p-EMBEDDINGS

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ABSTRACT. This is the sequel to [3] where the notion of a p-extension of commutative rings was investigated: a unital extension of commutative rings, say $R \hookrightarrow S$, is a p-extension if for every $s \in S$ there is an $r \in R$ such that $rS = sS$. In this article we apply the theory of p-extensions to rings of continuous functions. We show that this concept lays between the concepts of C^*-embeddings and z-embeddings.

1. INTRODUCTION

The history of investigating C-embedded and C^*-embedded subspaces can be traced back to Tietze [25] and Urysohn [26]. Tietze first showed that in a metric space every closed subset is C-embedded, then Urysohn generalized this statement to normal spaces. Engleking [9] calls this important theorem about normal spaces "The Tietze-Urysohn Theorem". Over the past 30 years many authors have looked at different ways for a space to embed in another that generalizes the notion of C^*-embedding.

Our main approach in this article is to view these kinds of embeddings from an algebraic point of view. In particular, given a Tychonoff space X we let $C(X)$ denote the ring of real-valued continuous functions on the domain X. Then for a given subspace Y of X we consider the ring homomorphism

$$\Psi : C(X) \to C(Y)$$

induced by restriction. It is known that Y is a dense subspace of X precisely when Ψ is injective. For the record we assume that all spaces are Tychonoff and all rings are commutative with identity.

Our starting point is the trivial case: when Ψ is an isomorphism. This leads to us recalling the definition of a C-embedding.

Definition 1.1. Let Y a subspace of X. Y is called a *C-embedded* subspace of X if for every $f \in C(Y)$ there is some $g \in C(X)$ such that for all $y \in Y$, $f(y) = g(y)$. Observe that this notion is equivalent to saying that $\Psi : C(X) \to C(Y)$ is a surjection. Thus a dense subspace of X is C-embedded if and only if Ψ is an isomorphism.

There is a similar situation regarding rings of bounded continuous functions. As is customary in the field, we let $C^*(X)$ denote the subring of $C(X)$ consisting of the bounded continuous functions. For a subspace $Y \subseteq X$ the map

$$\Psi^* : C^*(X) \to C^*(Y)$$

is the ring homomorphism induced by restriction. Continuing in this manner yields the definition of a C^*-embedded subspace and the result that a dense subspace is C^*-embedded if and only if Ψ^* is an isomorphism. It is straightforward to show that any C-embedded subspace

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is necessarily C^*-embedded. The converse does not hold as is well-known. (Notationally, for $f \in C(X)$ (resp. $f \in C^*(X)$) we prefer to write $f|_Y$ instead of $\Psi(f)$ (resp. $\Psi^*(f)$).)

As for an example, recall that βX denotes the Stone-Čech compactification of X. It is well-known that X is a dense C^*-embedded subspace of βX, and so the induced homomorphism of $C(\beta X) = C^*(\beta X)$ into $C^*(X)$ is an isomorphism. The real-compactification of X will be denoted as is customary by υX.

The most well-known generalization of a C^*-embedding is that of a z-embedding, which we define next.

Definition 1.2. For $f \in C(X)$ the *zeroset of* f is the set $Z(f) = \{x \in X : f(x) = 0\}$. The set-theoretic complement of $Z(f)$ is denoted by $\operatorname{coz}(f)$ and is called the *cozeroset of* f. A subspace of X is called a *zeroset* (resp. *cozeroset*) if it is of the form $Z(f)$ (resp. $\operatorname{coz}(f)$) for some $f \in C(X)$. We let $\mathcal{Z}(X)$ denote the collection of all zerosets of X. Continuity of functions ensures that zerosets are closed. Moreover, $\mathcal{Z}(X)$ forms a base for the topology of closed sets in X.

Definition 1.3. Let Y a subspace of X. Y is said to be *z-embedded* in X if for every $Z \in \mathcal{Z}(Y)$ there is a $W \in \mathcal{Z}(X)$ such that $W \cap Y = Z$. This is equivalent to the condition that for each cozeroset T of Y there is a cozeroset D of X such that $D \cap Y = T$.

Clearly, for each $Z \in \mathcal{Z}(X)$, $Z \cap Y \in \mathcal{Z}(Y)$ since $Z(f|_Y) = Z(f) \cap Y$. It follows then that a C^*-embedded subspace is a z-embedded subspace. It is also known that cozerosets are z-embedded. The concept of a z-embedding was originally introduced in [4]. The list of references on z-embeddings is vast, so we encourage the reader to peruse the literature on this subject. We are now in a position to state the Tietze-Urysohn Theorem involving the just-discussed embeddings.

Theorem 1.4. *Suppose X is a Tychonoff space. The following statements are equivalent.*

(1) *X is a normal space.*
(2) *Every closed set is C-embedded.*
(3) *Every closed set is C^*-embedded.*
(4) *Every closed set is z-embedded.*

Since we are interested in ring-theoretic properties concerning $C(X)$, we conclude this section with a discussion of some ring theory and its connection to rings of continuous functions. For more information on rings of continuous functions, we urge the reader to consult the most comprehensive text on the subject: [13]. Our main resource for topological information is [9].

Let R be a commutative ring with identity. The set of units of R will be denoted by $\mathfrak{U}(R)$. The annihilator of a subset $S \subseteq R$ is denoted by $\operatorname{Ann}_R(S)$. When $S = \{a_1, \dots, a_n\}$ we instead write $\operatorname{Ann}_R(a_1, \dots, a_n)$. The lattice of all ideals of R is denoted by $\mathcal{L}(R)$. The collection of all prime ideals of R is denoted by $\operatorname{Spec}(R)$. The subspace of $\mathcal{L}(R)$ consisting of all maximal (resp. minimal prime) ideals, will be denoted by $\operatorname{Max}(R)$ (resp. $\operatorname{Min}(R)$). The Gelfand-Kolmogorov Theorem asserts that $\operatorname{Max}(C(X))$ is homeomorphic to βX.

