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Neat rings

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Abstract

A ring is called clean if every element is the sum of a unit and an idempotent. Throughout the last 30 years several characterizations of commutative clean rings have been given. We have compiled a thorough list, including some new equivalences, in hopes that in the future there will be a better understanding of this interesting class of rings. One of the fundamental properties of clean rings is that every homomorphic image of a clean ring is clean. We define a neat ring to be one for which every proper homomorphic image is clean. In particular, the ring of integers, $\mathbb{Z}$, and any nonlocal PID are examples neat rings which are not clean. We characterize neat Bézout domains using the group of divisibility. In particular, it is shown that a neat Bézout domain has stranded primes, that is, for every nonzero prime ideal the set of primes either containing or contained in the given prime forms a chain under set-theoretic inclusion.

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1. A history of clean rings

In this section, we give an account of the class of clean rings. Over the past 25 years many authors in several different contexts have investigated clean rings. Our focus is on commutative clean rings. For a detailed reference on not necessarily commutative clean rings and exchange rings, see [29]. In Theorem 1.7, we give a list of characterizations of commutative clean rings. Most are old theorems but some new ones are also included. The theorem includes a collage of different kinds of rings. For the sake of completeness, we

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shall define most of the necessary concepts. To that end, we begin with the definition of a clean ring.

The ring A is called *clean* if every element is the sum of an idempotent and a unit. If every element of A is clean, then we say A is a *clean ring*. Some examples of clean rings include all commutative von Neumann regular rings, all local rings, any ring $M_n(R)$ of $n \times n$ matrices over a clean ring, and semiperfect rings. Furthermore, the class of clean rings is closed under products and homomorphic images. The definition of clean ring is due to Nicholson [27] and was shown to be a strengthening of the next condition which is due to Crawly and Jónsson [10] and, then later, Warfield [31]. An A-module M is said to have the *finite exchange property* if for any module N and decompositions

$$N = M' \oplus P = \bigoplus_{i \in I} Q_i,$$

where $M' \cong M$ and I is finite, then there exist submodules $Q_i' \subseteq Q_i$, for each i, such that

$$N = M' \oplus (\oplus Q_i').$$

A is called an *exchange ring* if it is a finite exchange A-module. As mentioned above Nicholson showed that a clean ring is an exchange ring and when idempotents are central in A the reverse holds. In general, there is an example of an exchange ring which is not clean (see [6]).

Example 1.1. As mentioned above the class of clean rings is closed under arbitrary products and homomorphic images. Furthermore, it is known that every commutative zero-dimensional ring, and hence boolean rings, are clean. An integral domain is clean precisely when it is local.

A clean ring is a Gelfand ring. Recall that a ring is called a *Gelfand ring* if whenever $a + b = 1$ there are $r, s \in A$ such that $(1 + ar)(1 + bs) = 0$. A ring is called a *pm-ring* if every prime ideal is contained in a unique maximal ideal. Commutative *pm*-rings were first studied in [13] and later in [9]. It is known that for any topological space X the ring $C(X)$ consisting of all real-valued continuous functions on X under the pointwise operations is always a *pm*-ring. It had been asserted that a commutative ring is a Gelfand ring if and only if it is a *pm*-ring. Recently, Banaschewski [3] has shown that whether this is true actually depends on one's set theoretic axioms. In particular, every commutative *pm*-ring is a Gelfand ring if and only if the Prime Ideal Theorem holds. In this present article, we shall not delve into set theoretic matters and simply work within the confines of ZFC.

For a ring A, $Spec(A)$ denotes the collection of prime ideals of A. The *hull-kernel topology* (or *Zariski* topology) on $Spec(A)$ is the topology obtained by taking the collection of sets of the form

$$\mathcal{U}(a) = \{P \in Spec(A) : a \notin P\}$$

for arbitrary $a \in A$ as a base for the open sets. Observe that $\mathcal{U}(a) \cap \mathcal{U}(b) = \mathcal{U}(ab)$. We write $\mathcal{V}(a)$ for the complement of $\mathcal{U}(a)$. The collection of maximal ideals of A is denoted by $\mathrm{Max}(A)$. The *hull-kernel topology* on $\mathrm{Max}(A)$ is simply the subspace topology $\mathrm{Max}(A)$ inherits from the hull-kernel topology on $Spec(A)$. In particular, we let

$$U(a) = \mathrm{Max}(A) \cap \mathscr{U}(a)$$

and

$$V(a) = \mathrm{Max}(A) \cap \mathscr{V}(a).$$

Since A has an identity both $Spec(A)$ and $\mathrm{Max}(A)$ are compact spaces. $Spec(A)$ is always T_0 and $\mathrm{Max}(A)$ is always T_1. Our standard reference for topological concepts is [15]. A topological space X is called *zero-dimensional* if it has a base of clopen sets. (For a Tychonoff space X, this corresponds to the small inductive dimension being 0, i.e, $ind\, X = 0$.) If X is a compact Hausdorff space, then it is known that X is zero-dimensional precisely when it is totally disconnected. Furthermore, when X is a compact, zero-dimensional Hausdorff space, then X is called a *boolean space*. We denote the nilradical and Jacobson radical of A by $n(A)$ and $\mathfrak{J}(A)$, respectively.

Recall that an ideal I of A is called a *radical ideal* (or a *semiprime ideal*) if $a^n \in I$ implies $a \in I$. The collection of radical ideals of A is denoted by $Rad(A)$ and it is a complete lattice when partially-ordered under inclusion. The ring A is said to be *semiprime* if the zero ideal is a semiprime ideal. A semiprime commutative ring is often referred to as *reduced*.

Definition 1.2. Let complete distributive L be a lattice with top element 1 and bottom element 0. An element $a \in L$ is said to be *complemented* if there exists a $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. When a is complemented a^* denotes its unique complement. L is called *normal* if $a \vee b = 1$ implies there exist $c, d \in L$ such that $a \vee c = 1 = b \vee d$ and $c \wedge d = 0$. Observe that the lattice of open sets of a topological space is a normal lattice if and only if the space is normal.

A lattice L is called *weakly zero-dimensional* if whenever $u \vee v = 1$ there exists a complemented element a such that $a \leqslant u$ and $a^* \leqslant v$. This definition is due to Banaschewski [3].

Proposition 1.3. *For a commutative ring A with* 1 *the following are equivalent*:

(i) *A is a pm-ring*.
(ii) *$Spec(A)$ is a normal space*.
(iii) *$Max(A)$ is a retract of $Spec(A)$*.
(iv) *A is a Gelfand ring*.
(v) *For each pair of distinct maximal ideals M and N there exist $a \notin M$, $b \notin N$ such that $ab = 0$*.
(vi) *$Rad(A)$ is a normal lattice*.

Proof. The equivalences of (i), (ii), and (iii) are proved in [13]. The equivalences of (i), (iv), and (v) are shown in [9], and finally that (iv) and (vi) are equivalent is proved in [3]. □

Definition 1.4. A ring is called a *topologically boolean ring* (or *tb-ring* for short) if for every pair of distinct maximal ideals there is an idempotent belonging to exactly one of them. This definition is due to Contessa [9].

Definition 1.5. Vasconcelos [30] defined a ring A to be an $\mathfrak{f}$-ring if every pure ideal is generated by idempotents. (Recall that the ideal I is said to be *pure* if for each $a \in I$ there is a $b \in I$ such that $ab = a$.) Jøndrup [22] was the first to look at the class of $\mathfrak{f}$-rings and showed the following:

Proposition 1.6. *The following are equivalent for a commutative ring A.*

(i) *A is an $\mathfrak{f}$-ring.*
(ii) *Every projective ideal is a direct sum of finitely generated ideals.*
(iii) *Given any sequence $\{a_n\}$ in A such that $a_n = a_n a_{n+1}$ for all n, the ideal generated by $\{a_n\}$ is generated by idempotents.*

We are now ready for the main theorem of this section. This theorem captures most of the known characterizations of clean rings. We let $Id(A)$ denote the set of idempotents of A. Recall that a ring is said to be *indecomposable* when the only idempotents of A are 0 and 1. Otherwise, the ring is called *decomposable*.

