Local Types of Classical Rings

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Abstract Motivated by recent results on commutative rings with zerodivisors ([2], [11]), we investigate the difference between the three notions of locally classical, maximally classical, and classical rings. Motivated also by results in [12], we explore these notions when restricted to certain subsets of the prime spectrum of the ring. As an application, we examine the case of locally classical rings of continuous functions, the case of maximally classical and classical rings having already been considered ([1], [14]).

1 Introduction and Main Results

Throughout, we shall assume that R denotes a commutative ring with identity. We denote the classical ring of quotients (also known as the total quotient ring) of R by $q(R)$. When a ring equals its classical ring of quotients, the ring is said to be *classical*. Classical rings are characterized by the simple condition that all regular elements are units. For any ring R, $q(R)$ is a classical ring.

A ring-theoretic property is said to be *local* if an arbitrary ring R satisfies the property if and only if the localization R_P satisfies the property for every prime ideal P of R. For example, “reduced” is a well-known local property, where a ring is *reduced* if it has no nonzero nilpotent elements. As a second example, we note the useful fact that “regular” is also a local property, which we show in the following lemma.

Lemma 1. *For an element $r \in R$, the following statements are equivalent.*

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1. *r is regular.*
2. *$\frac{r}{1}$ is regular in R_P for every prime ideal $P \subset R$.*
3. *$\frac{r}{1}$ is regular in R_M for every maximal ideal $M \subset R$.*

Proof. For (1) implies (2), if $\frac{r}{1} \cdot \frac{a}{s} = 0$ in R_P, then $ra \cdot t = 0$ for some $t \in R \setminus P$; if r is regular, then $at = 0$ in R forces $\frac{a}{s} = 0$ in R_P. (2) implies (3) is trivial. For (3) implies (1), if $ra = 0$ for some $a \in R$, then $\frac{r}{1} \cdot \frac{a}{1} = 0$ and $\frac{r}{1}$ regular in R_M for every maximal ideal $M \subset R$ implies that the annihilator $\mathrm{Ann}_R(a)$ of a in R is contained in no maximal ideal of R; that is, R annihilates a, so that $a = 0$.

The goal of this paper is to investigate the extent to which "classical" is a local property. The first motivation comes from [2] and [11], in which it is shown that "Prüfer " is not a local property. (Recall that a ring R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible. Note that a classical ring is Prüfer, because it has no proper regular ideals.) The ring R is said to be *locally* (respectively, *maximally*) *Prüfer* if the localization R_P is a Prüfer ring for every prime (respectively, maximal) ideal $P \subset R$. Boynton showed in [2] that locally Prüfer implies Prüfer but not conversely. Klingler, Lucas, and Sharma showed in [11] that maximally Prüfer implies Prüfer but not conversely, and that maximally Prüfer does not imply locally Prüfer. These results are summarized by the diagram

$$\text{locally Prüfer} \Rightarrow \text{maximally Prüfer} \Rightarrow \text{Prüfer}$$

in which neither arrow is reversible. Moreover, examples show that these implications are not reversible even under the extra hypothesis that the ring is reduced.

In this paper we establish the corresponding results for "classical." We shall say that a ring R is *locally* (respectively, *maximally*) *classical* if the localization R_P is a classical ring for every prime (respectively, maximal) ideal $P \subset R$. Clearly, locally classical implies maximally classical; our main theorem of this section establishes the second implication.

Theorem 1. *If R is maximally classical, then it is classical.*

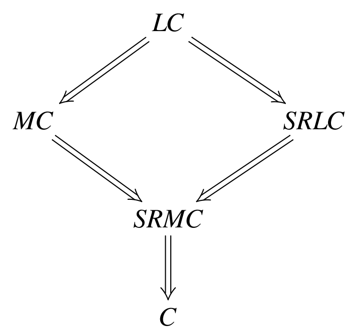
Proof. It is easier to prove the contrapositive, so suppose that R is not classical, say $r \in R$ is regular but not a unit. Then there is a maximal ideal $M \subset R$ such that $r \in M$, so $\frac{r}{1}$ is not a unit in the localization R_M. By Lemma 1, $\frac{r}{1}$ is regular, so R_M is not classical, and hence R is not maximally classical.