The ring R is called *reduced* if it has no nonzero nilpotent elements. For any space X, $C(X)$ is reduced. An element of R is called *regular* if it is not a zero-divisor. For $f \in C(X)$, f is regular if and only if $\operatorname{coz}(f)$ is a dense subspace of X. The principal ideal of R generated by r will be denoted by rR. An *idempotent* of R is an element $e \in R$ for which $e^2 = e$. For $C(X)$ the idempotents are precisely the characteristic functions of clopen subsets of X.

R is called a *von Neumann regular ring* if for every $a \in R$ there is an $x \in R$ such that $a^2x = a$. This is known to be equivalent to the condition that every principal ideal of R is generated by an idempotent of R. Alternatively, R is von Neumann regular if and only if R is reduced and every prime ideal is maximal. In the case of rings of continuous functions, X is called a *P-space* if $C(X)$ is a von Neumann regular ring. That X is a P-space is equivalent to the condition that every zeroset of X is open. Equivalently, every cozeroset of X is C-embeddded. This class of spaces was originally studied in [11].

R is called a *Bézout ring* if every finitely generated ideal is principal. A ring R is called a *chained ring* if its set of ideals is totally ordered by inclusion. A chained domain is called a *valuation domain*. A chained ring is Bézout, and every von Neumann regular ring is also Bézout. A space X is called an *F-space* if $C(X)$ is a Bézout ring. Equivalently, every cozeroset in X is C^*-embedded. This class of spaces was originally studied in [12].

R is called a *Prüfer ring* if every finitely generated regular ideal is invertible. Every Bézout ring is a Prüfer ring. A space X is called a *quasi F-space* if $C(X)$ is a Prüfer ring. Equivalently, every dense cozeroset in X is C^*-embedded. This class of spaces was originally studied in [7].

We conclude this section by recalling the main definitions from [3].

Definition 1.5. 1) We say the extension of rings $R \hookrightarrow S$ is a *p-extension* if it satisfies the following property:

$$\text{for every } s \in S \text{ there is an } r \in R \text{ such that } sS = rS.$$

In other words for each $s \in S$ there is an $r \in R$ and $t_1, t_2 \in S$ such that $r = st_1$ and $s = rt_2$.

2) We say the extension $R \hookrightarrow S$ is an *associate p-extension* if it satisfies the following property:

$$\text{for every } s \in S \text{ there is an } r \in R \text{ and a unit } u \in \mathfrak{U}(S) \text{ such that } r = su.$$

Condition 2) is saying that every element of S is associate to an element of R. Clearly, an associate p-extension is a p-extension. The converse holds whenever S is an integral domain.

3) A stronger condition than 2) is the following.

$$\text{For every } s \in S \text{ there are } r, u \in R \text{ such that } u \in \mathfrak{U}(S) \text{ and } r = su.$$

Condition 3) is a well-understood property in the theory of rings. Such an extension is called a *regular localization of* R (or, as it is also known, a *quotient ring* of R). If U is a multiplicatively closed subset of regular elements, then R_U denotes the quotient ring of R by U; specifically $R_U = \{\frac{a}{u} : u \in U\}$. When U is the set of all regular elements, we instead write $q(R)$, the classical ring of fractions of R. $R \hookrightarrow R_U$ is a regular localization.

Observe that an element $u \in R$ which is a unit of R_U is a regular element of R. Thus, if S is a regular localization of R, then every element $s \in S$ can be written in the form $\frac{r}{u}$ for some $r \in R$ and u a regular element of R. It follows that a regular localization of R is an overring.

For $C(X)$, The Representation Theorem of Fine, Gillman, and Lambek (see [10] Theorem 2.6) states that we can view the classical ring of quotients of $C(X)$, denoted $q(X)$, in the following useful way. Denote the collection of dense cozerosets of X by $\mathfrak{G}_0(X)$. Then

$$q(X) = \lim_{O \in \mathfrak{G}_0(X)} C(O).$$

Moreover, the direct limit on the right side of the equality translates into saying that $q(X)$ is the union of $\{C(O) : O \in \mathfrak{G}_0(X)\}$ modulo the equivalence relation defined as follows: for $f_1 \in C(O_1), f_2 \in C(O_2)$, f_1 and f_2 are equivalent if they agree on the intersection of their domains.

2. $C(X)$

We would like to investigate p-extensions in the context of rings of continuous functions. As mentioned in the first section, whenever Y is a dense subspace of X, we get an embedding of rings $\Psi : C(X) \hookrightarrow C(Y)$ induced by restriction. In general, if Y is a subspace (not necessarily dense), then the map induced by restriction is not necessarily injective. We can however still discuss whether the ring $C(Y)$ is a (associate) p-extension of $\Psi(C(X))$. It is in this sense that we say that the subspace Y is *(associate) p-embedded* in X. Furthermore, if $C^*(Y)$ is a (associate) p-extension of $\Psi(C^*(X))$, then we will say Y is a *(associate) p^*-embedded* subspace of X. If $C(Y)$ is a regular localization of $\Psi(C(X))$, then we shall say that Y is *localized* in X. It should be clear that a C-embedded subspace is associate p-embedded. As mentioned before we write $f|_Y$ instead of $\Psi(f)$ or $\Psi^*(f)$.

We begin by showing that a p^*-embedded subspace is C^*-embedded. One of the secondary goals of this section is to show how different the notion of p-embedded is from C^*-embedded.

Theorem 2.1. *Let Y be a subspace of X. The following statements are equivalent.*

(1) *Y is a C^*-embedded subspace of X.*
(2) *Y is an associate p^*-embedded subspace of X.*
(3) *Y is a p^*-embedded subspace of X.*

Proof. If Y is C^*-embedded, then the map Ψ^* from $C^*(X)$ into $C^*(Y)$ is surjective and hence $C^*(Y)$ is trivially an associate p-extension of $\Psi^*(C^*(X))$. Thus (1) implies (2).

Clearly, (2) implies (3).

