}(n)g(n) \cdot E(m,n) \\ &= \frac{\chi_{S_m}(n)g(n)}{m} \cdot \big[\frac{f(n)}{g(n)} - m \cdot E(m,n)\big].\end{aligned}$$

Since $FH - GE \in Q$, by definition $\sigma^{\mathcal{V}}_{FH-GE} \in \mathcal{U}$, so that there exists a natural number $m \in \sigma^{\mathcal{V}}_{FH-GE}$. Thus, $\sigma^{\mathcal{V}}_{FH-GE,m} \in \mathcal{V}$; that is, $\{n \in \mathbb{N} : (FH-GE)(m,n) = 0\} \in \mathcal{V}$. But $S_m \in \mathcal{V}$, so that $\{n \in \mathbb{N} : \chi_{S_m}(n)g(n) = 0\} \notin \mathcal{V}$, and hence

$$\{n \in \mathbb{N} : \frac{f(n)}{g(n)} = m \cdot E(m,n)\} \in \mathcal{V}.$$

For this natural number m, define $e : \mathbb{N}^* \to \mathbb{R}$ by $e(n) = E(m,n)$, so that $e \in C(\mathbb{N}^*)$ (since $E \in C^*(X)$). Now $\{n \in \mathbb{N} : f(n) = m \cdot g(n) \cdot e(n)\} \in \mathcal{V}$ implies $f - m \cdot g \cdot e \in P_{\mathcal{V}}$, and therefore $\bar{f}/\bar{g} = m \cdot \bar{e} \in A_{P_{\mathcal{V}}}$. But in Example 4.1 it was shown that $\bar{f}/\bar{g} \in K^0_{P_{\mathcal{V}}}$, so it follows that $\bar{f}/\bar{g} \in A_{P_{\mathcal{V}}} \cap K^0_{P_{\mathcal{V}}} = M_{P_{\mathcal{V}}}$, contradicting Example 4.1. It follows that $(\bar{F}/\bar{G})\bar{H} \notin M_Q$, as claimed.

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