Examples 2 and 3 below show that these two implications are not reversible, even under the additional hypothesis that the ring is reduced. That is, Example 2 gives a (reduced) ring R which is classical but not maximally classical, and Example 3 gives a (reduced) ring R which is maximally classical but not locally classical.

The second motivation for this paper comes from [12], in which the authors considered restricting the Prüfer property locally to only the regular or semiregular, prime or maximal ideals. (Recall that an ideal is called *semiregular* if it contains a finitely generated dense ideal, that is, a finitely generated ideal whose annihilator is zero.) Restricting the locally (or maximally) classical property to regular prime (or maximal) ideals would be pointless, however, since by Lemma 1, a regular nonunit in a prime ideal will remain a regular nonunit in the localization at that prime ideal. Thus, "regular locally classical" and "regular maximally classical" are equivalent to classical; that is, a ring is classical if and only if it has no regular prime ideals if and only if it has no regular maximal ideals. On the other hand, restricting the locally (or maximally) classical property to semiregular prime (or maximal) ideals does produce a weaker condition, as we shall show. Therefore, we shall say that a ring R is *semiregular locally* (respectively *semiregular maximally*) *classical* if the localization R_P is a classical ring for every semiregular prime (respectively semiregular maximal) ideal $P \subset R$.

Obviously locally classical implies semiregular locally classical, and semiregular locally classical and maximally classical each implies semiregular maximally classical. Moreover, a regular element generates a finitely generated ideal with zero annihilator, so the proof of Theorem 1 shows that, if the ring R is not classical, then it is not semiregular maximally classical. Therefore, semiregular maximally classical implies classical, and we obtain the following diagram of implications.

Diagram 2



where LC abbreviates locally classical, SRLC abbreviates semiregular locally classical, etc. Examples 5 and 6 below (together with the examples already mentioned) show that none of the arrows is reversible, and indeed, that there are no other implications than those implied by the diagram, even under the added hypothesis that the ring is reduced.

The third motivation for this paper comes from the theory of rings of (real-valued) continuous functions. For a topological space X, let $C(X)$ denote the set of continuous functions from the space X to the real field $\mathbb{R}$; $C(X)$ is a commutative ring with identity under pointwise addition and multiplication. One area of particular research interest has been determining conditions on the space X equivalent to some desired property of the ring $C(X)$. (See [8] as a good general reference for the theory of rings of continuous functions.) For example, Levy [14] gave necessary and sufficient conditions on X that $C(X)$ be classical, and Banerjee, Ghosh, and Henriksen [1] gave necessary and sufficient conditions on X that $C(X)$ be maximally classical. We devote section 2 of this paper to reviewing the necessary terminology for these results from the theory of rings of continuous functions, and to constructing two examples to sharpen Theorem 1. We also characterize the topological spaces X for which the ring $C(X)$ is locally classical (Theorem 7).

Note that the ring $C(X)$ has *Property A*, that is, every semiregular ideal of $C(X)$ is regular. This follows from the fact that in $C(X)$, the annihilator of f, g equals the annihilator of f^2+g^2. Thus, for $C(X)$, semiregular locally (respectively maximally) classical is equivalent to regular locally (respectively maximally) classical, both of which, as noted above, are equivalent to classical. Therefore, to establish the claim made about the completeness of the implications in diagram (2), we need to look for examples beyond rings of continuous functions. This we do in section 3, where we also collect some miscellaneous results on classical rings.

We finish the current section by developing a useful condition on the prime ideal P which, for a reduced ring R, is equivalent to the localization R_P being classical.