Suppose (3) is true. To show that Y is C^*-embedded in X, we recall Urysohn's Extension Theorem (see Theorem 1.17 [13]) which states that a subspace Y of X is C^*-embedded if and only if any pair of completely separated subsets of Y are completely separated in X. So let A, B be completely separated subsets of Y. This means there is an $f \in C^*(Y)$ such that $f(a) = 0$ for all $a \in A$ while $f(b) = 1$ for all $b \in B$. By hypothesis there is a $g \in C^*(X)$ such that $fC^*(Y) = g|_Y C^*(Y)$. Choose $s, t \in C^*(Y)$ for which $f = sg|_Y$ and $g|_Y = tf$. We want to show that A and B are completely separated in X.

First of all observe that $Z(g) \cap Y = Z(f)$ (see Proposition 2.11), so $g(a) = t(a)f(a) = 0$ for all $a \in A$ and $g(b) \neq 0$ for all $b \in B$. If $Z(g)$ is clopen in X, then A and B are completely separated in X. Otherwise, suppose that for each $n \in \mathbb{N}$ there is an $x_n \in B$ such that $|g(x_n)| < \frac{1}{n}$. Then for each $n \in \mathbb{N}$, $f(x_n) = 1$ and so

$$|s(x_n)| = |\frac{f(x_n)}{g(x_n)}| = |\frac{1}{g(x_n)}| \geq n.$$

But this forces s to be an unbounded function on Y, contradicting that $s \in C^*(Y)$. Therefore, there is some $N \in \mathbb{N}$ such that $|g(b)| \geq \frac{1}{N}$ for all $b \in B$. Then the function $h = N|g| \wedge 1 \in C^*(X)$ satisfies the conditions that $h(a) = 0$ for all $a \in A$ while $h(b) = 1$ for all $b \in B$, whence A and B are completely separated in X. $\square$

We recall another useful type of embedding.

Definition 2.2. Let R be a subring of S. In [2] the authors define S to be a *rigid extension* of R if for each $s \in S$ there is an $r \in R$ such that $\mathrm{Ann}_S(s) = \mathrm{Ann}_S(r)$. Here, a subspace Y of X is said to be *R-embedded* if $C(Y)$ is a rigid extension of $\Psi(C(X))$. This follows the convention used in [14]. A topological characterization of R-embedding follows. The result is known but we include a proof for completeness sake.

Lemma 2.3. *Let X be a space and $f, g \in C(X)$. Then $\mathrm{Ann}_{C(X)}(f) = \mathrm{Ann}_{C(X)}(g)$ if and only if $cl_X \operatorname{coz}(f) = cl_X \operatorname{coz}(g)$. Consequently, Y is R-embedded in X if and only if for every $f \in C(Y)$, there exists some $g \in C(X)$ such that $cl_Y \operatorname{coz}(f) = cl_Y(Y \cap \operatorname{coz}(g))$.*

Proof. The second statement follows from the first together with the fact that $\operatorname{coz}(g|_Y) = Y \cap \operatorname{coz}(g)$.

Let $f, g \in C(X)$. Notice that

$$\begin{aligned}
\mathrm{Ann}_{C(X)}(f) &= \{h \in C(X) | fh = 0\} \\
&= \{h \in C(X) | \operatorname{coz}(f) \cap \operatorname{coz}(h) = \emptyset\} \\
&= \{h \in C(X) | \operatorname{coz}(f) \subseteq Z(h)\} \\
&= \{h \in C(X) | cl_X \operatorname{coz}(f) \subseteq Z(h)\}
\end{aligned}$$

Suppose that $\mathrm{Ann}_{C(X)}(f) = \mathrm{Ann}_{C(X)}(g)$. If $cl_X \operatorname{coz}(f) \neq cl_X \operatorname{coz}(g)$ then without loss of generality there is an $x \in cl_X \operatorname{coz}(f) \smallsetminus cl_X \operatorname{coz}(g)$. Since X is Tychonoff there is some $h \in C(X)$ such that $h(x) = 1$ while $h(y) = 0$ for every $y \in cl_X \operatorname{coz}(g)$. It follows that $hg = 0$, i.e. $h \in \mathrm{Ann}_{C(X)}(g)$. Thus, by hypothesis, $hf = 0$ and hence $cl_X \operatorname{coz}(f) \subseteq Z(h)$. However, $x \in cl_X \operatorname{coz}(f)$ implies that $h(x) = 0$, a contradiction. Therefore, $cl_X \operatorname{coz}(f) \subseteq cl_X \operatorname{coz}(g)$. Similarly, $cl_X \operatorname{coz}(g) \subseteq cl_X \operatorname{coz}(f)$.

Conversely, suppose that $cl_X \operatorname{coz}(f) = cl_X \operatorname{coz}(g)$. By the above string of equalities it follows that $\mathrm{Ann}_{C(X)}(f) = \mathrm{Ann}_{C(X)}(g)$. □

Some history is in order. In [15] the authors defined an embedding of a *dense* subspace T of X as a $\mathcal{Z}^\sharp$-embedding if for each zeroset Z of T there is a zeroset Z' of X such that $cl_T int_T Z = T \cap cl_X int_X Z'$. In [14] the authors state that a dense subspace Y of X is R-embedded if and only if for every cozeroset of Y, say W, there is a cozeroset of X, say T, such that $cl_Y W = Y \cap cl_X T$. For our purposes we shall call an R-embedding with this property a *strong R-embedding* (or say that a subspace is strongly R-embedded). In both [15] and [14] as well as [16] the terms $\mathcal{Z}^\sharp$-embedding, R-embedding, and strongly R-embedding are used interchangeably. This is true because they were interested in either dense or open subspaces, and because Lemma 2.3 of [16] states that for an open or dense subspace, the three conditions are in fact equivalent.

Example 2.4. For general subspaces the strongly R-embedded property is in fact very strong. It should be apparent that a z-embedded subspace is always R-embedded. However, consider the space $X = \beta\mathbb{N}$ and subspace $Y = \beta\mathbb{N} \smallsetminus \mathbb{N}$. Y is C-embedded in X (by the Tietze-Urysohn Theorem) and hence z-embedded in X, yet it is not strongly R-embedded. To see this, observe that a strongly R-embedded subspace of a basically disconnected space is again basically disconnected. It is well known that X is extremally disconnected (and hence basically disconnected) yet Y is not basically disconnected.