Theorem 1.7. *For a commutative ring A with* 1, *the following statements are equivalent*:

1. *A is an exchange ring.*
2. *$End_A\, A$ is an exchange ring.*
3. *Idempotents can be lifted modulo every ideal of A.*
4. *A is a Gelfand ring and* $\mathrm{Max}(A)$ *is zero-dimensional.*
5. *A is a pm-ring and* $\mathrm{Max}(A)$ *is zero-dimensional.*
6. $\mathrm{Max}(A)$ *is a retract of $Spec(A)$ and* $\mathrm{Max}(A)$ *is zero-dimensional.*
7. *$Rad(A)$ is weakly zero-dimensional.*
8. *A is a clean ring.*
9. *$A/\mathfrak{J}(A)$ is clean and idempotents can be lifted modulo $\mathfrak{J}(A)$.*
10. *$A/n(A)$ is clean.*
11. *A is a tb-ring, that is, for any pair of distinct maximal ideals there is an idempotent in exactly one of them.*
12. *For every $m, m' \in A$ with $1 = a + b$ there is an idempotent e such that $e \in Am$ and $1 - e \in Am'$.*
13. *The collection $\mathscr{E} = \{U(e) : e \in Id(A)\}$ forms a base for the Zariski topology on* $\mathrm{Max}(A)$.
14. *For each $a \in A$, there exists an $e \in Id(A)$ such that $V(a) \subseteq U(e)$ and $V(a-1) \subseteq V(e)$.*
15. *A is pm-ring and an $\mathfrak{f}$-ring.*

Proof. 1 and 2. This was proved by Warfield [31]. This equivalence is true for noncommutative rings as well.

1, 3, and 8. The equivalence of these three conditions was proved by Nicholson [27]. In general, 1 is equivalent to 3 and as mentioned before a clean ring is always an exchange ring and the converse holds when idempotents are central.

1 and 9. This was first shown for commutative rings by Shutters [28] and then, in general, by Nicholson [27].

1 and 7. This can be found in [3].

4, 5, and 6 follows from Proposition 1.3.

8 and 10. This is shown in [1].

4 and 8. This was shown by Johnstone [21] using sheaf-theoretic techniques.

11 and 12. This can be found in [9].

That 15 and 5 are equivalent is proved in [12].

We now show that 8, 11, 13, 14 are all equivalent.

Lemma 1.8. *The collection $\mathscr{E}$ is closed under finite intersections*, *finite unions*, *and complements*.

Proof. It is clear that for $e, f \in Id(A)$, $\text{Max}(A)\backslash U(e) = U(1-e)$, $U(e) \cap U(f) = U(ef)$ and $U(e) \cup U(f) = U(e + f - ef)$. □

Let A be a clean ring and suppose that M and N are distinct maximal ideals. Choose $a \in M\backslash N$. There is an $x \in A$ such that $ax - 1 \in N$. Observe that $r = ax \in M\backslash N$. Since A is clean there is a unit u and an idempotent e such that $r = u + e$. If $e \in M$ then is $u = r - e$ contradicting that u is a unit. Therefore, $e \notin M$. If $e \notin N$, then $u = r - e + N = r - 1 + N = N$ again a contradiction. Thus, e is an idempotent belonging to exactly one of M or N and so A is a tb-ring which shows that 8 implies 11.

Suppose A is a tb-ring. It follows that the points of $\text{Max}(A)$ can be separated by clopen sets belonging to $\mathscr{E}$. Let $K \subseteq \text{Max}(A)$ be a closed subset and $M \notin K$. For each $N \in K$ there exists a clopen set $U(e_N) \in \mathscr{E}$ separating M and N, say $N \in U(e_N)$. The collection $\{U(e_N) : N \in K\}$ is an open cover of the compact set K and so is contained in a finite union of them. By the lemma it follows there is a clopen subset $C \in \mathscr{E}$ separating M from K. Therefore, $\mathscr{E}$ forms a base for the Zariski topology on $\text{Max}(A)$ and so 11 implies 13.

Suppose the collection $\mathscr{E}$ forms a base for the Zariski topology on $\text{Max}(A)$. Observe that since $\text{Max}(A)$ is a T_1 space then zero-dimensionality implies that $\text{Max}(A)$ is a boolean space. Moreover, it follows by compactness that every clopen subset belongs to $\mathscr{E}$. Let $a \in A$. Since $V(a)$ and $V(a-1)$ are disjoint closed subsets we can choose a clopen set, and hence a set of the form $U(e)$, for some $e \in Id(A)$, separating them. Therefore, 13 implies 14.

Suppose that 14 holds and let $a \in A$. Let $e \in Id(A)$ such that $V(a) \subseteq U(e)$ and $V(a-1) \subseteq V(e)$. We claim $a - e$ is a unit. Let M be a maximal ideal of A. If $a \in M$, then $e \notin M$ and so $a - e \notin M$. Next, assume that $a \notin M$ and by means of contradiction that $a - e \in M$. Then $a + M = e + M$ is a nonzero idempotent of the field A/M. It follows that

$$a + M = e + M = 1 + M$$

and that $e \notin M$. It also follows that $M \in V(a-1) \subseteq V(e)$, the desired contradiction. Thus, $a - e$ belongs to no maximal ideal, and so it is a unit. Whence, A is clean and so 14 implies 8 completing the cycle. □

Finally, we show that if A is a *pm*-ring and that $\text{Max}(A)$ is zero-dimensional, then $\mathscr{E}$ is the collection of all clopen subsets of $\text{Max}(A)$. From this we conclude that A is clean if and only if A is a *pm*-ring and $\mathrm{Max}(A)$ is zero-dimensional. Our main reason for including this is to supply a nonsheaf theoretic proof of the fact. Let $K \subseteq \mathrm{Max}(A)$ be a clopen subset. First of all, without loss of generality, we assume that $n(A) = 0$. Set

$$\mathscr{K} = \{P \in Spec(A) : P \leqslant M \text{ for some } M \in K\}.$$

Observe that since A is a *pm*-ring $\mathscr{K}$ is a clopen subset of $Spec(A)$. Let $J = \cap\{P : P \in \mathscr{K}\}$ and $H = \cap\{P : P \in Spec(A)\backslash\mathscr{K}\}$. Both are semiprime ideals. It also follows that $J \cap H = \{0\}$. Now, we claim that $H + J = A$. Since $\mathscr{K}$ is clopen and compact $\mathscr{K} = \mathscr{U}(a_1) \cup \cdots \cup \mathscr{U}(a_n)$ (where the a_i are not necessarily idempotents). Similarly, $Spec(A)\backslash\mathscr{K} = \mathscr{U}(b_1) \cup \cdots \cup \mathscr{U}(b_m)$. It follows that $\mathscr{K} = \mathscr{V}(b_1) \cap \cdots \cap \mathscr{V}(b_m)$ and $Spec(A)\backslash\mathscr{K} = \mathscr{V}(a_1) \cap \cdots \cap \mathscr{V}(a_n)$. Thus, every prime belonging to $\mathscr{K}$, and hence J, contains the elements $b_1, \ldots, b_m$. Similarly, H contains all the $a_1, \ldots, a_n$. If $J + H < A$ then it is contained in some prime ideal M. Since $Spec(A) = \mathscr{U}(a_1) \cup \cdots \cup \mathscr{U}(a_n) \cup \mathscr{U}(b_1) \cup \cdots \cup \mathscr{U}(b_m)$ it follows that either some a_i or some b_j does not belong to M, contradicting that $J + H < M$.

What we have shown is that J and H are complements of the lattice $Rad(A)$. Lemma 1 of [3] shows that in fact $J = Ae$ for some idempotent e. Hence, $\mathscr{K} = \mathscr{V}(e)$ and $K = V(e) \in \mathscr{E}$.