For prime ideal $P \subset R$, we denote by $O(P)$ the set of elements of R annihilated by an element of $R \setminus P$:

$$O(P) = \{a \in R : \text{ there is an } x \in R \setminus P \text{ such that } ax = 0\}$$

Note that $O(P) \subseteq P$; we easily obtain the following alternative description of R_P, since $O(P)$ is the kernel of the natural map from R to the localization R_P. (See Section 4 of [9] for details.)

Proposition 1. *Let P be a prime ideal of R, and set $\bar{R} = R/O(P)$ and $\bar{P} = P/O(P)$. Then $\bar{R}_{\bar{P}} \cong R_P$.*

If the ring R is reduced, the minimal prime ideals of R play a crucial role in determining whether or not R is locally or maximally classical. The following characterization of minimal primes in reduced rings will prove useful.

Lemma 2. *[10, Corollary 2.2] Let R be a reduced ring and suppose P is a prime ideal of R. Then P is a minimal prime ideal if and only if for each $a \in P$ there exists an element $x \in R \setminus P$ such that $ax = 0$.*

For a reduced ring R and prime ideal $P \subset R$, the following proposition gives an elegant description of the ideal $O(P)$.

Proposition 2. *If R is a reduced ring and P is a prime ideal of R, then*

$$O(P) = \cap\{Q \subseteq P : Q \text{ is a minimal prime}\},$$

the intersection of the minimal primes of R contained in P.

Proof. Let $a \in O(P)$ and Q be a minimal prime contained in P. By definition, there is an element $x \in R \setminus P$ such that $ax = 0$. Since $Q \subseteq P$, it follows that $x \notin Q$, and hence $a \in Q$. This demonstrates one containment.

For the opposite containment, if a is in all of the minimal primes contained in P, then $\frac{a}{1}$ is in all of the minimal primes of the localization R_P, so in the nilradical of R_P. Since R is assumed to be reduced, R_P is also reduced (as noted above), and hence $\frac{a}{1} = 0$ in R_P. This implies that the annihilator of a is not contained in P, so $a \in O(P)$.

For a reduced ring R and prime ideal $P \subset R$, we can now determine conditions on P which guarantee that the localization R_P is classical.

Theorem 3. *Let R be a reduced ring and P be a prime ideal of R. The following statements are equivalent.*

1. *The localization R_P is classical.*
2. *For every $a \in P$, the annihilator of a in R is not contained in $O(P)$.*
3. *$P = \cup\{Q \subseteq P : Q$ is a minimal prime$\}$, the union of the minimal primes of R contained in P.*

Proof. (1) implies (2). Let $a \in P$. If $a \in O(P)$, then by definition there is an element $x \in R \setminus P$ such that $ax = 0$, so the annihilator of a is not contained in P, and so not contained in $O(P)$. Assume instead that $a \in P \setminus O(P)$. By hypothesis, the localization R_P is classical, so $\frac{a}{1}$ must be a zerodivisor, and hence there is a nonzero $\frac{b}{s} \in R_P$ such that $\frac{a}{1} \cdot \frac{b}{s} = 0$ in R_P. This means that there is some $t \notin P$ such that $abt = 0$. If $bt \in O(P)$, then $btx = 0$ for some $x \in R \setminus P$, but then $tx \in R \setminus P$ would imply $\frac{b}{s} = 0$ in R_P, contrary to assumption. Therefore, $abt = 0$ with $bt \notin O(P)$, as required.

(2) implies (3). Clearly, $\cup\{Q \subseteq P : Q$ is a minimal prime$\} \subseteq P$. Conversely, suppose $a \in P$, so that, by hypothesis, $ab = 0$ for some $b \notin O(P)$. By Proposition 2, there is a minimal prime $Q \subseteq P$ such that $b \notin Q$. Then $ab = 0$ and Q prime implies $a \in Q$, proving the opposite containment.

(3) implies (1). Let $\frac{a}{s} \in PR_P$, so that $a \in P$. By hypothesis, $a \in Q$ for some minimal prime $Q \subseteq P$. By Lemma 2, there is an element $x \notin Q$ such that $ax = 0$. By Proposition 2, $x \notin O(P)$, so $\frac{x}{1}$ is a nonzero element of R_P, whence $\frac{a}{s}$ is a zerodivisor of R_P.