Moving on, it is well-known that every C-embedded subspace is C^*-embedded and every C^*-embedded subspace is z-embedded. We place our newly defined notions in their proper context.

Proposition 2.5. *Suppose $Y \subseteq X$. If Y is C^*-embedded in X, then Y is localized in X.*

Proof. Let $f \in C(Y)$. The function $f^* = \frac{1}{1+f^2} \in C^*(Y)$ and therefore, by hypothesis, can be extended continuously to X. Call the extension $g \in C(X)$. Notice that $g|_Y = f^* \in \mathfrak{U}(C(Y))$.

Furthermore, $ff^* \in C^*(Y)$ and so by hypothesis can be continuously extended to all of X. Let $h \in C(X)$ be any such extension. Finally, $fg|_Y = h|_Y$, where $g, h \in C(X)$ and $g|_Y \in \mathfrak{U}(C(Y))$. $\square$

We can now obtain the first part of the next well-known theorem.

Theorem 2.6 (Theorem 2.3 [10], Proposition 3.1 [8])**.** *$C(X)$ is a regular localization of $C^*(X)$. Moreover, any intermediate algebra between $C^*(X)$ and $C(X)$ is a regular localization of $C^*(X)$. In particular, X is localized in βX.*

Proposition 2.7 (Theorem 2.6 [10])**.** *Suppose $Y \subseteq X$ is a cozeroset. Then $C(Y)$ is a regular localization of $\Psi(C(X))$. Hence, every cozeroset of a space is localized.*

Proof. We recall the proof in [10], although the authors assumed the subspace is dense. Choose $d \in C(X)$ such that $Y = \mathrm{coz}(d)$. Let $f \in C(Y)$. Set

$$d'(x) = \begin{cases} \frac{d(x)}{1+f^2(x)}, & \text{if } x \in Y \\ 0, & \text{if } x \notin Y. \end{cases}$$

Observe that $d' \in C(X)$ and since $\mathrm{coz}(d') = \mathrm{coz}(d) = Y$ it follows that $d'|_Y$ is a unit of $C(Y)$. We claim that the function $fd'|_Y \in C(Y)$ has a continuous extension to all of X, in particular,

$$r(x) = \begin{cases} \frac{f(x)d(x)}{1+f^2(x)}, & \text{if } x \in Y \\ 0, & \text{if } x \notin Y. \end{cases}$$

To show $r \in C(X)$, it suffices to show continuity of r on $cl_X Y \smallsetminus Y$. Let $x \in cl_X Y \smallsetminus Y$ and $\epsilon > 0$. Since d is continuous on all of X, there exists an open neighborhood O of x (in X) such that $|d(z)| < \epsilon$ whenever $z \in O$. Observe that for any $y \in O$ either $y \in O\smallsetminus$ and so $|r(y)| = 0 < \epsilon$, or $y \in Y$ and so $|r(y)| = |\frac{f(y)d(y)}{1+f^2(y)}| < |d(y)| < \epsilon$. Thus $r \in C(X)$ as needed.

Now in $C(Y)$ we have $fd'|_Y = r|_Y$ with $d'|_Y$ a unit of $C(Y)$. Consequently, $C(Y)$ is a regular localization of $\Psi(C(X))$. $\square$

The most useful characterization of localized subspaces is the following theorem, which can be found in Proposition 3.1 of [5]. The terminology used there is different so we supply an argument.

Definition 2.8. Let Y be a subspace of X. The element $h \in C(Y)$ is called a *quotient from* $\Psi(C(X))$ if there are $f, g \in C(X)$ such that $Y \subseteq \mathrm{coz}(g)$ and for all $y \in Y$, $h(y) = \frac{f(y)}{g(y)}$.

Theorem 2.9 (Proposition 3.1 [5])**.** *Let Y be a subspace of X. The element $h \in C(Y)$ is a quotient from $\Psi(C(X))$ if and only if $h \in C(Y)$ extends continuously to some cozeroset of X containing Y.*

Theorem 2.10. *Let Y be a subspace of X. Y is localized in X if and only if each $h \in C(Y)$ extends continuously to some cozeroset of X containing Y.*

Proof. Y being localized in X means that for every $h \in C(Y)$ there are $f, g \in C(X)$ such that $g|_Y$ is invertible in $C(Y)$ and $f|_Y = hg|_Y$. The invertibility condition is equivalent to $Y \subseteq \mathrm{coz}(g)$. The equality condition applied with the invertibility condition gives that for each $y \in Y$, $h(y) = \frac{f(y)}{g(y)}$. $\square$

We know localized implies p-embedded. Next we consider how p-embedded and z-embedded are related.

Proposition 2.11. *Suppose Y is p-embedded in X. Then Y is z-embedded in X. In particular, if Y is localized in X, then it is z-embedded in X.*

Proof. Let Z be a zeroset of Y. So there is an $f \in C(Y)$ such that $Z(f) = Z$. By assumption there is a $g \in C(X)$ such that $fC(Y) = g|_Y C(Y)$. This means there are $h_1, h_2 \in C(Y)$ such that $f = h_1 g|_Y$ and $g|_Y = h_2 f$. It follows that $Z(f) = Z(g|_Y) = Z(g) \cap Y$. □

Proposition 2.12. *Suppose Z is a localized (p-embedded) subspace of Y and that Y is a localized (p-embedded) subspace of X. Then Z is a localized (p-embedded) subspace of X.*

Proof. The statement can be easily checked for p-embeddings. As for localizations being transitive, this is listed as Exercise 2. of III.4. [17]. □

Consider the following conditions on a subspace Y of X. Each one implies the next one.

