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Topology and its Applications 153 (2006) 1838–1848
Topology and its Applications
www.elsevier.com/locate/topol

The m-topology on $C_m(X)$ revisited

Javier Gómez-Pérez [a,1], Warren W. McGovern [b,*]

[a] *Departamento de Matemáticas, Universidad de León, 24071 León, Spain*
[b] *Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, USA*

Received 18 February 2005; accepted 6 June 2005

Abstract

Hewitt [E. Hewitt, Rings of real-valued continuous functions, I, Trans. Amer. Math. Soc. 64 (1948) 45–99], generalizing work of E.H. Moore, defined the m-topology on $C(X)$. In his article he demonstrated that certain classes of topological spaces X can be characterized by topological properties of $C_m(X)$. For example, he showed that X is pseudocompact if and only if $C_m(X)$ is first countable. Others have also investigated topological properties of X via properties of $C_m(X)$, e.g., [G. Di Maio, L. Holá, D. Holý, R.A. McCoy, Topologies on the space of continuous functions, Topology Appl. 86 (2) (1998) 105–122] and [E. van Douwen, Nonnormality or hereditary paracompactness of some spaces of real functions, Topology Appl. 39 (1) (1991) 3–32]. We continue this practice in the second section and give some new equivalent characterizations. In the third section we prove the converse of a theorem of van Douwen [E. van Douwen, Nonnormality or hereditary paracompactness of some spaces of real functions, Topology Appl. 39 (1) (1991) 3–32] completing a characterization of when $C_m(X)$ is a weak P-space. In the fourth section we determine when $C_m(X)$ has no non-trivial convergent sequences.

MSC: primary 54C35; secondary 54G99

Keywords: m-topology; Weak P-space; $C(X)$

* Corresponding author.
E-mail addresses: demjgp@unileon.es (J. Gómez-Pérez), warrenb@bgnet.bgsu.edu, mcgovern@cl.uh.edu (W.W. McGovern).
1 First author partially supported by the Spanish DGICYT grant BFM2002-04125-C02-02, and Junta de Castilla y León grant LE 66/03.

0166-8641/$ – see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.topol.2005.06.016

1. Introduction

For a given a topological space X we denote the set of real-valued continuous functions on X by $C(X)$. It is well known that $C(X)$ is an $\mathbb{R}$-algebra under pointwise operations of addition, multiplication, and scalar multiplication. $C(X)$ becomes a topological space when endowed with a number of different topologies. For example, there is the topology of pointwise convergence which is defined as the subspace topology inherited from the product topology on $\mathbb{R}^X$ (and goes under the nickname C_p-theory). One may also consider the compact-open topology or the uniform topology. In this article we investigate the m-topology. Hewitt [6], generalizing work of E.H. Moore, defined the m-topology on $C(X)$ to be the one whose base is the collection of sets of the form

$$B(f,e) = \{g \in C(X): |f(x) - g(x)| < e(x) \text{ for every } x \in X\},$$

where $f \in C(X)$ and $e \in U(X)^+$ are arbitrary. Here $U(X)^+$ refers to the set of all positive multiplicative units of $C(X)$. (See the remark at the end of this section.) The notation $C_m(X)$ will be used when referring to $C(X)$ under the m-topology. This topology, in general, is finer than the uniform topology (denoted $C_u(X)$), and makes $C_m(X)$ into a Hausdorff topological ring, i.e., $+, \cdot$ are continuous operations. Generally speaking it is not a topological algebra although both translation by an element and multiplication by a multiplicative unit are autohomeomorphisms of $C_m(X)$, and so $C_m(X)$ is a homogeneous space. Observe that since the m-topology is finer than the uniform topology the proof of the first part of our first proposition is standard. The last statement is straightforward to check.