Quantifying over all prime or all maximal ideals of the reduced ring R, Theorem 3 yields criteria for R to be locally or maximally classical.

Corollary 1. *If R is reduced, then:*

1. *R is locally classical if and only if every prime ideal is a union of minimal prime ideals.*
2. *R is maximally classical if and only if every maximal ideal is a union of minimal prime ideals.*

2 Rings of Continuous Functions

We recall a useful classification of local Prüfer rings. First, recall that a ring R is called a *Bézout ring* if every finitely generated ideal is principally generated. In the weaker case that every finitely generated regular ideal is principally generated we say R is *quasi-Bézout*. Observe that a quasi-Bézout ring is a Prüfer ring. Of course, there are Prüfer domains which are not Bézout (and hence not quasi-Bézout). Theorem 2 of [15] states that for rings of continuous functions the notions of quasi-Bézout and Prüfer are equivalent. It is also the case that these conditions are equivalent for local rings. We let $Z(R)$ denote the set of zerodivisors of the ring R.

Theorem 4. *[15, Proposition] Let R be a local ring. The following statements are equivalent.*

1. *R is a Prüfer ring.*
2. *R is a quasi-Bézout ring.*
3. *$Z(R)$ is an (prime) ideal of R and $R/Z(R)$ is a valuation domain.*

Remark 1. Recall the domain is a valuation domain if and only if its ideals are linearly-ordered, i.e. the domain is a chained ring. When the ring has zerodivisors the distinction between valuation rings and chained rings becomes slightly more delicate. For more information on this we suggest the reader peruse [13].

Next, we give a brief account of the theory of rings of continuous functions. For a topological space X, we let $C(X)$ denote the ring of real-valued continuous functions on X. The subring of bounded continuous functions on X is denoted by $C^*(X)$. We shall assume that X is a Tychonoff space, that is, completely regular and Hausdorff.

Next, recall the following subsets of X that are useful in describing algebraic properties of $C(X)$. For $f \in C(X)$, we denote its *zeroset* by $Z(f) = \{x \in X : f(x) = 0\}$. The set-theoretic complement of $Z(f)$ in X is denoted by $\mathrm{coz}(f)$ and is called the *cozeroset* of f. A subset $V \subseteq X$ is called a zeroset (respectively, cozeroset) if there is some $f \in C(X)$ such that $V = Z(f)$ (respectively, $V = \mathrm{coz}(f)$). We shall use $\mathrm{cl}_X V$ and $\mathrm{int}_X V$ to denote the closure and interior of V in X, respectively. We shall also feel free to drop the subscripts on these operators when it is clear which space is being discussed.

For a Tychonoff space X, the Stone-Čech compactification of X, denoted βX, is the unique compact space (up to homeomorphism) containing X densely and C^*-embedded. Recall that a subspace Y of a space X is said to be *C^*-embedded* in X if every bounded continuous function on Y has a continuous extension to X. Our main reference for $C(X)$ is [8].

For $p \in \beta X$ we form two ideals of $C(X)$:

$$M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$$

and

$$\begin{aligned} O^p &= \{f \in C(X) : \mathrm{cl}_{\beta X} Z(f) \text{ is a neighborhood of } p\} \\ &= \{f \in C(X) : \text{ there is a } \beta X\text{-neighborhood } V \text{ of } p \text{ such that } V \cap X \subseteq Z(f)\}. \end{aligned}$$

It is known that each M^p is a maximal ideal of $C(X)$ and that every maximal ideal of $C(X)$ is of the form M^p for some (unique) $p \in \beta X$; this is known as the Gelfand-Kolmogoroff Theorem. The ring $C(X)$ is a *pm*-ring, that is, every prime ideal is contained in a unique maximal ideal. Furthermore, for any prime $P \in \mathrm{Spec}(C(X))$ the set of prime ideals containing P forms a chain, i.e. $\mathrm{Spec}(C(X))$ is a root system. Since O^p is a radical ideal, it follows that O^p is the intersection of the minimal prime ideals contained in M^p. (When $p \in X$ we instead write M_p and O_p and notice that $M_p = \{f \in C(X) : f(p) = 0\}$ and $O_p = \{f \in C(X) : p \in \mathrm{int}_X Z(f)\}$.)