Remark 1.9. It is interesting to note that it is possible for $\mathrm{Max}(A)$ to be zero-dimensional and the collection $\mathscr{E}$ not to form a base for the topology on $\mathrm{Max}(A)$, e.g. any domain with two maximal ideals. In particular, such an A is not a *pm*-ring.

2. Neat rings

As previously mentioned a basic property of clean rings is that any homomorphic image of a clean ring is again clean. This leads to our definition of a neat ring. We say a ring A is a *neat ring* if every nontrivial homomorphic image is clean.

Proposition 2.1. *The following are equivalent for a ring A.*

(1) *A is neat.*
(2) *A/aA is clean for every nonzero $a \in A$.*
(3) *For any collection of nonzero prime ideals $\{P_j\}_{j\in J}$ of A with $I = \bigcap_{j\in J} P_j$ different than 0 we have A/I is clean.*
(4) *A/aA is neat for every $a \in A$.*
(5) *A/I is clean for every nonzero semiprime ideal.*

Moreover, a homomorphic image of a neat ring is neat.

Proof. Using the fact that a homomorphic image of a clean ring is clean it is straightforward to check that (1) and (2) are equivalent. (5) is just a restatement of (3). That (1) and (5) are equivalent follows from the fact a ring A is clean if and only if $A/n(A)$ is clean. Finally, (4) implies (1) by using $a = 0$. Conversely, if a is nonzero, then A/aA is clean and hence neat. □

Corollary 2.2. *If A is a neat ring which is not clean, then A is semiprime.*

Proposition 2.3. *Let A be a decomposable ring. Then A is neat if and only if it is clean.*

Proof. If A is decomposable then there is are ideals I and J such that $A = I \oplus J$. Now, if A is neat, then by Proposition 2.1 $J \cong A/I$ is also clean. Similarly, I is clean. Thus, A being a direct product of clean rings is clean. □

It follows that in our investigation of neat rings which are not clean the indecomposable ones shall play a pivotal role. An indecomposable ring is clean if and only if it is local. Moreover, integral domains are a huge source of indecomposable rings and so we will classify certain neat integral domains. Our standard example is the domain of integers $\mathbb{Z}$. It is well-known that every nontrivial factor of $\mathbb{Z}$ is a product of local rings and hence clean. Thus, $\mathbb{Z}$ is neat. A nice generalization of this is the following theorem.

Proposition 2.4. *If A is a domain of* (*Krull*) *dimension equal to* 1, *then A is neat. In particular*, *PIDs are neat.*

Proof. This follows from the fact that zero-dimensional rings are clean. □

Example 2.5. If F is a field and $A = F[X, Y]$, then A is not neat as $A/YA \cong F[X]$ is not clean (see [1]). $F[X]$ is neat by the previous theorem. Moreover, if $A[X]$ is neat, then A is field.

It follows that if A is neat, then every nonzero prime ideal is contained in a unique maximal ideal. We call such a ring a *pm*-ring*. Obviously, a ring which is not an integral domain is a *pm*-ring precisely when it is a *pm**-ring. Therefore, our interest in *pm**-rings will take place in the class of integral domains. The proof of our next theorem is a simple adaptation of the proof of Theorem 1.2 of [13]. Let $Spec(A)^*$ denote the set of all nonzero prime ideals. For $M \in \mathrm{Max}(A)$ we let O_{M^*} denote the intersection of all nonzero prime ideals contained in M.

Theorem 2.6. *Let A be a commutative ring with identity. The following are equivalent*:

(a) *A is a pm*-ring.*
(b) $\mathrm{Max}(A)$ *is a retract of* $Spec(A)^*$.
(c) *For each* $M \in \mathrm{Max}(A)$, *M is the unique maximal ideal containing* O_{M^*}.

Note that $\mathrm{Max}(A)$ need not be Hausdorff if A is a *pm**-ring, e.g. $A = \mathbb{Z}$.

3. FGC rings

A ring A is called an *FGC ring* if every finitely generated module is isomorphic to a direct sum of cyclics. This class of rings dates back to Kaplansky [23] who was interested in classifying rings which satisfied the generalization of the Fundamental Theorem of Finitely Generated Abelian Groups. FGC rings are classified in [5]. Information on FGC rings can also be found in [16]. The classification states that an FGC ring is finite direct product of three types of rings. Before we classify neat FGC rings we recall a few definitions.

Definition 3.1. Let A be a ring and M an A-module. We say M is a *linearly compact A-module* if every collection of cosets with the finite intersection property has nonempty intersection. It is known that a homomorphic image of a linearly compact A-module is linearly compact (see [5]). If A is a linearly compact A-module, then we say A is *maximal*. Artinian rings are maximal. The ring of p-adic integers is a maximal ring. A is said to be *almost maximal* if A/I is a linearly compact A-module for every nonzero ideal I of A. If A_M is almost maximal for all maximal ideals M of A, then A is said to be *locally almost maximal*. For more information on maximal and almost maximal rings see [5,16]. For example, the next theorem may be found in both places.

Theorem 3.2 (*Zelinsky*). *If the ring A is maximal, then it is a finite direct product of local rings.*

Corollary 3.3. *A maximal ring is clean. Moreover, an almost maximal ring is neat.*

A ring A is called *h-local* if it is of finite character and every proper homomorphic image is a *pm*-ring. To be of finite character means that every element is contained in a finite number of maximal ideals. Recall that a ring is a *Bézout ring* if every finitely generated ideal is principal. The class of Bézout domains includes PIDs and valuation domains.

Definition 3.4. A ring A is called a *torch ring* if it satisfies the following conditions:

(1) A is not local.
(2) A has a unique minimal prime ideal P which is nonzero and whose A-submodule form a chain.
(3) A/P is an h-local domain.
(4) A is a locally almost maximal Bézout ring.

The interested reader should check [5] for an example of a torch ring. PIDs are almost maximal Bézout domains.

Theorem 3.5 (*Brandal [5, Theorem 9.1]*). *A ring is an FGC-ring if and only if it is a finite direct product of the following types of rings*:

1. *Maximal valuation rings.*
2. *Almost maximal Bézout domains.*
3. *Torch rings.*

Lemma 3.6. *A torch ring is never neat.*

Proof. Let A be a torch ring and P its unique minimal prime ideal. If A is neat then A/P is a clean domain and hence local. But, then so is A contradicting (i) of Definition 3.4. The result follows. □

Theorem 3.7. *Suppose A is an FGC ring. A is clean if and only if A is a finite direct product of local rings. In this case, it is a finite direct product of almost maximal valuation rings.*

Proof. It suffices to show that if A is clean then it is a finite direct product of local rings. Suppose, A is a clean FGC ring. Write $A = A_1 \times \cdots \times A_n$, where each A_i is one of the appropriate types of rings from Theorem 3.5. Since A is clean each A_i is clean. By the previous lemma it follows that none of the A_i are torch rings and hence each A_i is either a maximal valuation ring or an almost maximal Bézout domain. Zelinsky's theorem takes care of the maximal valuation rings and a clean domain is local. Since each local Bézout domain is a valuation domain we obtain that A is a finite direct product of almost maximal valuation rings. □

Theorem 3.8. *Suppose A is an FGC ring. A is neat if and only if A is either a clean ring or it is an almost maximal Bézout domain which is not local.*

Proof. First observe that the sufficiency is true by Proposition 2.3 and the last part of Zelinsky's Theorem. As for the necessity we suppose A is neat but not clean. Then first off A is indecomposable and it is not local. Now, A is either a maximal valuation ring or an almost maximal Bézout domain. But it cannot be a maximal ring since that would imply it is clean. Therefore, it follows that A is an almost maximal Bézout domain which is not local. □

Corollary 3.9. *An FGC-domain is neat.*

Almost maximal rings are neat. Almost maximal domains were classified by Brandal:

Proposition 3.10 (*Brandal [4]*). *A ring is an almost maximal domain if and only if it is h-local and locally almost maximal.*

At this point a natural question is whether *h*-local domains are neat. We presently answer in the affirmative. The result easily follows once we recite some results from [5]. The interested reader should consult [24,5]. For an ideal I of A it is useful to let $V(I)$ denote the set of maximal ideals of A containing I. We now may restate the definition of an *h*-local domain as a ring A that is a pm^*-ring and $V(I)$ is finite for every nonzero ideal I.