Proposition 1.1. *Let $\mathfrak{e}: X \to \mathbb{R}^{C_m(X)}$ be the evaluation map defined by $\mathfrak{e}(x)(f) = f(x)$ for $f \in C_m(X)$. Then for each $x \in X$, $\mathfrak{e}(x) \in C(C_m(X))$. The map $\mathfrak{e}: X \to C_m(C_m(X))$ is continuous precisely when X is discrete.*

Remark 1.2. For a given ring R it is standard to denote the set of multiplicative units of R by $U(R)$. Thus, we take the approach that $U(X)^+$ stands for the set of strictly positive multiplicative units of $C(X)$, that is, $e(x) > 0$ for all $x \in X$. We have found that some authors prefer to use $C(X)^+$ for this set, but in the context of the lattice structure on $C(X)$ (partially ordered pointwise) this notation denotes the set of functions for which $f \geqslant \mathbf{0}$. We hope this does not cause any confusion.

We shall assume that *all spaces considered in this article are Tychonoff.*

2. Pseudocompactness

A space X is said to be *pseudocompact* if every element of $C(X)$ is bounded. It is known that a space is pseudocompact if and only if there does not exist any C-embedded discrete sequence (see [5]).

Recall that a topological space is said to be first countable if it has countable *character*, that is, every point has a countable base of neighborhoods. In [6], Hewitt shows that the

m-topology on $C(X)$ is first countable precisely when X is pseudocompact, which in turn is equivalent to the m-topology coinciding with the uniform topology. Hewitt's proof can be slightly modified to obtain the stronger result that the m-topology is countably tight if and only if X is pseudocompact. Recently, other authors have expanded on Hewitt's Theorem to show that X being pseudocompact is equivalent to more topological conditions on $C(X)$. For example, in [2] the Čech-complete property is investigated and the authors show the following.

Theorem 2.1. *The following are equivalent*:

(i) $C_m(X)$ *is countably tight.*
(ii) $C_m(X)$ *is first countable.*
(iii) $C_m(X)$ *is completely metrizable.*
(iv) $C_m(X)$ *is Čech-complete.*
(v) X *is pseudocompact.*

In this section our aim is to show that three other properties may be added to this list.

Definition 2.2. Let X be a space and $p \in X$. A *π-base for p* is a collection $\mathcal{B}$ of nonempty open sets such that for each neighborhood U of p there exists an $O \in \mathcal{B}$ such that $O \subseteq U$. The smallest cardinal of a π-base at p is called the *π-character at p* and is denoted by $\pi\chi(p, X)$. The supremum of all π-characters over points in p is called the *π-character of X* and is denoted by $\pi\chi(X)$. Observe that a base of neighborhoods is always a π-base, and so the π-character is never greater than the character of a space.

Lemma 2.3. *Suppose S is a countable discrete C-embedded subset of X. Any strictly positive function on S can be extended to an element of $U(X)^+$.*

Proof. Start off by selecting homeomorphism $h : (0, \infty) \to (0, 1)$. Now, if $f \in U(S)^+$, then $g = \frac{1}{h \circ f} \in U(S)^+$. Since S is a C-embedded subset of X there exists a continuous extension of g to all of X, say $g' \in C(X)$. Without loss of generality we can assume that $\mathbf{1} \leqslant g'$. It is now straightforward to check that the function $h^{-1} \circ \frac{1}{g'}$ belongs to $U(X)^+$ and is continuous extension of f. □

Theorem 2.4. *X is pseudocompact if and only if $\pi\chi(C_m(X)) = \aleph_0$.*

Proof. The necessity follows from Hewitt's Theorem. If X is pseudocompact, then $C_m(X)$ is first countable and so has a countable $\pi\chi$-character.

To show the sufficiency suppose that X is not pseudocompact and let $S = \{x_n\}$ be a countable discrete C-embedded subset of X. Let $\{B(f_n, d_n)\}_{n \in \mathbb{N}}$ be any countable collection of basic open subsets of $C_m(X)$. We will construct a unit $e \in U(X)^+$ such that $B(f_n, d_n) \nsubseteq B(\mathbf{0}, e)$ for each natural number n from which it will follow that the π-character of $C_m(X)$ is greater than $\aleph_0$.

First of all for each natural number n if $f_n(x_n) \neq 0$ then let

$$e_n = d_n \wedge \left|\frac{f_n(x_n)}{4}\right|.$$

Otherwise let $e_n = d_n$. Notice that $B(f_n, e_n) \subseteq B(f_n, d_n)$ for each n. We will show that $B(f_n, e_n) \nsubseteq B(\mathbf{0}, e)$ for each natural number n which is sufficient to prove our claim.