Chronologically, Gilman and Henriksen [6] characterized when $C(X)$ is a von Neumann regular ring, calling such a space X a *P-space*. It is known that X is a P-space if and only if the topology of open sets is closed under countable intersections if and only if every zeroset is open. Next, the authors classified in [7] when $C(X)$ is a Bézout ring, calling such a space an *F-space*. X is an F-space if and only if every cozeroset is C^*-embedded if and only if $C(X)$ is an arithmetical ring. Then, in [4], the authors classified when $C(X)$ is quasi-Bézout calling such a space X a *quasi F-space*. X is a quasi F-space if and only if every dense cozeroset is C^*-embedded. In [15] the authors proved that for $C(X)$ (and in a more general situation), $C(X)$ is a quasi-Bézout ring if and only if $C(X)$ is a Prüfer ring. Formally:

Theorem 5. *[15, Theorem 2][4, Theorem 5.1] For a space X the following statements are equivalent.*

1. *$C(X)$ is a Prüfer ring.*
2. *Every dense cozeroset of X is C^*-embedded, that is, X is a quasi F-space.*
3. *βX is a quasi F-space.*
4. *$C^*(X)$ is a Prüfer ring.*

Levy [14] called a space X an *almost P-space* if the interior of every nonempty zeroset is nonempty. It follows that every P-space is an almost P-space. It also follows that in an almost P-space there are no non-trivial dense cozerosets of X. Since $f \in C(X)$ is regular precisely when $\mathrm{coz}(f)$ is dense we obtain:

Theorem 6. *The space X is an almost P-space if and only if $C(X)$ is a classical ring. In particular, an almost P-space is a quasi F-space.*

Example 1. The one-point compactification of an uncountable discrete space, αD, is an example of an almost P-space. If X is locally compact and real compact but not compact, then $\beta X \setminus X$ is an almost P-space ([5, Lemma 3.1]).

The question of when $C(X)_M$ is classical for every maximal ideal was addressed in [1], though not in these terms. The authors call X a *UMP-space* (pronounced U - M - P - space) if every maximal ideal of $C(X)$ is the union of minimal prime ideals. An application of Theorem 3 yields the following result.

Corollary 2. *[1, Theorem 2.2] The space X is a UMP-space if and only if $C(X)$ is maximally classical.*

Example 2. It is pointed out in [1] that a UMP-space is an almost P-space. This also follows from the fact that a maximally classical ring is classical. In Observation 1.6 of [1] it is pointed out that the space $\beta\mathbb{N} \setminus \mathbb{N}$ is an example of an almost P-space which is not a UMP-space. Therefore, $C(\beta\mathbb{N} \setminus \mathbb{N})$ is classical but not maximally classical.

Example 3. Let D be an uncountable discrete space and let αD denote its one-point compactification. The ring $C(\alpha D)$ is a maximally classical ring, equivalently, X is a UMP-space (see [1, Example 1.8]). However, $C(\alpha D)$ is not a locally classical ring. In $C(\alpha D)$ every non-maximal prime ideal (necessarily lying beneath M_α) has a unique minimal prime ideal beneath it ([3, Proposition 3]). Therefore, if P is any non-maximal prime, then $O(P)$ is a prime ideal. Thus, $C(X)_P$ is a domain. To be classical we would need $C(X)_P$ to be a field, which occurs precisely when $P = O(P)$, i.e. P is a minimal prime ideal. Since there are primes which are both non-maximal and non-minimal prime ideals of $C(\alpha D)$ it follows that $C(X)$ is not locally classical.

A recap is in order. $C(X)$ is classical if and only if X is an almost P-space, and $C(X)$ is maximally classical if and only if X is a UMP-space. We now come to the main theorem in this section, characterizing when $C(X)$ is locally classical.