Lemma 3.11 (*Brandal [5, Lemma 2.4]*). *Let I be an ideal of A which is contained in a finite number of maximal ideals. Then R/I is a direct sum of indecomposable modules of the form R/J, where $I \leqslant J$.*

Proposition 3.12 (*Brandal [5, Proposition 2.5]*). *Let I be an ideal of A such that $V(I)$ is finite. Then R/I is indecomposable if and only if for all nontrivial partitions V_1, V_2 of $V(I)$ there are $M_1 \in V_1$, $M_2 \in V_2$ and a prime ideal P of A such that $I \subseteq P \subseteq M_1 \cap M_2$.*

Proposition 3.13. *Suppose $V(I)$ is finite and R/I is a pm-ring. Then R/I is a finite direct product of local rings.*

Proof. Let $V(I)$ be finite. By Lemma 3.11, R/I is a direct sum of indecomposable modules of the form R/J and $I \leqslant J$. Now, each R/J is also a *pm*-ring and $V(J)$ is finite. Proposition 3.3 forces each R/J to be local otherwise we would be able to find a nontrivial partition of $V(J)$ and hence we could find a prime contained in two different maximal ideals. □

We are now in position to state our desired theorem whose proof is a consequence of the previous proposition.

Theorem 3.14. *An h-local domain is neat.*

Example 3.15. Divisorial and hence reflexive domains are *h*-local and therefore neat. A discussion of these domains may be found in Chapter IV of [16].

4. Groups of divisibility

In Section 5, we will give necessary and sufficient conditions for a Bézout domain to be neat. These conditions involve the domain's group of divisibility which we presently recall. For a domain A we denote by qA its classical field of fractions, and qA^* the set of nonzero elements of qA. qA^* is an abelian group under multiplication and $U(A)$ is a subgroup. We define $G(A) = qA^*/U(a)$ and call this the *group of divisibility of* A. $G(A)$ is partially ordered in the following manner. For any $aU(A), bU(A) \in G(A)$ we set $aU(A) \leqslant bU(A)$ if $b/a \in A$. This definition is well-defined and makes $G(A)$ into a partially-ordered group. The positive cone, that is, the set of elements $aU(A) \in G(A)$ for which $1U(A) \leqslant aU(A)$ is the set of cosets whose representatives belong to A. We denote the positive cone of a partially ordered group by G^+.

The partial-order defined above becomes a lattice-order and makes $G(A)$ into a lattice-ordered group (or ℓ-group for short) precisely when A is a GCD-domain. (A domain in which every pair of elements a, b has a greatest common divisor is called a GCD-domain.) In particular, domains in which every finitely generated ideal is principal, that is, Bézout domains are GCD-domains. The following well-known theorem states that every abelian ℓ-group may be realized as the group of divisibility of a Bézout domain. Some nice references for lattice-ordered groups are [11,2].

Theorem 4.1 (*Jaffard–Ohm–Kaplansky*). *Let G be an abelian ℓ-group. There exists a Bézout domain A for which $G(A) \cong G$.*

The next result characterizes the clean Bézout domains. Recall that a domain is clean if and only if it is local, then:

Corollary 4.2. *The following are equivalent for the domain D*:

(i) *D is a clean Bézout domain.*
(ii) *D is a valuation domain.*
(iii) *$G(D)$ is a totally-ordered group.*

As we mentioned before our aim is to classify neat Bézout domains. The reason for not considering GCD-domains in general is that for Bézout domains there is a nice correspondence between the prime ideals of A and the prime subgroups of $G(A)$. In particular, $\mathbb{Z}[x]$ is a GCD-domain that is not a Bézout domain. It is also not neat as previously mentioned. The group of divisibility of $\mathbb{Z}[X]$ is a direct sum of copies of $\mathbb{Z}$ and as we shall later see if A is Bézout domain whose group of divisibility is a direct sum of copies of the integers, then it is neat.

Definition 4.3. Let G be an abelian ℓ-group. A subgroup H is called an *ℓ-subgroup* if it is a sublattice of G. A subset $C \subseteq G$ is called *convex* if whenever $x \leqslant y \leqslant z$ and $x, z \in C$ then $y \in C$. The collection of convex ℓ-subgroups of G is denoted by $\mathscr{C}(G)$. Since the intersection of an arbitrary set of convex ℓ-subgroups is again a convex ℓ-subgroup it follows that $\mathscr{C}(G)$ is a complete lattice when partially-ordered under inclusion. It also follows that given any set $S \subseteq G$, there exists a least convex ℓ-subgroup containing S. We denote it by $G(S)$. When $S = \{g\}$ we simply write $G(g)$.

Let P be a convex ℓ-subgroup. P is said to be a *prime subgroup* if whenever $a \wedge b = e$ then either $a \in P$ or $b \in P$. Given a convex ℓ-group H, a relation is defined on G/H by setting $a + H \leqslant b + H$ if there is an $h \in H$ such that $a + h \leqslant b$. This relation is well-defined and makes G/H into an ℓ-group. It is then a fact that P is prime if and only if G/P is totally-ordered. We let $Spec_\ell(G)$ denote the collection of prime subgroups of G. It is a consequence of Zorn's Lemma that minimal prime subgroups exist. We use $\mathrm{Min}(G)$ to denote this collection.

The following theorem is well-known and can be found in several places, e.g. [26,18,2].

Theorem 4.4. *Let A be a Bézout domain. There is a one-to-one order-reversing correspondence between nonzero prime ideals of A and prime subgroups of $G(A)$. Furthermore, if A is a Bézout domain, then the set of nonzero prime ideals of A forms a tree, that is, for any prime ideal P the set of prime ideals contained in P forms a chain.*

Corollary 4.5. *Let A be a Bézout domain. The map v when restricted to* $\mathrm{Max}(A)$ *is a bijection onto* $\mathrm{Min}(G(A))$.

Let P be a nonzero prime ideal of A. If A is neat, then A/P is a clean Bézout domain and hence a valuation domain. It follows that the set of all ideals containing P and in particular the primes above P form a chain. Since we already know that the primes contained in P form a chain it follows that $Spec(A)^*$ is a disjoint union of chains. An abelian ℓ-group satisfying the property that the set of prime subgroups forms a disjoint union of chains is said to have *stranded primes*. Equivalently, the ℓ-group G has stranded primes if and only if every prime subgroup contains a unique minimal prime subgroup [11, Definition 18.2]. Examples of abelian ℓ-groups with stranded primes include Example 18.1 of [11] and also the set of integer-valued continuous functions on a topological space under pointwise operations. If A is a Bézout domain whose group of divisibility has stranded primes, then we shall say that A has stranded primes.

We have shown the following:

Proposition 4.6. *Suppose A is a neat Bézout domain. Then $G(A)$ has stranded primes.*

Definition 4.7. The ℓ-group G is said to be *hyper-archimedean* if every prime subgroup is minimal. This is not the usual definition (see [7]) but will suffice for our purposes here. Obviously, a hyper-archimedean ℓ-group has the stranded prime property. It should also be obvious that $G(A)$ is hyper-archimedean if and only if the Krull dimension of A is less than or equal to 1. Therefore, we obtain

Proposition 4.8. *If A is a Bézout domain for which $G(A)$ is hyper-archimedean, then A is neat.*

Corollary 4.9. *Suppose A is a Bézout domain and $G(A)$ is isomorphic to a direct sum of copies of $\mathbb{Z}$. Then A is neat.*

It is now a good time to characterize h-local Bézout domains. Let G be an abelian ℓ-group and $a \in G^+$. A Zorn's Lemma argument produces convex ℓ-subgroups of G which are maximal with respect to not containing a. Such subgroups are called *values of a* (or sometimes said to be regular) and a subgroup is said to be a *value* if it is the value of some positive element. Values are prime subgroups. The group G is said to be *finite-valued* if every positive element has only a finite number of values.