We define a function on the sequence $S = \{x_n\}$ as follows. Let $n \in \mathbb{N}$. If $f_n(x_n) = 0$, then let $e'(x_n)$ be any positive real number satisfying

$$0 < e'(x_n) < \frac{e_n(x_n)}{6}.$$

If $f_n(x_n) \neq 0$, then notice that $f_n(x_n) + \frac{e_n(x_n)}{4} \neq 0$, and so let $e'(x_n)$ satisfy

$$0 < e'(x_n) < \left|f_n(x_n) + \frac{e_n(x_n)}{4}\right|.$$

Since n is arbitrary this defines a function on S. Let e be any continuous extension of e' to a positive unit of X. Such an element exists by the previous lemma.

Now, $f_n + \frac{e_n}{4} \in B(f_n, e_n)$. However, by design $f_n + \frac{e_n}{4} \notin B(\mathbf{0}, e)$. To see this consider two cases. The first is that $f_n(x_n) = 0$. Then

$$0 < e(x_n) = e'(x_n) < \frac{e_n(x_n)}{6} < \frac{e_n(x_n)}{4} = \left|f_n(x_n) + \frac{e_n(x_n)}{4}\right|.$$

Next, if $f_n(x_n) \neq 0$, then

$$0 < e(x_n) = e'(x_n) < \left|f_n(x_n) + \frac{e_n(x_n)}{4}\right|.$$

We have shown that when X is not pseudocompact then any arbitrary countable collection of open sets $\{B(f_n, d_n)\}$ is not a π-base, and thus $\pi\chi(C_m(X)) > \aleph_0$. □

Recall that for an ordinal κ a map from κ into A is called a *κ-sequence in A*. A κ-sequence is often denoted by $\{x_\sigma\}_{\sigma<\kappa}$ where $x_\sigma \in A$. If $\{x_\sigma\}_{\sigma<\kappa}$ is a κ-sequence in a space X, then we say the *sequence converges to the point x* or, equivalently, that x is the *limit of the sequence* if for every neighborhood U of x there is a $\tau < \kappa$ such that $x_\sigma \in U$ for every $\sigma \geqslant \tau$.

Definition 2.5. A space X is called *radial* if whenever $A \subseteq X$ and $x \in \overline{A}$, then there is an ordinal κ and a κ-sequence $\{x_\sigma\}_{\sigma<\kappa}$ in A such that x is the limit of the sequence. All totally-ordered spaces are radial. This is an ordinal generalization of the Fréchet–Urysohn condition.

A space is called *pseudoradial* if whenever $A \subseteq X$ is not closed then there is an ordinal κ and a convergent κ-sequence $\{x_\sigma\}_{\sigma<\kappa}$ in A whose limit does not belong to A. It was proved in [4] that for $C_p(X)$ the conditions of being Fréchet–Urysohn, radial, and pseudoradial are all equivalent. We show that this holds for the m-topology as well. The proof of the next lemma is straightforward.

Lemma 2.6. *Suppose X is radial and $x \in \overline{A}$. Then the least ordinal κ for which there is a κ-sequence in A converging to x is a cardinal.*

Theorem 2.7. *The following are equivalent for a space X:*

(i) *X is pseudocompact.*
(ii) *$C_m(X)$ is radial.*
(iii) *$C_m(X)$ is pseudoradial.*

Proof. Clearly, by Theorem 2.1 (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Therefore, it suffices to suppose that $C_m(X)$ is pseudoradial and show X is pseudocompact. To that end let A be a nonclosed subset of $C_m(X)$. By hypothesis there is a cardinal κ and a κ-sequence in A, say $\{f_\sigma\}_{\sigma<\kappa}$, such that the sequence converges to some $f \notin A$. We claim that there is an $\aleph_0$-subsequence which converges to f. Once this is shown it will follow that $C_m(X)$ is a sequential space and hence countably tight. By Hewitt's Theorem X is pseudocompact.