Theorem 7. *Let X be a space. The following statements are equivalent.*

1. *The ring $C(X)$ is a locally classical ring.*
2. *The ring $C(X)$ is a von Neumann regular ring.*
3. *X is a P-space.*

Before we supply a proof we recall a needed definition. Recall that an ideal $I \subset C(X)$ is called a *z-ideal* if $f \in I$ and $Z(f) = Z(g)$ implies that $g \in I$. For example, each maximal ideal and each minimal prime ideal of $C(X)$ is a z-ideal.

Lemma 3. *If $\{I_\sigma\}_{\sigma\in\tau}$ is a collection of z-ideals such that the union $I = \cup_{\sigma\in\tau} I_\sigma$ forms an ideal, then I is a z-ideal.*

Proof. Let $f \in I$ and $Z(f) = Z(g)$. By hypothesis there is a $\sigma \in \tau$ such that $f \in I_\sigma$. Since I_σ is a z-ideal it follows that $g \in I_\sigma$, whence $g \in I$.

Remark 2. Lemma 3 can be generalized to the join of z-ideals being a z-ideal using [8, Lemma 14.8]. However, we do not need the full version here, and so we provided a proof for completeness sake.

We can now prove Theorem 7.

Proof. That (2) and (3) are equivalent has already been pointed out. If $C(X)$ is von Neumann regular, then it is locally a field, hence locally classical, so (2) implies (1).

For (1) implies (2), suppose that $C(X)$ is locally classical; we show that each point in X is a P-point. Let $p \in X$. If $O_p \subsetneq M_p$, then it is well-known that there exists a prime ideal P beneath M_p which is not a z-ideal ([8, Section 14.13]). Since $C(X)_P$ is classical, by Proposition 3 we know that P is the union of the minimal prime ideals beneath it. But minimal prime ideals are z-ideals, so it follows from Lemma 3 that P is a z-ideal, contradiction. Therefore, $O_p = M_p$, and hence p is a P-point. Consequently, X is a P-space.

For reduced rings, being von Neumann regular is equivalent to being zero-dimensional, so Theorem 7 means that a ring $C(X)$ is locally classical if and only if it is zero-dimensional. The next example shows that, for rings in general, this is not the case, even for reduced rings.

Example 4. Let k be a field and $D = k[[x]]$ a power series ring in one indeterminate over k; so D is a discrete valuation domain with unique maximal ideal $M = xk[[x]]$. Set $B = \oplus_{n\in\mathbb{N}} M$, the direct sum of countably many copies of M. One can define a ring structure on the cartesian product $R = D \times B$ using coordinatewise addition, and multiplication defined by $(a,b)(c,d) = (ac, ad+bc+bd)$; in [12], R is called a *ring of form* $A+ZB[[Z]]$. By [12, Theorem 3.7 (8)], R is a local ring of Krull dimension 1, and by [12, Theorem 3.7 (1) and (2)], R is a classical ring. One easily checks that R is reduced, so R is locally a field at all minimal primes and hence at all non-maximal primes. It follows that R is a locally classical, reduced ring with Krull dimension equal to 1.

3 Additional Examples and Further Results

The ring $R = C(\beta\mathbb{N} \setminus \mathbb{N})$ of Example 2 is classical but not maximally classical. As noted in the introduction, since R has property A, it is also semiregular locally (and hence semiregular maximally) classical. Thus, in the notation of diagram (2), no member of {C, SRMC, SRLC} implies a member of {MC, LC}. Similarly, the ring $R = C(\alpha D)$ of Example 3 is maximally classical but not locally classical; that is, MC does not imply LC. Moreover, both examples are reduced, so none of these implications holds even under the additional assumption that the ring is reduced.

To complete the claim following diagram (2) that no implications hold other than those implied by the diagram, we give additional (reduced) examples showing that C implies neither SRMC nor SRLC (Example 5), and that neither MC nor SRMC implies SRLC (Example 6). It is then straightforward to verify that the only necessary implications are the (downward) directed paths in diagram (2).