Proposition 4.10. *If A is a Bézout domain, then A is h-local if and only if A has stranded primes and $G(A)$ is finite valued. In this case, A is neat.*

Proof. If A is h-local, then A is neat by Theorem 3.14 and so A has stranded primes. Next, since every element of A belongs to only a finite number of maximal ideals it follows that each element $g \in G(A)^+$ is not contained in only a finite number of minimal prime subgroups. Since every value of g must contain one of these minimal prime subgroups and $Spec_\ell(G(A))$ is a root system it follows that g has only a finite number of values, whence $G(A)$ is finite-valued.

Conversely, if $a \in A$ is nonzero, then by hypothesis $aU(A)$ has only a finite number of values. Since $G(A)$ has stranded primes this forces the number of minimal prime subgroups not containing $aU(A)$ to be finite and hence the number of maximal ideals which contain a is finite. Thus, A has finite character. Since A is a pm^*-ring it follows that A is an h-local domain. □

Example 4.11. There are examples of abelian ℓ-groups which are finite valued but do not have the stranded primes property. Such an ℓ-group induces a Bézout domain which is not neat.

Our aim is to demonstrate that we can characterize neat Bézout domains via their groups of divisibility. In order to do so we shall make use of the space of minimal prime subgroups of an abelian ℓ-group. For any commutative ring with identity A, $\mathrm{Max}(A)$ is a compact T_1

space under the hull-kernel topology, and is not Hausdorff in general. On the other hand, for an arbitrary abelian ℓ-group $\mathrm{Min}(G)$ has a very rich structure. Sets of the form

$$M(a) = \{P \in \mathrm{Min}(G) : a \notin P\}$$

for arbitrary $a \in G^+$ form a basis for the open sets of $\mathrm{Min}(G)$ under the hull-kernel topology. The complement of this set shall be denoted by $N(a)$. Every open (closed) subset of $\mathrm{Min}(G)$ has the form $M(H)$ ($N(H)$) for some convex ℓ-subgroup $H \leqslant G$, where $M(H) = \bigcup\{U(h) : h \in H^+\}$ ($N(H) = \mathrm{Min}(G)\backslash M(H)$). Furthermore, if P and Q are distinct minimal prime subgroups, then there are disjoint $a, b \in G^+$ such that $a \in P\backslash Q$ and $b \in Q\backslash P$. It follows that $M(a) \cap M(b) = \emptyset$, where $P \in M(b)$ and $Q \in M(a)$. Thus, the hull-kernel topology on $\mathrm{Min}(G)$ is Hausdorff. For more information on this topic see [8].

Definition 4.12. Let G be an ℓ-group and $X \subseteq G$. The *polar of* X is defined as

$$X^{\perp} = \{g \in G : |g| \wedge |x| = 0\ \forall x \in X\}$$

and is a convex ℓ-subgroup of G. When $X = \{x\}$, we denote its polar by $x^{\perp}$. Polars are useful in distinguishing the minimal prime subgroups from the rest of the prime subgroups. The next lemma is usually known as the Lemma on Ultrafilters.

The following lemma shall play a pivotal role in what follows. A discussion concerning the lemma can be found in [8].

Lemma 4.13 (*Lemma on Ultrafilters*). *Let G be an ℓ-group. For each minimal prime subgroup P, the set $\mathscr{V} = \{g \in G^+ : g \notin P\}$ is an ultrafilter of G^+. Conversely, if $\mathscr{V}$ is an ultrafilter of G^+, then the set $P = \bigcup\{x^{\perp} : x \in \mathscr{V}\}$ is a minimal prime subgroup. Moreover, a prime subgroup P is a minimal prime subgroup if and only if $P = \bigcup\{x^{\perp} : x \notin P\}$.*

One of the main things we can conclude using the Lemma on Ultrafilters is that for each $a \in G$, $M(a) = M(a^{\perp\perp}) = V(a^{\perp})$. Thus, each of the basic open sets is clopen, whence the hull-kernel topology on $\mathrm{Min}(G)$ is zero-dimensional. Unlike $\mathrm{Max}(A)$, $\mathrm{Min}(G)$ is only compact in certain instances.

Next, let $\tau : \mathrm{Min}(G(A)) \to \mathrm{Max}(A)$ be the inverse of the bijection v. Since $\tau^{-1}(U(a)) = N(a)$ we obtain that τ is a continuous function and so the topology on $\mathrm{Min}(G(A))$ is, in general, finer than the topology on $\mathrm{Max}(A)$. We also conclude that τ is a homeomorphism if and only if v is continuous. We end this section by discussing when this situation occurs.

Definition 4.14. Let G be an ℓ-group and $u \in G^+$. We say u is a (*weak*) *order unit* if $u^{\perp} = \{0\}$. It follows that a positive element of G is a weak order unit precisely when it does not belong to any minimal prime subgroup of G. Not every ℓ-group possesses a weak order unit, e.g. the group of divisibility of the integers. Observe that the corresponding definition for an element a of a domain A is that the element belong to the Jacobson radical of A. For $x \in G^+$ if there exists a $y \in G^+$ such that $x \wedge y = 0$ and $x \vee y$ is an order unit, then x is said to be a *complemented element of* G. If every positive element of G is complemented, then G is said to be a *complemented* ℓ-group. G is called *locally complemented* if for each $g \in G^+$ the subgroup $G(g)$ is complemented.

Lemma 4.15. *Let G be an ℓ-group and let $x, y \in G^+$. If $x \wedge y = 0$ and $x \vee y$ is a weak order unit, then $M(x) = N(y)$. The converse also holds.*

Proof. Consider $M(x) \cup M(y) = M(x \vee y)$. It follows that $x \vee y$ is a weak order unit if and only if $M(x) \cup M(y) = \mathrm{Min}(G)$. Furthermore, since for each positive element there is a minimal prime subgroup not containing it we conclude that $M(x) \cap M(y) = \emptyset$ if and only if $x \wedge y = 0$. □

Theorem 4.16. *Let A be a Bézout domain and $G = G(A)$. The map*

$$v : \mathrm{Max}(A) \to \mathrm{Min}(G(A))$$

is continuous (and hence a homeomorphism) if and only if G is complemented. In this case, $\mathrm{Max}(A)$ *is a boolean space.*

Proof. We first show that v is continuous if G is complemented. Since sets of the form $M(a)$ form a basis for $\mathrm{Min}(G)$ we need to show that each $v^{-1}(M(a)) = V(a)$ is open for each nonzero $a \in A$. By the previous lemma there is a $b \in A$ such that $M(a) = N(b)$. Therefore, $V(a) = U(b)$ which is open.

Conversely, if v is a homeomorphism, then $\mathrm{Min}(G) \cong \mathrm{Max}(A)$ is a compact space. Theorem 2.2 of [8] easily shows that G must be complemented. □

Corollary 4.17. *Let A be a Bézout domain and suppose that $G(A)$ is a complemented ℓ-group. Then A is neat if and only if A has stranded primes.*

Proof. The necessity is clear. If A has stranded primes then for any nonzero element $a \in A$, the ring A/aA has stranded primes and hence is a *pm*-ring. It is straightforward to check that $\mathrm{Max}(A/aA) \cong V(a)$ where the latter space is a subspace of a zero-dimensional space, and hence zero-dimensional. By Johnstone's Theorem, A/aA is clean. □

The fact that G is complemented if and only if the hull-kernel topology on $\mathrm{Min}(G)$ is compact is proved in Theorem 2.2 of [8]. For abelian ℓ-groups our proof here drastically simplifies their proof of the necessity, which uses transfinite induction and therefore we emphasize this result.

Theorem 4.18. *Let G be an abelian ℓ-group. Under the hull-kernel topology* $\mathrm{Min}(G)$ *is compact if and only if G is complemented.*

Obviously a boolean space is zero-dimensional. Corollary 4.17 can be strengthened by requiring that $\mathrm{Max}(A)$ be zero-dimensional instead of $G(A)$ be complemented. Therefore, in the next section we consider when $\mathrm{Max}(A)$ is zero-dimensional for a Bézout domain.