For each natural number n we can choose an ordinal $\sigma_n < \kappa$ such that $\sigma_n > \sigma_{n-1}$ and that for every $\sigma_n < \tau < \kappa$, $f_\tau \in B(f, \frac{\mathbf{1}}{\mathbf{n}})$. The sequence $\{\sigma_n\}$ converges to κ. Otherwise there is an ordinal $\tau < \kappa$ such that $\sigma_n < \tau$ for each n, whence $f = f_\tau \in A$; a contradiction.

Next, for any $e \in U(X)^+$ there is an ordinal σ such that for every $\sigma < \tau < \kappa$ we have $f_\tau \in B(f, e)$. Now, there is a natural number n such that $\sigma < \sigma_m < \kappa$ for each $m \geqslant n$. This forces the convergence of the sequence $\{f_{\sigma_n}\}$ to f, and as mentioned before we obtain that X is pseudocompact. □

Now that we have shown that X is pseudocompact precisely when $C_m(X)$ is either radial, pseudoradial, or has countable π-character we conclude this section by showing that there is a general property that $C_m(X)$ has when X is not pseudocompact.

Proposition 2.8. *If X is not pseudocompact, then $C_m(X)$ is not locally compact.*

Proof. Suppose X is not pseudocompact and that, in order to obtain a contradiction, $C_m(X)$ is a locally compact space. Let U be a compact neighborhood of $\mathbf{0}$. There is some unit $u \in U(X)^+$ such that $B(\mathbf{0}, u) \subseteq U$. Since multiplication by a unit is a homeomorphism we can, without loss of generality, assume that $u = 1$. Consider the sequence of constant functions $S = \{\frac{\mathbf{1}}{\mathbf{n}}\}_{n\in\mathbb{N}}$. This set is discrete in the uniform topology whose closure is obtained by adjoining $\mathbf{0}$. It follows that S is a discrete subset of $C_m(X)$. Since X is not pseudocompact there is a discrete C-embedded sequence of distinct points, say $T = \{x_n\}_{n\in\mathbb{N}}$. By Lemma 2.3, there is an $e \in U(X)^+$ such that $e(x_n) = \frac{1}{n}$. Since $B(\mathbf{0}, e) \cap S = \emptyset$ it follows that S is a closed subset of $C_m(X)$ as well. Furthermore, since $S \subseteq U$ is closed it is compact forcing a contradiction. □

3. Weak P-spaces

The aim of this section is to classify when $C_m(X)$ is a weak P-space, that is, every countable subset is closed. In [3] the author gives a sufficient condition for $C_m(X)$ to be a weak P-space and for countable spaces he shows it is also necessary. One of the main theorems of this article shows that, in general, it is necessary.

The concept of a weak P-space is a generalization of the well-known class of P-spaces (see [5]) and was originally coined by Kunen [7]. Recall that for any cardinal κ, a space X is called a *P_κ-space* if the intersection of less than many κ open sets is again open. It is common to write P-space for $P_{\aleph_1}$-space. It is straightforward to show that a (Hausdorff) P_κ-space is a weak P_κ-space and there are several standard examples which show that the converse is not true. The class of weak P-spaces is not as well behaved as the class of P-spaces. For example, an infinite weak P-space may be pseudocompact or it may contain non-isolated G_δ-points. Every P-space is zero-dimensional, but the density topology on the real numbers is an example of a connected weak P-space.

Since the m-topology is finer than the uniform topology it follows that $\mathbf{0}$ remains a G_δ-point in $C_m(X)$. Thus, it is impossible for $C_m(X)$ to ever be a P-space. The fact that there are spaces for which $C_m(X)$ is a weak P-space is somewhat surprising as $C(X)$ is never a weak P-space when it is endowed with any of the other topologies mentioned above. Our main theorem generalizes Proposition 5.12 [3]. The main class of spaces we deal with are DRS-spaces and R-spaces. For completeness sake we define these concepts presently.

Definition 3.1. The sequence $\{O_n\}_{n\in\mathbb{N}}$ is said to be *disjoint* if $O_n \cap O_m = \emptyset$ whenever $n \neq m$. The sequence is called *discrete* if for every $x \in X$ there is some neighborhood of x which intersects at most one of the O_n.