Example 5. The ring R of the form $A+B$ in [12, Example 3.5] is classical ($R = q(R)$, the total quotient ring of R), while the maximal ideal $N+B$ of R is semiregular, and $R_{N+B} = D$ is an integral domain but not a field. Therefore, R is classical but not semiregular maximally classical, and hence (in the notation of diagram (2)), C implies neither SRMC nor SRLC. Moreover, the ring R is reduced.

Example 6. The ring $Q(R) = R_{N+B} = \hat{D}+B$ (where $R = D+B$) is a ring of the form $A+B[[Z]]$ in [12, Example 3.11]. The ring $\hat{D}+B$ is classical (because $Q(R)$ is the total quotient ring of R), and $N+B$ is the unique maximal ideal of $\hat{D}+B$ by [12, Theorem 3.7 (8)] (because N is the unique maximal ideal of $\hat{D}$). Therefore, $\hat{D}+B$ is a local classical ring and hence maximally (and semiregular maximally) classical. The ideal $P+B$ (where $P = (X_2, X_3)\hat{D}$) is a semiregular prime ideal of both R and $\hat{D}+B$ (so that, incidentally, the maximal ideal $N+B$ of $\hat{D}+B$ is semiregular as well), and $(\hat{D}+B)_{P+B} = R_{P+B} = \hat{D}_P$ is an integral domain but not a field. Therefore, $\hat{D}+B$ is (semiregular) maximally classical but not semiregular locally classical, and hence (in the notation of diagram (2)), neither MC nor SRMC implies SRLC. Again, the ring $\hat{D}+B$ is reduced.

We conclude this section with a few miscellaneous results and examples concerning classical rings. We start by noting that "classical" *lifts* modulo the nilradical but does not *pass* modulo the nilradical.

Proposition 3. *If $R/\mathfrak{N}$ is classical, where $\mathfrak{N}$ is the nilradical of R, then R also classical.*

Proof. Let r be a regular element of R. We claim that $r+\mathfrak{N}$ is a regular element of $R/\mathfrak{N}$. If $(r+\mathfrak{N})(s+\mathfrak{N}) = 0+\mathfrak{N}$ for some $s \in R$, then $(rs)^n = 0$ for some $n \in \mathbb{N}$, so regularity of r (and thus of r^n) implies that $s \in \mathfrak{N}$, proving the claim. Now by hypothesis, $r+\mathfrak{N}$ is a unit in $R/\mathfrak{N}$, from which it follows that r is a unit in R. Therefore, R is classical.

Example 7. The converse of Proposition 3 is not true. Let K be a field and set $R = K[x,y]_{(x,y)}/(xy,y^2)$. Observe that R is a local ring whose maximal ideal consists of zerodivisors, whence R is classical. Moreover, the nilradical of R is $\mathfrak{N} = (y+(xy,y^2))$, and $R/\mathfrak{N}$ is isomorphic to $K[x]_{(x)}$, which is a domain but not a field. Consequently, $R/\mathfrak{N}$ is not classical.

Finally, we show that a "trivial extension" of a classical ring is classical. Recall that, for ring R and R-module M, we can form the *trivial extension* $R \propto M$ (also called the *idealization*) starting with the additive group $R \times M$, and defining multiplication by $(r,m)(s,n) = (rs, rn+sm)$ (see [10, Section 25] for details). In the following theorem, we collect together some important (known) facts about trivial extensions.

Theorem 8. *If R is a ring and M is an R-module, then:*

1. *$(r,m) \in R \propto M$ is a unit if and only if $r \in R$ is a unit.*
2. *$(r,m) \in R \propto M$ is regular if and only if $r \in R$ is regular and r acts faithfully on M (that is, $rn = 0$ implies $n = 0$ for $n \in M$).*
3. *J is a prime ideal of $R \propto M$ if and only if $J = P \propto M$ for some prime ideal P of R, in which case $(R \propto M)_J \cong R_P \propto M_P$.*
4. *If P is a prime ideal of R such that $P \propto M$ is a semiregular prime ideal of $R \propto M$, then P is a semiregular prime ideal of R.*

Proof. (1) is [10, Theorem 25.1 (6)]; (2) is [10, Theorem 25.3]; and (3) is [10, Theorem 25.1 (3) and Corollary 25.5 (2)].