(1) C-embedded
(2) C^*-embedded
(3) localized
(4) associate p-embedded
(5) p-embedded
(6) z-embedded
(7) R-embedded

One of our goals is to create a list of spaces that show none of the conditions in the list implies the previous one. Some examples are easier than others. For example, every space is C^*-embedded in its Stone-Čech compactification but not necessarily C-embedded. Every cozeroset of a space is localized in said space. However, saying that every cozeroset is C^*-embedded is saying that the space is an F-space. Therefore, we are left with finding a space X with a subspace Y such that i) Y is associate p-embedded in X but not localized, ii) Y is p-embedded in X but not associate p-embedded, and iii) Y is z-embedded in X but not p-embedded.

Unfortunately, finding spaces that satisfy (i) or (ii) has been difficult. The minor difference in associate p-embedded and p-embedded has been intractable. In the final section of the article, we will turn our attention to the class of totally ordered spaces and discuss some interesting examples, including ones that satisfy (iii).

3. Peculiar Spaces

In this section we consider the passage of topological properties via our special embeddings. We are interested classes of spaces which relate to nice ring-theoretic properties. In [3] we looked at the passage of ring theoretic properties between extensions of rings. We will liberally use those results here. It is interesting to us is some properties do not pass in the general context but they do in the context of rings of continuous functions.

As a way to illustrate the passage of topological properties, consider the following. Theorem 2.2 of [12] states that any C-embedded subspace of an F-space is an F-space. In fact, this is true for C^*-embedded subspaces. Theorem 2.2 was later (partially) generalized in Proposition 3.2 of [15] which states that any dense R-embedded subspace of a quasi F-space is again a quasi F-space. In particular, they showed that a dense subspace of a quasi F-space is R-embedded if and only if it is C^*-embedded. Thus, it follows that a dense R-embedded subspace of an

F-space is C^*-embedded, hence an F-space. We do not know whether an arbitrary subspace of an F-space is R-embedded if and only if it C^*-embedded, or whether z-embedded subspaces of F-spaces are F-spaces. However, we can say something of interest. We give a generalization of Theorem 2.2 of [12].

Proposition 3.1. *Suppose X is an F-space and $Y \subseteq X$. If Y is p-embedded in X, then Y is an F-space. Therefore, a cozeroset of an F-space is an F-space.*

Proof. The second statement is known but its proof involves the fact that a cozeroset of a cozeroset is a cozeroset. One can easily prove this from the fact that a cozeroset is localized and hence a p-embedded subspace.

Consider $\Psi : C(X) \to C(Y)$. Any homomorphic image of a Bézout ring is again a Bézout ring. Thus, $\Psi(C(X)$ is a Bézout ring. By assumption $C(Y)$ is a p-extension of $\Psi(C(X))$ so that by Proposition 4.2 of [3] we gather that $C(Y)$ is a Bézout ring. □

Remark 3.2. The proof of the previous proposition can be modified to prove that any associate p-embedded subspace (e.g. localized) of a T-space is again a T-space. Instead of Proposition 4.2, use Proposition 4.7 of [3]. (By a T-space we mean a space X for which $C(X)$ is a Hermite ring.)

Also, we do not know whether we can generalize these two results to R-embedded subspaces since our method of proof breaks down. In particular, there are rigid ring extensions $R \to S$ for which R is a Bézout ring, yet S is not (see Proposition 2.18 of [2]).

We can say something more about p-embeddings when Y is a P-space.

Proposition 3.3. *The following are equivalent for $Y \subseteq X$, when Y is a P-space.*

(1) *associate p-embedded*
(2) *p-embedded*
(3) *z-embedded*
(4) *R-embedded*

Proof. The result follows from Proposition 5.1 of [3]. □

Example 3.4. With regards to Prüfer rings, a homomorphic image of a Prüfer ring need not be a Prüfer ring. In particular, a C-embedded subspace of a quasi F-space need not be a quasi F-space, e.g. there is a copy of $\alpha\mathbb{N}$ inside of αD.

Next we turn to clean rings. Recall that a ring is called *clean* if every element can be written as the sum of a unit and idempotent. Clean rings possess many idempotents which, in the context of $C(X)$, correspond to clopen subsets of X.

Recall that X is called *zero-dimensional* if X has a base of clopen subsets. X is called *strongly zero-dimensional* if βX is zero-dimensional. Since zero-dimensionality is a hereditary property, it follows that a strongly zero-dimensional space is zero-dimensional but not conversely. In fact, strongly zero-dimensionality is not a hereditary property. A useful characterization is the following: X is strongly zero-dimensional precisely when each cozeroset of X can be written as the union of a countable number of clopen subsets. We recall Johnstone's Theorem (see: Theorem 2.5 of [1], Theorem 13 of [22], [18]).

Theorem 3.5. *$C(X)$ is clean if and only if X is strongly zero-dimensional if and only if $C^*(X)$ is clean.*

In the context of commutative rings, it is possible to have a clean (reduced) ring R for which $q(R)$ is not clean (see [6] and [19]). Thus, a localized extension of a clean ring need not be a clean ring. Things are much nicer in the context of $C(X)$. If $C(X)$ is clean, then so is $q(X)$.

Proposition 3.6 (Lemma 5.3 [7])**.** *Every z-embedded subspace of a strongly zero-dimensional space is strongly zero-dimensional.*

A natural question is whether an R-embedded subspace of a strongly zero-dimensional space is again strongly zero-dimensional. In [20] the authors defined a space X to be a *qsz*-space if for each cozero set of X, say T, there is a countable sequence of clopen subsets of X, say $\{K_n\}_{n\in\mathbb{N}}$ for which $cl_X T = cl_X \cup_{n\in\mathbb{N}} K_n$. (The term qsz-space is short for quasi strongly zero-dimensional.) Such spaces possess clopen π-bases, and the class of qsz-spaces is incomparable to the class of zero-dimensional spaces. Our next result was proved in [20]

Proposition 3.7 (Lemma 4.3 [20])**.** *Suppose X is a qsz-space. If $Y \subseteq X$ is dense (or open) and R-embedded, then Y is a qsz-space.*

Restricting to the class of zero-dimensional spaces, we can use the Banaschewski (zero-dimensional) compactification to characterize qsz-spaces. For a zero-dimensional space X, let $\beta_0 X$ denote its Banaschewski compactification. Recall that $\beta_0 X$ is the unique (up to homeomorphism) zero-dimensional compact space containing X as a dense 2-embedded subset. Saying that X is strongly zero-dimensional is equivalent to stating that βX and $\beta_0 X$ are homeomorphic. Equivalently (Proposition 3.6), the embedding of X into $\beta_0 X$ is a z-embedding. We can say more.