A space is called a *Discrete Refining Sequence space* (or *DRS-space* for short) if for every sequence $\{O_n\}$ of nonempty, open subsets there is a discrete sequence $\{V_n\}$ of nonempty, open subsets such that $V_n \subseteq O_n$ for every n. Note that since we are assuming that our spaces are Tychonoff we could replace open set with cozeroset.

Some basic facts about a DRS-space X include that X is a *crowded space*, that is, a space with no isolated points, and that X is not pseudocompact. If X is a DRS-space, then so is every dense subspace and so is $X \times Y$ for any space Y. (The proof of these results is in [3].)

Proposition 3.2. *X is a DRS-space if and only if for each sequence of open subsets $\{O_n\}_{n\in\mathbb{N}}$ of X there exists a C-embedded, discrete subset $S = \{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in O_n$. Moreover, such a set is closed.*

Proof. If $\{O_n\}$ is a sequence of open sets and $\{V_n\}$ is a discrete sequence refining it, then we may find a function $f \in C(X)$ for which $f(x_n) = n$ for some $x_n \in V_n$. The set $S = \{x_n\}$ is then homeomorphic to the set of natural numbers and by Theorem 1.19 of [5], S is C-embedded in X. The converse is patent. □

Some other interesting facts about DRS-spaces follow.

Lemma 3.3. (See [3].) *Let X be a DRS-space. Then $\pi\chi(x, X) > \aleph_0$ for every $x \in X$. Moreover, if X is countable, then the converse is true.*

Lemma 3.4. *Let X be a DRS-space and U a nonempty, open subset. Then U is a DRS-space.*

Proposition 3.5. *Suppose X is a DRS-space. Then X is nowhere locally compact.*

Proof. Suppose by way of contradiction there is a compact subset of X, say K, with nonempty interior. Let O be the interior of K. Considering the sequence $\{O\}_{n\in\mathbb{N}}$ we can find a distinct sequence, say $\{x_n\}_{n\in\mathbb{N}}$, which is discrete, C-embedded and $\{x_n\}_{n\in\mathbb{N}} \subseteq O \subseteq K$. Thus there is an unbounded continuous function on K, a contradiction. □

Definition 3.6. For a cardinal κ, a space X is called an *R_κ-space* if there exists a base of open sets $\mathcal{U}$ such that $\bigcup \mathcal{A}$ is closed for each $\mathcal{A} \subseteq \mathcal{U}$ with $|\mathcal{A}| < \kappa$. Such a base will be called an *R_κ-base*. Observe that R_κ-spaces are necessarily zero-dimensional and that they are also weak P_κ-spaces. It is straightforward to show that every P_κ-space is an R_κ-space. As before when $\kappa = \aleph_1$ we call them R-spaces. In [3] it is shown that R-spaces are weak P-spaces. It is clear that a P-space is an R-space. We show that R_κ-spaces have similar properties to P_κ-spaces.

It is known that the product of two (weak) P_κ-spaces is again a (weak) P_κ-space. The same holds for R_κ-spaces as our next result demonstrates.

Proposition 3.7. *Suppose X and Y are R_κ-spaces. Then so is $X \times Y$. Conversely, if $X \times Y$ is an R_κ-space, then so are X and Y.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be R_κ-bases of X and Y, respectively. Consider

$$\mathcal{B} = \{O \times P\colon\ O \in \mathcal{B}_1,\ P \in \mathcal{B}_2\}.$$

Clearly, this is a base of clopen sets for $X \times Y$. We proceed to show that for any $T \subseteq \mathcal{B}$ with $|T| < \kappa$, $\bigcup T$ is clopen. Enumerate T as a τ-sequence for some cardinal $\tau < \kappa$, say $T = \{O_\alpha \times P_\alpha\}_{\alpha<\tau}$. Observe that $\bigcup T \subseteq (\bigcup_{\alpha<\tau} O_\alpha) \times (\bigcup_{\alpha<\tau} P_\alpha)$, where the latter set is clopen. Our aim is to show that no element of the complement of these two sets lies in the closure of $\bigcup T$. To that end let $(x, y) \in (\bigcup_{\alpha<\tau} O_\alpha) \times (\bigcup_{\alpha<\tau} P_\alpha) \setminus \bigcup T$.

