COMMUTATIVE NIL CLEAN GROUP RINGS

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ABSTRACT. In [5] and [6], a nil clean ring was defined as a ring for which every element is the sum of a nilpotent and an idempotent. In this short article we characterize nil clean commutative group rings.

1. INTRODUCTION

Throughout this article we assume that all rings are commutative and possess an identity. Furthermore, we assume that all groups are abelian and multiplicative. Unless otherwise noted $(R, +, \cdot, 1_R)$ denotes such a ring while $(G, \cdot, 1_G)$ denotes such a group. We are interested in classifying when the group ring $R[G]$ is nil clean. The notion of a nil clean ring was first defined in [6]. In [5] some fundamental results are proved concerning this class of rings. One can see that this concept is a modification of Nicholson's clean ring. Cleanliness was originally defined in [12]. For a history of clean rings we urge the reader to examine [11] as well as the plethora of articles on the subject. Some recent papers on nil clean rings are [3], [4], [1], and [2].

Definition 1.1. The element $r \in R$ is said to be *nil clean* if it may be written as the sum of a nilpotent and an idempotent. If every element in R is nil clean then we call R a *nil clean* ring.

We now recall a classification of commutative nil clean rings. We use $n(R)$ to denote the nilradical of R, while we use $\mathfrak{J}(R)$ to denote its Jacobson radical.

Proposition 1.2 (Diesl's Proposition: Proposition 3.15 [5])**.** *Let I be a nil ideal of R. R is nil clean if and only if R/I is nil clean.*

Proposition 1.3 (Corollary 3.20 [5])**.** *Suppose R is a commutative ring with identity. R is nil clean if and only if $R/\mathfrak{J}(R)$ is boolean and $\mathfrak{J}(R)$ is nil.*

Corollary 1.4. *Suppose R is a commutative ring. R is nil clean if and only if $R/n(R)$ is boolean. In particular, if R is nil clean, then 2 is nilpotent.*

Proof. The second statement follows from the first since a boolean ring has characteristic 2.

Suppose R is nil clean. Then by Proposition 1.3, $R/\mathfrak{J}(R)$ is boolean and $\mathfrak{J}(R)$ is nil. Since the Jacobson radical is nil it follows that $\mathfrak{J}(R) = n(R)$ and hence $R/n(R) = R/\mathfrak{J}(R)$ is boolean.

Conversely, if $R/n(R)$ is boolean then R is zero-dimensional and hence $\mathfrak{J}(R) = n(R)$ is nil. $\quad\square$

Remark 1.5. The class of nil clean rings is closed under homomorphic images and finite products. This follows from the fact that a ring homomorphism takes nilpotent (resp. idempotent) elements to nilpotent (resp. idempotent) elements. This will be useful in the next section.

2. GROUP RINGS

There are many references for the topic of group rings; our preferences are [15] and [7].

Definition 2.1. Recall that a group G is said to be *torsion* if every element has finite order. For a prime p we denote by G_p the subset of G consisting of those elements whose orders are a power of p. Recall that the group G is said to be *p-torsion free* if G_p is trivial, that is $G_p = \{e_G\}$. If G is abelian, then G_p is always a subgroup of G.

Throughout we let C_n be the multiplicative cyclic group of order n. When understood by context we shall assume that $C_n = <g>$. We use $\mathbb{Z}_n$ to denote the ring of integers modulo the ideal $n\mathbb{Z}$. Also, $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at the prime p.

Given a ring R and group G we denote the group ring of R over G by $R[G]$. A monomial in the group ring is written as rg for $r \in R$ and $g \in G$. An arbitrary element of $R[G]$, say $\alpha \in R[G]$, is of the form $\alpha = \sum_{g \in G} r_g g$ where it is assumed that the set $\mathrm{supp}(\alpha) = \{g \in G : r_g \neq 0\}$, called the *support of α*, is finite. Recall that the map $\Psi : R[G] \to R$ defined by $\Psi(\sum_{g \in G} r_g g) = \sum_{g \in G} r_g$ is a surjective ring homomorphism called the *trace map*. The kernel of this map is known as the *augmentation ideal of $R[G]$* denoted by $\omega(R[G])$. It is a fact that $\omega(R[G])$ is generated as an ideal by the set $\{1 - g : g \in G\}$.