Proposition 3.8. *Let X be a zero-dimensional space. X is a qsz-space if and only if $X \subseteq \beta_0 X$ is an R-embedding.*

Proof. The sufficiency follows from Proposition 3.7. As for the necessity, suppose that X is a qsz-space and let W be a cozero set of X. There is a sequence of clopen subsets of X, say $\{T_n\}_{n\in\mathbb{N}}$, such that $cl_X W = cl_X \cup_{n\in\mathbb{N}} T_n$. Since X is 2-embedded in $\beta_0 X$, for each $n \in \mathbb{N}$ there is a clopen subset of $\beta_0 X$, say K_n, such that $T_n = X \cap K_n$.

Then

$$\begin{aligned} cl_X W &= cl_X \bigcup_{n\in\mathbb{N}} T_n \\ &= cl_X \bigcup_{n\in\mathbb{N}} (X \cap K_n) \\ &= cl_X (X \cap \bigcup_{n\in\mathbb{N}} K_n). \end{aligned}$$

Since a clopen set is a cozero set and a countable union of cozero sets is again a cozero set, it follows that X is R-embedded in $\beta_0 X$. □

It follows that even a strongly R-embedded subspace of a strongly zero-dimensional space need not be strongly zero-dimensional. For example, consider the example from [24] of a zero-dimensional space X which is not strongly zero-dimensional. In [20] it is shown that X is a qsz-space. By the previous proposition, X is R-embedded in $\beta_0 X$. Since it is also dense, X is strongly R-embedded in $\beta_0 X$. The latter is a compact zero-dimensional space and hence strongly zero-dimensional. It thus appears that Proposition 3.6 is a sharp result.

4. Linearly-Ordered Spaces

In this final section we are interested in discussing what occurs in the case of a linearly-ordered space. By a linearly-ordered space we mean a pair $(X, \leq)$ where $\leq$ is a linear-order on X and X is equipped with the interval topology. That is, the topology on X is given by the base of open intervals $(a, b) = \{x \in X : a < x < b\}$. We denote open rays by $(a, \infty) = \{x \in X : a < x\}$ and $(-\infty, b) = \{x \in X : x < b\}$.

Dedekind completeness plays an important role in the theory of linearly-ordered spaces. A linearly-ordered space X is said to be *Dedekind complete* if every bounded set has a least upper bound and a greatest lower bound. Compact linearly-ordered spaces are characterized by being bounded and Dedekind complete, that is, complete. Every ordered space X has a Dedekind completion via Dedekind cuts. We shall denote the Dedekind completion of X by $\hat{X}$. Observe that $\hat{X}$ is compact precisely when X is bounded.

In [12] the authors studied linearly-ordered spaces and proved some important results about them. Over time the names of certain concepts have changed, and we shall do our best and making the translation to present day terms. In Section 9 of [12] the authors characterized paracompact linearly-ordered spaces. Then in Section 10, they investigated when a linearly-ordered space is a real-compact space (there called a Q-space). Theorem 10.6 of [12] will play a pivotal role in our work. We now recall some useful definitions.

Definition 4.1. Let $(X, \leq)$ be a linearly-ordered space and $p \in X$.

(1) An element of $\hat{X} \smallsetminus X$ is called a *gap* of X.
(2) An *endpoint* of X, if it exists, is either the largest element of X or the least element of X.
(3) An element $p \in X$ is called a *predecessor* if the set (p, ∞) has a least element. An element $p \in X$ is called a *successor* of X if the set $(-\infty, p)$ has a largest element.
(4) We call a point of X an *embedded point* if it is neither an endpoint, a predecessor, nor a successor. It follows that a point is an embedded point if and only if it belongs to the closure of both (p, ∞) and $(-\infty, p)$.
(5) A point p is called a *P-point* if it is not the limit of any ascending or descending sequence of points in X (see 5O.1 of [13]).

Lemma 4.2. *Let X be a linearly-ordered space. The point $p \in X$ is a G_δ-point of X if and only if one of the following conditions hold:*

- *p is an embedded point that is both the limit of an (strictly) increasing sequence of points of X and the limit of a (strictly) decreasing sequence of points of X.*
- *p is a successor and the limit of a (strictly) decreasing sequence of points of X.*
- *p is a predecessor and the limit of an (strictly) increasing sequence of points of X.*
- *p is both a successor and a predecessor, i.e. p is isolated in X.*

Remark 4.3. In Section 6 of [12] the authors associate to each point $u \in \hat{X}$ an ordered pair (ρ, σ), where ρ designates the co-finality of $\{x \in X : x < u\}$ and σ denotes the co-initiality of $\{x \in X : u < x\}$. The ordered pair is called the*character of u*. Note, if $u \in X$ and is a predecessor or successor, then instead of saying that $\rho = 1$ we will say $\rho = \omega_0$. Thus, Lemma 4.2 has been restated to say that a point $p \in X$ is a G_δ-point if and only it the character of p is (ω_0, ω_0).

Definition 4.4. We are interested in studying points in X. For a point $p \in X$, we denote its co-singleton set by $X_p = X \smallsetminus \{p\}$. The point $p \in X$ is called a *C^*-point* (resp. C-point) if X_p is C^*-embedded in X.

Our next theorem is a characterization of C^*-points in linearly-ordered spaces. It shall be useful to note that if p is an embedded point of X, then p is a gap of X_p (with respect to the interval topology on X_p). On the other hand, if p is either a successor or predecessor of X, then p is not a gap of X_p. However, in [12], the authors define an *end gap* in the case that X does not have a largest or smallest element. An end gap, if needed, is simply an extra element which is the supremum or infimum of X.