Set $I = \{\alpha < \tau\colon\ x \notin O_\alpha\}$ and $J = \{\alpha < \tau\colon\ y \notin P_\alpha\}$ and observe that $I \cup J = \tau$. Let α, β be such that $x \in O_\alpha \setminus O_\beta$, $y \in P_\beta \setminus P_\alpha$.

Now,

$$(x, y) \in \left(O_\alpha \setminus \left(\bigcup_{\sigma\in I} O_\sigma\right)\right) \times \left(P_\beta \setminus \left(\bigcup_{\sigma\in J} P_\sigma\right)\right),$$

where the right-hand side is a clopen set (by hypothesis). If (x, y) belongs to the closure of $\bigcup T$, then there is some $(w, z) \in (O_\alpha \setminus (\bigcup_{\sigma\in I} O_\sigma)) \times (P_\beta \setminus (\bigcup_{\sigma\in J} P_\sigma)) \cap O_\gamma \times P_\gamma$ for some $\gamma < \tau$. Since $I \cup J = \tau$ it follows that either $\gamma \in I$ or $\gamma \in J$. In the first case we obtain that $w \notin O_\alpha \setminus (\bigcup_{\sigma\in I} O_\sigma)$ and in the second case we have $z \notin P_\beta \setminus (\bigcup_{\sigma\in J} P_\sigma)$. Either case leads to a contradiction and therefore we have that $\bigcup T$ is clopen, whence $X \times Y$ is an R_κ-space.

For the converse note that a subspace of an R_κ-space is again an R_κ-space. It follows that if $X \times Y$ is an R_κ-space then so are X and Y. □

Corollary 3.8. *X is an R_κ-space if and only if $X \times X$ is an R_κ-space.*

Proposition 3.9. *If X is an R_κ-space, then every subset of size less than κ is C-embedded. Hence, if X is an R-space, then X is a weak P-space and every countable set is C-embedded.*

Proof. The second statement clearly follows from the first. Let X be an R_κ-space and $\mathcal{U}$ an R_κ-base. For any subset $S \subseteq X$ of size less than κ we use the fact that X is a weak P_κ-space to obtain S is a discrete closed subset, and thus for each $s \in S$ we may find an open neighborhood $O_s \in \mathcal{U}$ of s which contains no other point of S. Since each element of $\mathcal{U}$ is clopen it follows that X is a topological sum of the O_s (and the complement of the union of them). It is patent to now show that S is C-embedded in X. □

Corollary 3.10. *A pseudocompact R-space is finite.*

Definition 3.11. A space is said to satisfy (γ) whenever it has the property that every countable subset is C-embedded. The subclass of weak P-spaces which satisfy (γ) is an interesting class of spaces and has been shown to be useful in the area of free topological groups (see [8]).

Theorem 3.12. *Let X be a weak P-space satisfying (γ). If X is crowded, then X is a DRS-space. In particular, every crowded R-space is a DRS-space.*

Proof. This is clear as given any sequence of open sets $\{O_n\}$ we can inductively select a sequence $\{x_n\}$ such that $x_n \in O_n$. The hypothesis together with Proposition 3.2 does the rest. □

Remark 3.13. There exist crowded, zero-dimensional weak P-spaces which are not *DRS*-spaces (and hence not R-spaces). For example, the set of weak P-points of $\beta\mathbb{N} \setminus \mathbb{N}$ is a pseudocompact, zero-dimensional weak P-space (see [1] or [9]).

We are ready for our main theorem.

Theorem 3.14. *Let X be a Tychonoff space.*

(i) *X is a DRS-space.*
(ii) *$C_m(X)$ is an R-space.*
(iii) *$C_m(X)$ is a weak P-space satisfying (γ).*
(iv) *$C_m(X)$ is a weak P-space.*

Proof. As previously mentioned it is shown in [3, Theorem 5.12] that (i) ⇒ (ii) ⇒ (iv). Therefore, since (ii) ⇒ (iii) ⇒ (iv) we need only demonstrate the direction (iv) ⇒ (i).