To prove (4), suppose that $P \subset R$ is a prime ideal and $(r_1,m_1),\ldots,(r_t,m_t) \in P \propto M$ generate a subideal J with zero annihilator. If $x \in R$ annihilates the subideal I of P generated by $r_1,\ldots,r_t$, then for each index i, $(0,xm_i)$ annihilates J, so that $xm_i = 0$. Thus, $(x,0)$ annihilates J, which forces $x = 0$, and hence I has zero annihilator. Therefore, P is also semiregular.

As an immediate consequence, we get the following characterization of "classical" for trivial extensions.

Corollary 3. *For a ring R and R-module M, $R \propto M$ is classical if and only if every regular element of R that acts faithfully on M is a unit in R.*

Note that $(0) \propto M$ is a nilpotent ideal of $R \propto M$ and hence contained in the nilradical. Although $(0) \propto M$ need not equal the nilradical of $R \propto M$, the result of Proposition 3 still holds.

Corollary 4. *If R is a classical ring and M is an R-module, then $R \propto M$ is classical.*

Proof. If $(r,m) \in R \propto M$ is regular, then $r \in R$ is regular by Theorem 8 (2), so r is a unit by assumption, and hence (r,m) is a unit by Theorem 8 (1).

In fact, we can extend this result to both "locally classical" and "maximally classical," and to their semiregular analogs.

Corollary 5. *Let R be a ring and M an R-module.*

1. *If R is (semiregular) locally classical, then so is $R \propto M$.*
2. *If R is (semiregular) maximally classical, then so is $R \propto M$.*

Proof. By Theorem 8 (3), the prime (respectively maximal) ideals of $R \propto M$ have the form $P \propto M$ as P ranges over the prime (respectively maximal) ideals of R, and $(R \propto M)_{P \propto M} \cong R_P \propto M_P$. Moreover, by Theorem 8 (4), if $P \propto M$ is a semiregular prime ideal of $R \propto M$, then P is a semiregular prime ideal of R, so both statements (and their semiregular analogs) follow immediately from Corollary 4.

Note that the converse to Theorem 8 (4) does not hold. For example, if $p \in \mathbb{Z}$ is prime, then $p\mathbb{Z} \propto \mathbb{Z}/p\mathbb{Z}$ is not a semiregular ideal of $\mathbb{Z} \propto \mathbb{Z}/p\mathbb{Z}$ (being annihilated by $(0,1)$), even though $p\mathbb{Z}$ is a regular prime ideal of $\mathbb{Z}$. We conclude by adapting this example to show that the converses of the statements in Corollaries 4 and 5 do not hold either.

Example 8. If $p \in \mathbb{Z}$ is prime, then $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is a classical ring by Corollary 3, because only the elements of $\mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$ act faithfully on $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$. On the other hand, clearly $\mathbb{Z}_{(p)}$ is not classical, so that the converse of Corollary 4 fails. Since $\mathbb{Z}_{(p)}$ (and hence also $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$) is local, the converse of Corollary 5 (2) also fails. In fact, $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is also locally classical, because, by Theorem 8 (3), its only non-maximal prime ideal is $(0) \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$, and

$$(\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)})_{(0)\propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}} \cong (\mathbb{Z}_{(p)})_{(0)} \propto (\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)})_{(0)} \cong \mathbb{Q}.$$

Thus, Corollary 5 (1) fails as well. Finally, since $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is locally classical, it is semiregular locally and semiregular maximally classical, but $\mathbb{Z}_{(p)}$ is neither, so the converse of the semiregular variations of Corollary 5 fail too.

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