Theorem 4.5 (Theorem 10.6 [12]). *Let u be a gap of X (possibly a left end gap). Then u is a non-Q-gap from the left if and only if every function $f \in C(X)$ is constant on left tail at u. Analogously, for the right.*

Definition 4.6. An increasing or decreasing α-sequence $S = \{x_\beta\}_{\beta<\alpha}$ of points from $\hat{X}$ is called a *Q-sequence* if for every nonzero limit ordinal $\lambda < \alpha$, the limit (in $\hat{X}$) of the segment $\{x_\beta\}_{\beta<\lambda}$ of S is a gap of X. A gap u is called a *Q-gap* from the left if there exists an increasing Q-sequence at u.

Theorem 4.7 ([21]). *Let $(X, \leq)$ be a linearly-ordered space and $p \in X$. The following statements are equivalent.*

1. *The element $p \in X$ is a C^*-point of X.*
2. *The element $p \in X$ satisfies $p \in \upsilon X_p$.*
3. *The element $p \in X$ is a P-point of X which is not an embedded point nor the supremum or infimum of gaps in X.*
4. *The element $p \in X$ is not an embedded point and every function $f \in C(X_p)$ is constant on a tail at p.*

Proof. The proof of the theorem follows from Theorem 4.5 together with the note from Definition 4.4 that any point $p \in X$ can be viewed as a gap (possibly an endcap) of X_p.

However, we provide the steps that are either not explicitly stated in [12] or are straightforward enough to include here. For example, consider statements (1) and (2). If $p \in \upsilon X_p$, then we gather that p is a C-point of X_p and hence a C^*-point. Conversely, we shall see that a C^*-point must be a P-point of X and hence is not a G_δ-point of X. That a C^*-point which is not a G_δ-point is a C-point can be seen from Theorem 4.2 of [23].

To see that a C^*-point of X cannot be an embedded point, nor be a supremum nor infimum of gaps in X. If p is embedded, then the characteristic function of the interval $(-\infty, p)$ in the union $X_p = (-\infty, p) \cup (p, \infty)$ cannot be extended to p. So if p is a C^*-point then it cannot be embedded. Since it is not an embedded point the options are that it is a P-point or a non-isolated G_δ-point. The latter is not C^*-embedded. Thus, if p is a C^*-point of X, then it must be a P-point.

Next, suppose, without loss of generality, that p is a supremum of gaps $(p_i)_{i\in I}$; then it must be a predecessor or else the largest element of X, by the previous paragraph. We may then just as well assume that it is the largest, and also suppose – by choosing a cofinal sequence – that I is well ordered, with $i < j$ implying that $p_i < p_j$. Let $T_i = \{x \in X : p_i < x < p_{i+1}\}$. Then X_p is the disjoint union of the clopen subspaces T_i. Let k be the characteristic function of the T_i for all odd $i \in I$. This is continuous on X_p, but cannot be extended to p. Thus, p is not a C^*-point. It follows that if p is a C^*-point, then p is a non-embedded P-point which is neither the supremum nor infimum of gaps. □

We would like to consider the other ways X_p can be embedded in X. We break it up into two cases: whether q is an embedded point or not.

Proposition 4.8. *Let q be an embedded point of a linearly-ordered space X. Then the following are equivalent.*

(1) *q is a G_δ-point of X.*
(2) *X_q is localized in X.*
(3) *X_q is associate p-embedded in X.*
(4) *X_q is p-embedded in X.*
(5) *X_q is z-embedded in X.*
(6) *X_q is R-embedded in X.*

Proof. We only need to show (6) implies (1). To that end suppose that X_q is R-embedded in X. Consider the characteristic function $\chi_{(q,\infty)} \in C(X_q)$. Since we are assuming that X_q is R-embedded in X, it follows that there is some cozeroset of X, say $T = \mathrm{coz}(f)$ for $f \in C(X)$, such that

$$(*) \qquad cl_{X_q}(T \cap X_q) = cl_{X_q}(q,\infty) = (q,\infty).$$

Observe that if $q \in T$, then since q is an embedded point there is some $x < q$ belonging to T and hence, by $(*)$, $x \in (q,\infty)$, a contradiction. Therefore $q \notin T$, and so $f(q) = 0$. Now if q is not the limit of a descending (countable) nontrivial sequence of points, then f vanishes on a right-sided neighborhood of q. But this cannot happen since otherwise the closure of $T \cap X_q$ would be a proper subset of (q,∞). Therefore, q must be a limit of a descending sequence of points.

Similarly, q must be the limit of an increasing sequence of points. Therefore, q is a G_δ-point. □

Proposition 4.9. *Let $q \in X$ be a point of a linearly-ordered space X that is not embedded. Then either q is a G_δ-point of X (hence X_q is localized in X) or else the following statements are equivalent.*

(1) *q is a C^*-point of X.*
(2) *X_q is localized in X.*
(3) *X_q is p-embedded in X.*
(4) *X_q is z-embedded in X.*
(5) *X_q is R-embedded in X.*

Proof. We begin by pointing out that since q is not an embedded point, then either q is an endpoint, $[q,\infty)$ is clopen, or $(-\infty,q]$ is clopen. We can assume, without loss of generality, that q is an endpoint of X, say q is the largest point of X.

There are two cases to consider: either q is the limit of an increasing sequence of points or it is not. In the first case q is a G_δ-point of X. Therefore, we assume that q is not the limit of a non-trivial increasing sequence of points. Therefore, q is a P-point of X.

We suppose that X_q is R-embedded in X. We aim to show that q is a C^*-point of X, which by Theorem 4.7, is equivalent to q not being a supremum of gaps in X.