Suppose that $C_m(X)$ is a weak P-space. X cannot have any isolated points. Now, let $\{O_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty, open subsets of X. Without loss of generality we assume that

$$O_n = \operatorname{coz}(f_n)$$

for some $f_n \in C(X)$ satisfying $0 < f_n \leqslant \frac{\mathbf{1}}{\mathbf{n+1}}$. Our hypothesis allows us to choose a unit $e \in U(X)^+$ for which

$$B(\mathbf{0}, e) \cap \{f_n\}_{n\in\mathbb{N}} = \emptyset.$$

In other words for each natural number n there is an $x_n \in O_n$ for which $e(x_n) < f_n(x_n)$. We may without much effort suppose that $x_n \neq x_m$ whenever $n \neq m$. Set $S = \{x_n\}$. We claim that S is a closed, discrete subset of X. To see that S is closed observe that if x is an accumulation point of S then since $e(x_n) \leqslant \frac{1}{n+1}$ it follows by continuity that $e(x) = 0$, contradicting that e is a unit.

Now since e is a unit set $g = e^{-1}$ and observe that $\{g(x_n)\}_{n\in\mathbb{N}} \cap (m, m + \frac{3}{2})$ is a finite subset for each natural number m. So S is C-embedded. □

Corollary 3.15. *If $Y \subseteq X$ is a dense C-embedded subspace, then X is a DRS-space if and only if Y is a DRS-space.*

Remark 3.16. Putting Theorems 3.12 and 3.14 together we see that if X is a DRS-space, then $C_m(X)$ is a crowded R-space and hence a DRS-space. The converse is easily seen to be false. Notice that if X is the topological sum of two spaces Y and Z, then $C_m(X)$ and $C_m(Y) \times C_m(Z)$ are homeomorphic. Thus, if we let Y be a DRS-space and Z be any space which is not a DRS-space, then $C_m(X)$ is a DRS-space yet X is not.

4. Convergent sequences in $C_m(X)$

Recall that if X is a DRS-space, then no countable subset of $C_m(X)$ has an accumulation point. In this section we consider a weaker property. Namely, we are interested in determining when $C_m(X)$ has no convergent sequences. At the end of this section we supply an example showing that this property is in fact weaker than $C_m(X)$ being a weak P-space.

Definition 4.1. We call a space X an *almost DRS-space* if for every sequence of nonempty open subsets of X, say $\{O_n\}$, there exists an infinite subset $T \subseteq \mathbb{N}$ and a discrete sequence of nonempty open sets $\{V_n\}_{n\in T}$ such that $V_n \subseteq O_n$ for each $N \in T$. It is again easily seen that an almost DRS-space has no isolated points and is not pseudocompact. Since the proofs of Lemma 3.4 and Proposition 3.5 can be slightly modified for almost DRS-spaces we obtain that an almost DRS-space is nowhere locally compact.

Lemma 4.2. *X is an almost DRS-space if and only if for every sequence of open sets $\{O_n\}_{n\in\mathbb{N}}$ there exists an infinite subset T of $\mathbb{N}$ and a sequence of distinct points of X, say $S = \{x_n\}_{n\in T}$ such that $x_n \in O_n$ for each $n \in T$ and S is a discrete C-embedded subset of X.*

Proposition 4.3. *Let X be an almost DRS-space. Then for every $x \in X$, $\chi(x, X) > \aleph_0$.*

Remark 4.4. In our example below we will show that an almost DRS-space can have countable π-character.

Theorem 4.5. *The following are equivalent for a Tychonoff space X:*

(i) *X is an almost DRS-space.*
(ii) *$C_m(X)$ has no nontrivial convergent sequences.*
(iii) *The only compact subspaces of $C_m(X)$ are the finite ones.*

Proof. Suppose by means of contradiction that X is an almost DRS-space yet there is a nontrivial convergent sequence, say $\{f_n\}_{n\in\mathbb{N}}$, in $C_m(X)$. By translating and taking absolute values, we can assume that $f_n \to \mathbf{0}$ and that $\mathbf{0} \leqslant f_n$ for each $n \in \mathbb{N}$. Set $O_n = \operatorname{coz}(f_n)$ and by hypothesis choose a countable subset $T \subseteq \mathbb{N}$ and a set $S = \{x_n\}_{n\in T}$ such that $x_n \in O_n$ and S is discrete and C-embedded in X. Let $e \in U(X)^+$ so that $e(x_n) = \frac{1}{2} f_n(x)$. It follows that for every $n \in T$, $f_n \notin B(\mathbf{0}, e)$. Therefore, the sequence $\{f_n\}$ cannot converge to $\mathbf{0}$ which demonstrates that (i) implies (ii).