5. The inverse topology on $\mathrm{Min}(G)$

It should be apparent that studying the topological structure of the maximal ideal space of a Bézout domain is equivalent to studying the topological structure of the space of minimal prime subgroups of its group of divisibility endowed with the topology generated by sets of the form $N(g)$ for arbitrary positive g. This topology is known as the *inverse topology*. (It is also known as the cotopology on a structure space.) To save time and space we shall forego the transfer of information between these two homeomorphic spaces and simply work in $\mathrm{Min}(G)$. As we saw previously if $a \in G^+$ has a complement b, then $M(a) = N(b)$ and so this set is clopen in the inverse topology. Furthermore, since the inverse topology is a weaker topology than the hull-kernel topology it follows that if a set is clopen in the inverse topology then it is clopen in the hull-kernel topology.

Lemma 5.1. *Let $K \subseteq \mathrm{Min}(G)$ and suppose that K is clopen with respect to the inverse topology. Then $K = N(g)$ for some complemented element $g \in G^+$.*

Proof. Let K be as in the hypothesis. Let $\{N(a_i)\}$ be an open cover of K by basic open sets with $a_i \in G^+$ and $N(a_i) \subseteq K$. Since the inverse topology on $\mathrm{Min}(G)$ is compact it follows that K is compact. Therefore,

$$K = N(a_{i_1}) \cup \cdots \cup N(a_{i_n}) = N(g),$$

where $g = a_1 \wedge \cdots \wedge a_n$. Similarly, $X \backslash K = N(b)$ for some b. It follows from Lemma 4.15 that g is a complemented element. □

Corollary 5.2. *Let $g \in G^+$. $M(g)$ is open in the inverse topology if and only if g is complemented.*

Corollary 5.3. *G has no complemented elements if and only if the inverse topology on $\mathrm{Min}(G)$ is connected.*

In [8] it is shown that if $P \in \mathrm{Min}(G)$ is an isolated point in the hull-kernel topology, then it is of the form $P = b^{\perp}$ for some positive b. In general, such a minimal prime subgroup need not be isolated in the inverse topology. This leads us to our next result.

Proposition 5.4. *Suppose G is an ℓ-group and $P \in \mathrm{Min}(G)$. Then the following are equivalent*:

(i) *P is an isolated point with respect to the inverse topology on $\mathrm{Min}(G)$.*
(ii) *$P = b^{\perp}$ for some complemented element $b \in G^+$.*
(iii) *$P = b^{\perp}$ for some complemented, basic element $b \in G^+$.*

We now classify those abelian lattice-ordered groups for which the inverse topology on $\mathrm{Min}(G)$ is zero-dimensional.

Theorem 5.5. *The following are equivalent for an abelian lattice-ordered group G.*

(i) *The inverse topology on $\mathrm{Min}(G)$ is zero-dimensional.*
(ii) *For each $0 < g$ and each minimal prime containing g there is a complemented element x above g which is also in P.*

(iii) *For each pair of distinct minimal prime subgroups there exists a positive complemented element belonging to exactly one of them.*
(iv) *The inverse topology on* Min(G) *is totally disconnected.*
(v) *Whenever* $a, b \in G^+$ *and* $a \wedge b = 0$ *there is a complementary pair of elements, say* x *and* y, *such that* $a \leqslant x$ *and* $b \leqslant y$.

Proof. (i) $\Rightarrow$ (ii) Suppose that the inverse topology on Min(G) is zero-dimensional and let $P \in N(g)$ with $g \in G^+$. Let K be a clopen set satisfying $P \in K \subseteq N(g)$. By Lemma 5.1, $K = N(a)$ for some complemented a. Now, $N(a \vee g) = N(a) \cap N(g) = N(a)$ so that Lemma 5.2 forces $x = a \vee g$ to be a complemented element above g and belonging to P.

(ii) $\Rightarrow$ (iii) Let P and Q be distinct minimal prime subgroups of G and let $g \in P^+ \backslash Q$. By hypothesis there is a complemented element $x \geqslant g$ belonging to P. Since $g \leqslant x$, $g \notin Q$, and Q is convex it follows that $x \notin Q$.

(iii) $\Rightarrow$ (iv) Let P and Q be distinct minimal prime subgroups of G. By hypothesis, there is a positive complemented element belonging to exactly one of them, say $x \in P$. The clopen subset $N(x)$ separates P and Q. Therefore, Min(G) is totally disconnected.

(iv) $\Rightarrow$ (i) This is patent.

(i) $\Rightarrow$ (v) Suppose, $a \wedge b = 0$. Then $M(a) \cap M(b) = \emptyset$. This means that $M(a)$ and $M(b)$ are disjoint closed subsets of the compact zero-dimensional space Min(G). It follows that they can be separated by a clopen set. In particular, there are complements x and y such that $M(a) \subseteq M(x)$ and $M(b) \subseteq M(y)$ with $a \leqslant x$ and $b \leqslant y$.

(v) $\Rightarrow$ (iii) Let P and Q be distinct minimal prime subgroups of G. Choose disjoint positive elements a and b such that $a \in P \backslash Q$, $b \in Q \backslash P$. By hypothesis, there is a pair of complements x and y such that $a \leqslant x$ and $b \leqslant y$. Since $a \notin Q$ it follows that $x \notin Q$. It also follows that $x \in P$ since x is complemented. $\square$

Definition 5.6. We call an abelian ℓ-group G satisfying the equivalent conditions of Theorem 5.5 *weakly complemented*. If G has the property that $G(g)$ is weakly complemented for each $g \in G^+$, then G is called *locally weakly complemented*. Note that since complemented ℓ-groups are weakly complemented it follows that locally complemented ℓ-groups are locally weakly complemented. Clearly, a weakly complemented ℓ-group is locally weakly complemented. If G has a weak order unit, then the converse holds. We will later show that not all weakly complemented ℓ-groups are complemented.

Theorem 5.7. *Let A be a Bézout domain. A is a neat ring if and only if* $G(A)$ *has stranded primes and is locally weakly complemented.*

Proof. Let $G = G(A)$ and $g = aU(a) \in G^+$. The main point of the proof is that

$$\mathrm{Min}(G(g)) \cong M(g) \cong V(A/aA).$$

If A is neat, then it has stranded primes and each $V(A/aA)$ is zero-dimensional, hence $G(g)$ is weakly complemented. Since g was arbitrary it follows that G is locally weakly complemented. The converse is similar. $\square$

Theorem 5.8. *Suppose A is a Bézout domain. The following are equivalent*:

(i) *A is neat and* $\mathfrak{J}(A) \neq 0$.
(ii) *A has stranded primes and* $\mathrm{Max}(A)$ *is zero-dimensional*.
(iii) *A has stranded primes and* $G(A)$ *is weakly complemented*.

6. r^*-extensions

Recall that an extension of ℓ-groups, say $G \leqslant H$, is called a *rigid extension* if for each $h \in H^+$ there is a $g \in G^+$ such that $g^{\perp\perp} = h^{\perp\perp}$. The most common example of a rigid extension $G(u) \leqslant G$, where $u \in G^+$ is a weak order unit. In [8] the authors generalized this notion by defining an extension $G \leqslant H$ of ℓ-groups to be an *r-extension* if for every $0 < h \in H$ and each $P \in \mathrm{Min}(H)$ not containing h there is a $0 < g \in G \backslash P$ such that $g^{\perp\perp} \subseteq h^{\perp\perp}$. Clearly, a rigid extension is an r-extension. It is then proved that $G \leqslant H$ is an r-extension precisely when $P \in \mathrm{Min}(H)$ implies $P \cap G \in \mathrm{Min}(G)$ and the contraction mapping $C \to C \cap G$ restricts to a homeomorphism of $\mathrm{Min}(H)$ onto $\mathrm{Min}(G)$ with respect to the hull-kernel topology. This leads us to another generalization of rigid extension.