Assume, by way of contradiction, that q is the supremum of gaps in X. Since X_q is dense in X, it will suffice to show that there is a cozeroset C of X_q such that for all cozerosets V of X, $cl_{X_q}C \neq X_q \cap cl_X V$. The point q is the supremum of gaps in X, thus we can write $X_q = K_1 \cup K_2$ where K_1 and K_2 are disjoint clopen sets and q is the supremum of both sets. Note that $\chi_{K_1} \in C(X_q)$. However, since q is a P-point of X, there cannot exist a cozeroset V of X such that $clX_qK_1 = X_q \cap cl_X V$, contradicting our assumption that X_q is R-embedded in X. Therefore, q is not the supremum of gaps and so q is a C^*-point by Theorem 4.7.

□

The point of these results is not just to characterize when in a linearly-ordered space removing a point yields certain embedded subspaces. Our goal is to convince the reader that in order to construct z-embedded subspaces that are not localized we need to remove more than a finite number of points from a linearly-ordered space. We conclude with a general construction that yields a space with a z-embedded subspace that is not p-embedded.

Construction 4.10. Let X be a linearly-ordered space. We assume that X is not locally compact. Hence there is a point with no compact neighborhood, say p. It follows that p is the supremum or infimum of gaps. Without loss of generality, we may assume that p is the supremum of gaps and also that p is the largest element of X. Finally, we assume that the cofinality of the gaps in X is $\aleph_0$. We prove that such a space is not localized in its Dedekind completion.

Choose an increasing sequence of gaps, say $\{\hat{x}_n\}_{n\in\mathbb{N}}$, whose limit is p. Denote $T_0 = \{x \in X : x < \hat{x}_0\}$ and for each $1 \leq n \in \mathbb{N}$ define

$$T_n = \{x \in X : \hat{x}_n < x < \hat{x}_{n+1}\}.$$

Observe that for each $n = 0, 1, \ldots$, T_n is a clopen subset of X. Also, $\bigcup_{n\in\mathbb{N}} T_n = X_p$.

Define $f : X \to \mathbb{R}$ as follows.

$$f(x) = \begin{cases} \frac{1}{n+1}, & \text{if } x \in T_n \\ 0, & x = p. \end{cases}$$

Obviously $f \in C(X)$ and notice that $Z(f) = \{p\}$. We claim that f has no extension to a cozeroset of $\hat{X}$ which contains X. If so, then let K be a cozeroset of $\hat{X}$ such that $X \subseteq K \subseteq \hat{X}$. Choose $g \in C(K)$ such that $g(x) = f(x)$ for all $x \in X$. Since K is open in $\hat{X}$ and $p \in K$ it follows that there is some $\hat{X}$-neighborhood around p contained in K, say $(q, p] \subseteq K$. However, this implies that there is some $n \in \mathbb{N}$ such that the gap $\hat{x}_n \in (q, p]$ and hence $\hat{x}_n \in K$. By construction and the fact that $\hat{x}_n$ is a gap, the function f does not have a continuous extension to $\hat{x}_n$. This contradiction forces us to conclude that X is not localized in $\hat{X}$.

Example 4.11. By the above construction neither the space of rationals, $\mathbb{Q}$, nor the space of irrationals, $\mathbb{R} \smallsetminus \mathbb{Q}$, is localized in $\mathbb{R}$. Since $\mathbb{R}$ is perfectly normal it follows that every subspace is z-embedded. What we will gather from our next result is that neither $\mathbb{Q}$ nor $\mathbb{R} \smallsetminus \mathbb{Q}$ are p-embedded in $\mathbb{R}$.

Proposition 4.12. *$C(\mathbb{R}) \hookrightarrow C(\mathbb{Q})$ is not a p-embedding.*

Proof. Let $T = \{r_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of irrationals which converge to 0. Define $F \in C(\mathbb{R} \smallsetminus T)^+$ by

$$F(x) = \begin{cases} 0 \text{ if } x \in (-\infty, 0] \cup (r_1, \infty) \\ 0 \text{ if } x \in (r_{2n+1}, r_{2n}) \text{ for } n \in \mathbb{N} \\ x \text{ if } x \in (r_{2n}, r_{2n-1}) \text{ for } n \in \mathbb{N}. \end{cases}$$

Let $f = F|_{\mathbb{Q}}$ and assume, by way of contradiction, that there exists a $g \in C(\mathbb{R})$ such that $fC(\mathbb{Q}) = g|_{\mathbb{Q}}C(\mathbb{Q})$. In particular, there exist $h_1, h_2 \in C(\mathbb{Q})$ such that $f = g|_{\mathbb{Q}}h_1$ and $g|_{\mathbb{Q}} = fh_2$. Observe that continuity of g on $\mathbb{R}$ and $Z(f) = Z(g) \cap \mathbb{Q}$ forces $g(r_n) = 0$, for every $n \in \mathbb{N}$.

Fix $n, N \in \mathbb{N}$. By continuity of $g(x) \in C(\mathbb{R})$ there is some δ_1 such that if $|r_{2n+1} - x| < \delta_1$, then $|g(x)| = |g(r_{2n+1}) - g(x)| < \frac{r_{2n+1}}{2N}$. Also, there is a $\delta_2 > 0$ such that if $0 < r_{2n+1} - x < \delta_2$, then $f(x) > \frac{r_{2n+1}}{2}$. Set $\delta = \min\{\delta_1, \delta_2\}$. Choose a $q \in Q$ such that $0 < r_{2n+1} - q < \delta$ and observe that

$$|h_1(q)| = \frac{f(q)}{|g(q)|} > \frac{f(q)}{\frac{r_{2n+1}}{2N}} = \frac{2Nf(q)}{r_{2n+1}} > N.$$

This holds for every $n \in \mathbb{N}$.

In other words for each $n \in \mathbb{N}$, near r_{2n+1}, $|h_1(x)|$ gets arbitrarily large. However, $r_n \to 0$, so this means h_1 cannot be continuous at 0, a contradiction. Therefore $C(\mathbb{R}) \hookrightarrow C(\mathbb{Q})$ is not a p-embedding. $\square$

Unresolved Questions.

(1) Does there exist a space X with a subspace Y which is p-embedded but not associate p-embedded?
(2) Does there exist a space X with a subspace Y which is associate p-embedded but not localized?
(3) Is normality equivalent to the condition that every closed subspace is R-embedded?

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