Next, let $\{O_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty open subsets of X. Without loss of generality, we assume that $O_n = \operatorname{coz}(f_n)$ for each natural number n where $f_n \leqslant \frac{\mathbf{1}}{\mathbf{n}}$. By hypothesis we know that the sequence $\{f_n\}$ does not converge in $C_m(X)$; in particular, they do not converge to $\mathbf{0}$. By definition this means there is some $e \in U(X)^+$ for which $B(\mathbf{0}, e)$ is disjoint to some subsequence $\{f_{n_k}\}$. What this means is that for each natural number k there is some $x_{n_k} \in O_{n_k}$ such that $0 < e(x_{n_k}) < f_{n_k}(x_{n_k}) \leqslant \frac{1}{n_k}$. It easy to show that we can take the x_{n_k} to be distinct and hence $S = \{x_{n_k}\}$ is a discrete and C-embedded subset of X. This shows that (ii) implies (i).

Clearly, since a convergent sequence is compact it follows that (iii) implies (ii). So suppose that A is a compact subspace of $C_m(X)$. If A is infinite then we can choose a countable discrete subspace of A, say S, and let $f \in \overline{S} \setminus S$. Since translation by an element is a homeomorphism it follows that without loss of generality $f = \mathbf{0} \in \overline{S} \setminus S$. Therefore, for each $n \in \mathbb{N}$ there is an $f_n \in S$ such that $f_n \in B(\mathbf{0}, \frac{1}{n})$. We claim that $f_n \to \mathbf{0}$.

First of all, observe that the closure of the set $\{f_n\}$ is a compact subspace and since $\{f_n\}$ is a discrete subspace there exists a function in the closure. But the f_n converge to $\mathbf{0}$ pointwise, and so

$$\overline{\{f_n\}} = \{f_n\} \cup \{\mathbf{0}\}.$$

The fact that this set is compact and that the f_n form a discrete subspace implies that $f_n \to \mathbf{0}$. □

Example 4.6. We now construct an example of an almost DRS-space which is not a DRS-space. Moreover, the space will be countable and will have countable π-character. Consider E the absolute of the space $[0, 1]$ (see [10]). It is known that E is a crowded extremally disconnected space and has countable π-weight. Therefore, E is separable. Let X be a countable dense subset of E. The density of X in E implies that X is crowded and has countable π-weight. We show that X is an almost DRS-space. To that end let $\{O_n\}$ be a collection of nonempty open subsets of X. It is straightforward to check that there is a subsequence T of $\mathbb{N}$ and a sequence $S = \{x_n\}_{n \in T}$ with $x_n \in O_n$ and S discrete. Our goal is to show that S contains an infinite subset which is closed. Since X is countable and hence normal this set will also be C-embedded.

Let $\mathcal{A}$ be an uncountable collection of infinite subsets of S which are pairwise almost disjoint (see 12B, [5]). Since X is extremally disconnected and S is discrete it follows that S is a C^*-embedded subset of X. Therefore, the collection of sets $\{\mathrm{cl}_X A \setminus S\}_{A \in \mathcal{A}}$ is pairwise disjoint and so one must be empty, say $A \in \mathcal{A}$. It follows that A is a desired infinite closed (and hence C-embedded) subset of S.

Remark 4.7. We would like to thank the referee for supplying us with the space in Example 4.6. We were able to construct an example of a countable almost DRS-space containing exactly one point of countable π-character. The above example is a much more elegant example of an almost DRS-space demonstrating that Proposition 4.3 cannot be strengthened.

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