Definition 6.1. We define an extension $G \leqslant H$ to be an $r^\flat$-*extension* whenever the contraction mapping $P \to P \cap G$ takes a minimal prime subgroup of H to a minimal prime subgroup of G in a bijective manner. It follows from what was said above that an extension of ℓ-groups is an r-extension if and only if it is an $r^\flat$-extension and contraction is a homeomorphism.

Next, we call the extension $G \leqslant H$ an r^*-*extension* if for every $0 < h \in H$ and $P \in N_H(h)$, there is a $0 < g \in G \cap P$ such that $h^{\perp\perp} \subseteq g^{\perp\perp}$. As mentioned previously, a rigid extension is an r^*-extension. We now describe the remaining connections between r^*-extensions and the others defined. We note that when working with hulls and kernels we shall make explicit use of subscripts to denote which group's collection of minimal primes are being dealt with, e.g. $N_G(g)$.

Lemma 6.2. *Suppose* $G \leqslant H$ *is an* $r^\flat$-*extension. The contraction map of* $\mathrm{Min}(H)$ *onto* $\mathrm{Min}(G)$ *is continuous with respect to both the hull-kernel and inverse topologies*.

Proof. Let $\sigma : \mathrm{Min}(H) \to \mathrm{Min}(G)$ denote the contraction map which by hypothesis is well-defined. Let $g \in G^+$. Then

$$\begin{aligned}\sigma^{-1}(N_G(g)) &= \{P \in \mathrm{Min}(H) : g \in P \cap G\}\\ &= \{P \in \mathrm{Min}(H) : g \in P\}\\ &= N_H(g).\end{aligned}$$

Since sets of this form a base for the topology of open (resp., closed) sets on the inverse (resp., hull-kernel) topology on $\mathrm{Min}(G)$ it follows that σ is continuous with respect to both topologies. □

We leave the verification of the next lemma to the interested reader. The Lemma on Ultrafilters (Lemma 4.13) is useful.

Lemma 6.3. *$G \leqslant H$ is an r^*-extension if and only if for each $0 < h \in H$ and $P \in N_H(h)$ there exists a $0 < g \in G$ such that $g \in P$ and $N_H(g) \subseteq N_H(h)$.*

Proposition 6.4. *Suppose H is complemented and G is an r^*-subgroup. Then G is a rigid-subgroup. Therefore, the set of r^*-subgroups of a complemented group equals the set of rigid-subgroups.*

Proof. By Theorem 4.16 it follows that $\mathrm{Min}(H)$ is a compact Hausdorff space and so is every set of the form $N_H(h)$. For each $P \in N_H(h)$, we can select a $g \in G$ such that $P \in N_H(g) \subseteq N_H(h)$ so that the collection of these forms an open cover. A finite subcover will give rise to an element $g \in G$ such that $N_H(g) = N_H(h)$. From here we gather that $g^{\perp\perp} = h^{\perp\perp}$. □

Proposition 6.5. *An r^*-extension is an $r^\flat$-extension. Furthermore, an extension is an r^*-extension precisely when the contraction mapping is a homeomorphism of* $\mathrm{Min}(H)$ *onto* $\mathrm{Min}(G)$ *with respect to the inverse topologies.*

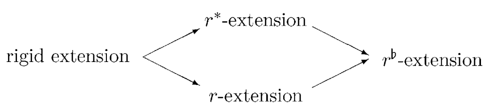
Proof. *Note*: a reader familiar with [8] may find this proof similar to the proof of 2.3. Also, it is proved there that if *N* is a minimal prime subgroup of *G*, then there is a minimal prime subgroup of *H* which contracts to *N*. Hence, the map is surjective.

Suppose, $G \leqslant H$ is an r^*-extension. Let $P \in \mathrm{Min}(H)$. Without loss of generality, there exists a different minimal prime subgroup *Q* of *H*. Otherwise, the group is totally ordered and every positive element of *H* and *G* is a weak order unit, and hence the extension is an *r*-extension. Therefore, choose an $0 < h \in Q \backslash P$ and select a $0 < g \in G$ such that $g \in Q$ and $h^{\perp\perp} \subseteq g^{\perp\perp}$. If $g \in P$, then $g^{\perp\perp} \subseteq P$ and so $h \in P$; a contradiction. Therefore, $g \notin P$ and it follows that $G \cap P < G$.

Next, let *P* and *Q* be distinct minimal prime subgroups of *H* and choose $h_1 \in P \backslash Q$ and $h_2 \in Q \backslash P$ with $h_1 \wedge h_2 = 0$. Select a $0 < g \in G$ such that $g \in P$ and $h_1^{\perp\perp} \subseteq g^{\perp\perp}$. If $g \in Q$ it follows that $h_1 \in Q$. This contradiction implies that $P \cap G \neq Q \cap G$. Thus, the contraction map is injective. It also follows that the contraction of a minimal prime of *H* is a minimal prime of *G*.

Finally, the condition that the contraction map be an open map is precisely that of an r^*-extension. We leave the proof of this fact to the interested reader. □

Example 6.6. We thus have the following diagram of implications.



The examples given in [8] are useful in showing that none of the above arrows reverse. Let K be the ℓ-group of eventually constant integer-valued sequences ordered pointwise (where **0** is the constant value 0). For each element $k \in K$, we use k_∞ to denote the limit of the sequence. Next, let $H = K \times Z$ and order H by defining $(k, n) \geqslant (\mathbf{0}, 0)$ if either (1) $k \geqslant \mathbf{0}$ and $k_\infty > 0$, or (2) $k > \mathbf{0}$, $k_\infty = 0$, and $n \geqslant 0$. Define $G_1 = \{(k, n) \in H : k_\infty = 0\}$ and $G_2 = K \times \{0\}$. Both $G_1 \leqslant H$ and $G_2 \leqslant H$ are $r^\flat$-extensions.

It is straightforward to show that the hull-kernel topologies on $\mathrm{Min}(H)$ and $\mathrm{Min}(G_1)$ are the discrete topologies on $\mathbb{N}$, and that the hull-kernel topology on $\mathrm{Min}(G_2)$ is the one-point compactification of $\mathbb{N}$ (it is complemented). Under the inverse topologies $\mathrm{Min}(H)$ and $\mathrm{Min}(G_2)$ are copies of the one-point compactification of $\mathbb{N}$ and $\mathrm{Min}(G_1)$ is equipped with the cofinite topology. Hence, we can conclude that $G_1 \leqslant H$ is an r-extension which is not an r^*-extension, and $G_2 \leqslant H$ is an r^*-extension which is not an r-extension.

We now can generalize Proposition 2.3 of [8]. We leave its proof to the interested reader.

Proposition 6.7. *Suppose $G \leqslant H$ is an extension and H is complemented. If the extension is either an r-extension or an r^*-extension, then it is a rigid extension.*

7. Applications to $C(X)$

Recall that for a topological space X, $C(X)$ ($C^*(X)$) denotes the set of all (bounded) real-valued continuous functions on X. $C(X)$ is an abelian ℓ-group under the pointwise operations. We shall assume that all of our topological spaces are Tychonoff, that is, completely regular and Hausdorff. Ref. [17] still is the ultimate source for rings and groups of continuous functions.

For any point $p \in X$

$$M_p = \{f \in C(X) : p \in Z(f)\}$$

and

$$O_p = \{f \in C(X) : p \in int\, Z(f)\}.$$

M_p is always a maximal convex ℓ-subgroup of $C(X)$ and hence it is prime. It is straightforward from the definition that O_p is a convex ℓ-subgroup of $C(X)$ contained in M_p. Furthermore, it is known that O_p is precisely the intersection of all minimal prime subgroups of $C(X)$ contained in M_p.

Definition 7.1. A space X is called *cozero complemented* if for each cozero set C there is a disjoint cozero set C' so that $C \cup C'$ is a dense subset of X. It is well-known that X is cozero complemented if and only if $C(X)$ is a complemented ℓ-group. We call a space X *weakly cozero complemented* if for each pair of disjoint cozero sets C_1, C_2 there exists a pair of disjoint cozerosets T_1, T_2 such that $C_i \subseteq T_i$ and the union of T_1 and T_2 is a dense subset of X. The next result should not be surprising.