Remark 2.2. As we shall see the classification of when a group ring is nil clean is much nicer then any current attempt at classifying when a group ring is clean. With regards to clean rings, in [8] it is shown that the group ring $\mathbb{Z}_{(7)}[C_3]$ is not clean. They also demonstrate that if B is a boolean ring and G is a torsion group then $B[G]$ is clean. Recently, in [9] and [10], the author classifies when $\mathbb{Z}_{(p)}[C_n]$ is a clean ring:

Theorem 2.3 (Proposition 2.7 [10]). *Suppose p is a prime and $n = p^k m$ with $\gcd(p, m) = 1$. The group ring $\mathbb{Z}_{(p)}[C_n]$ is a clean ring if and only if p is a primitive root of d for each positie divisor d of m.*

In the two papers [13] and [14], the authors investigate uniquely clean rings, that is, a ring for which every element has a unique clean expression. One of the interesting results in [14], Proposition 26, is that a commutative ring with identity R is uniquely clean if and only if it is semi-boolean (that is, every element is the sum of a idempotent and an element belonging to the Jacobson radical). It follows that a nil clean ring is uniquely clean. In Proposition 24 of [13] it is shown that for a commutative uniquely clean ring R, the group ring $R[C_{2^k}]$ is uniquely clean. As the title of our article suggests we classify when $R[G]$ is a nil clean group ring. To our knowledge there is no current classification of when $R[G]$ is uniquely clean. However, we can say that there is a uniquely nil clean group ring that is not nil clean, e.g. $\mathbb{Z}_{(2)}[C_2]$.

Lemma 2.4. *If $R[G]$ is nil clean, then R is nil clean.*

Proof. The statement is a consequence of the following two facts: i) R is a homomorphic image of $R[G]$ via the trace map and ii) the class of nil clean rings is closed under homomorphic images. □

Lemma 2.5. *Let R be a commutative ring and G be an abelian group. Suppose $2 \in n(R)$ and $g \in G_2$. Then the element $1 - g \in R[G]$ is nilpotent.*

Proof. Since $g \in G_2$ we know that the order g is a power of 2, say $|g| = 2^k$. Then the Binomial Theorem yields that

$$(1-g)^{2^k} = \sum_{i=0}^{2^k} \binom{2^k}{i}(-1)^i g^i = 2 + \sum_{i=1}^{2^k-1} \binom{2^k}{i}(-1)^i g^i.$$

By Lucas' Theorem in number theory it follows that each of $\binom{2^k}{i}$ is divisible by 2 and hence $(1-g)^{2^k}$ is divisible by 2 (for $i = 1, \ldots, 2^k - 1$). By hypothesis, 2 is nilpotent in R and hence nilpotent in $R[G]$. It follows that $(1-g)^{2^k}$ and in particular, $1-g$ is nilpotent in $R[G]$. $\qquad\square$

We are now ready for our characterization of commutative unital nil clean group rings.

Theorem 2.6. *Suppose R is a commutative ring with identity and G is an abelian group. The group ring $R[G]$ is nil clean if and only if R is nil clean and G is a torsion 2-group.*

Proof. Suppose first that R is nil clean and G is a torsion 2-group. Since R has characteristic a power of 2 it follows that 2 is nilpotent in R. By Lemma 2.5 since G is 2-torsion it follows that for each $g \in G$ the element $1-g$ is nilpotent. Thus, the augmentation ideal, $\omega(R[G])$, is a nil ideal of $R[G]$. Since $R[G]/\omega(R[G])$ is isomorphic to R, which is assumed to be nil clean, we apply Diesl's Proposition 1.2 to conclude that $R[G]$ is nil clean.

Conversely, suppose $R[G]$ is nil clean. Then we know that R is nil clean. Recall that for any subgroup of H of G, the group ring $R[G]$ maps naturally onto $R[G/H]$ via the map $\Psi_H : R[G] \to R[G/H]$ defined by

$$\Psi_H\left(\sum_{g \in G} r_g g\right) = \sum_{g \in G} r_g(g+H).$$

In particular, $R[G/G_2]$ is a homomorphic image of $R[G]$ and hence is nil clean since we are assuming that $R[G]$ is nil clean. Note that G/G_2 is 2-torsion free.

Corollary 9.3 of [7] states that if D is an integral domain of characteristic $p \neq 0$, then $D[G]$ is reduced if and only if G is p-torsion free. It follows that since $\mathbb{Z}_2$ is a homomorphic image of R that $\mathbb{Z}_2[G/G_2]$ is nil clean as well as reduced, whence boolean. But if G/G_2 is non-trivial then $\mathbb{Z}_2[G/G_2]$ has more than one unit. Therefore, $G = G_2$ and we conclude that G is a torsion 2-group.

$\qquad\square$

Our final result should be considered in relation to Proposition 24 of [13].

Corollary 2.7. *Suppose R is nil clean. Then for any natural $k \in \mathbb{N}$, $R[C_{2^k}]$ is nil clean.*

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