Proposition 7.2. *For a space X, the following are equivalent*:

(i) $\mathrm{Min}(C(X))$ *is zero-dimensional with respect to the inverse topology.*
(ii) *X is weakly cozero complemented.*
(iii) *βX is weakly cozero complemented.*

Proof. Since $C^*(X) \leqslant C(X)$ is a rigid extension it follows that $\mathrm{Min}(C^*(X)) \cong \mathrm{Min}(C(X))$ with respect to the inverse topologies. □

The following can be found in [14]. We include its proof for completeness sake.

Proposition 7.3. *Let X be an F-space. Then βX is homeomorphic to* $\mathrm{Min}(C(X))$ *under the inverse topology*.

Proof. As we mentioned before $\mathrm{Min}(C(X)) \cong \mathrm{Min}(C^*(X))$. Therefore, it is enough to show that the proposition is true for compact F-spaces X. If X is a compact F-space, then every minimal prime subgroup of $C(X)$ is of the form O_p for some $p \in X$, and so there is an obvious bijection between X and $\mathrm{Min}(C(X))$. Now, for $f \in C(X)$

$$\begin{aligned} N(f) &= \{O_p : f \in O_p\} \\ &= \{O_p : p \in \mathit{int}\, Z(f)\} \end{aligned}$$

and therefore the inverse topology on $\mathrm{Min}(C(X))$ is homeomorphic to the topology on X generated by basic sets of the form $\mathit{int}\, Z(f)$ for arbitrary $f \in C(X)$. It is straightforward to show that this latter topology is equal to the original topology on X. □

Theorem 7.4. *Let X be an F-space. Then the following are equivalent*:

(i) *X is weakly cozero complemented.*
(ii) *βX is weakly cozero complemented.*
(iii) *X is strongly zero-dimensional.*
(iv) *βX is zero-dimensional.*

Proof. This follows directly from Proposition 7.3. Since X is an F-space, $\mathrm{Min}(C(X))$ and βX are homeomorphic. Therefore, (ii) and (iv) are equivalent. The proofs of the other equivalences are known. □

Example 7.5. It is known that an F-space X is cozero complemented precisely when X is basically disconnected. It follows by Theorem 7.4 that if X is a compact zero-dimensional F-space which is not basically disconnected, e.g. $\beta\mathbb{N}\backslash\mathbb{N}$, then $C(X)$ is a weakly complemented ℓ-group with stranded primes which is not complemented.

We conclude this section with an interesting result. Recall that a topological space is called *extremally disconnected* if the closure of every open set is open. A space which is compact, Hausdorff, and extremally disconnected is called a *Stone space*. In [19] the authors determine when $\mathrm{Min}(C(X))$ under the hull-kernel topology is a Stone space. The authors show that this occurs precisely when the space X has the property that every regular closed subset is the closure of a cozero set. Such a space X is called a *fraction-dense space*. (Recall that a closed subset is said to be regular closed if it is the closure of its interior.)

Observe that if X is a fraction dense space, then X is complemented and hence the inverse topology on $\mathrm{Min}(C(X))$ results in a Stone space. It is the converse of this statement that intrigues us and we shall prove its validity in 7.10. First, a remark and some needed lemmas.

Remark 7.6. We should point out that if G is a cardinal summand of denumerably many copies of $\mathbb{Z}$ then the inverse topology on $\mathrm{Min}(G)$ is the cofinite topology which is compact and extremally disconnected as every nonempty open subset is dense. But this space is not a Stone space as it is not Hausdorff.

Lemma 7.7. *Let G be an abelian ℓ-group and consider* $\mathrm{Min}(G)$ *equipped with the inverse topology. Let $g \in G^+$ and $P \in \mathrm{Min}(G)$. Then $P \in cl\, N(g)$ if and only if whenever $f \in P^+$ then $f \vee g$ is not a weak order unit.*

Proof. Observe that $P \in cl\, N(g)$ precisely when $P \in N(x)$ implies

$$N(x \vee g) = N(x) \cap N(g) \neq \emptyset. \qquad \square$$

Lemma 7.8. *Let G be an abelian ℓ-group and consider* $\mathrm{Min}(G)$ *equipped with the inverse topology. Let $0 < h \leqslant g \in G^+$ and suppose that $cl\, N(g) = N(h)$. Then h is a complemented element and it is connected to g in the following manner. For any $f \in G^+$, $f \vee g$ is a weak order unit if and only if $f \vee h$ is a weak order unit.*

Proof. Since $cl\, N(g) = N(h)$ it follows that $N(h)$ is a clopen subset and hence by Lemma 5.2 h is complemented. Since $0 < h \leqslant g$ it follows that if $h \vee f$ is a weak order unit then so is $g \vee f$. So suppose that $h \vee f$ is not a weak order unit. Then $h \vee f$ and hence both h and f belong to some minimal prime subgroup, say P. Thus, $P \in N(h) = cl\, N(g)$ and so by the previous lemma $f \vee g$ is not a weak order unit. $\square$

Lemma 7.9. *Let X be a Tychonoff space and $0 < g \in C(X)$. Then $N(g)$ is not a dense subset of* $\mathrm{Min}(C(X))$ *with respect to the inverse topology.*

Proof. Suppose $g > 0$ and let $p \in co\, z(g)$. Choose a function $f \geqslant 0$ such that $p \in int\, Z(f) \subseteq Z(f) \subseteq co\, z(g)$. Let P be a minimal prime ideal of $C(X)$ such that $O_p \leqslant P$. Note that $g \notin P$ and thus $P \notin N(g)$. Now, by our choice $f \in O_p$ and hence $f \in P$. Observe that $(f \vee g)(x) \neq 0$ for all $x \in X$ and hence $f \vee g$ is a weak order unit. It follows from Lemma 7.7 that $P \notin cl\, N(g)$. $\square$

Theorem 7.10. *For a Tychonoff space X, the following are equivalent:*

(i) *X is fraction dense.*
(ii) $\mathrm{Min}(C(X))$ *is a Stone space with respect to the hull-kernel topology.*
(iii) $\mathrm{Min}(C(X))$ *is a Stone space with respect to the inverse topology.*
(iv) *The inverse topology on* $\mathrm{Min}(C(X))$ *is extremally disconnected.*

Proof. As noted before the equivalence of (i) and (ii) is shown in [19]. If the hull-kernel topology on $\mathrm{Min}(C(X))$ results in a Stone space, then $C(X)$ is complemented and hence the hull-kernel and inverse topologies coincide. Therefore, (ii) implies (iii). Clearly, (iii) implies (iv).

(iv) implies (ii). Suppose that inverse topology on $\mathrm{Min}(C(X))$ is extremally disconnected. We shall show that every positive element of $C(X)$ is complemented. From this it follows that the two topologies coincide. To that end, let $g \in C(X)^+$. By hypothesis, $cl\, N(g)$ is a clopen subset of $\mathrm{Min}(G)$ and by Lemma 7.9 it is a proper subset. By Proposition 5.1, there exists a complemented element $h \in C(X)^+$ such that $cl\, N(g) = N(h)$. Without loss of generality, we may assume that $0 < h \leqslant g$. Our aim is to show that $cl_X\, co\, z(g) = cl_X\, co\, z(h)$ from which it will follow that g is complemented since any complement for h will also be a complement for g.

Since $0 \leqslant h \leqslant g$ it follows that $co\, z(h) \subset co\, z(g)$. Suppose there is an $p \in co\, z(g) \setminus cl_X co\, z(h)$. This set is open and so there is an $f \in C(X)^+$ such that $p \in int_X\, Z(f) \subseteq Z(f) \subseteq co\, z(g)$. Now, $f \vee g$ is a weak order unit. Observe that $f, h \in O_p$ and so for any minimal prime subgroup P containing O_p, $P \in N(h) \notin cl\, N(g)$, a contradiction. Therefore, $co\, z(g) \subseteq cl_X\, co\, z(h)$, whence $cl_X\, co\, z(g) = cl_X\, co\, z(h)$. $\square$

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