

LAMRON ℓ -GROUPS

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ABSTRACT. The article introduces a new class of lattice-ordered groups. An ℓ -group G is *lamron* if $\text{Min}(G)^{-1}$ is a Hausdorff topological space, where $\text{Min}(G)^{-1}$ is the space of all minimal prime subgroups of G endowed with the inverse topology. It will be evident that lamron ℓ -groups are related to ℓ -groups with stranded primes. In particular, it is shown that for a \mathbf{W} -object (G, u) , if every value of u contains a unique minimal prime subgroup, then G is a lamron ℓ -group; such a \mathbf{W} -object will be said to have \mathbf{W} -stranded primes. A diverse set of examples will be provided in order to distinguish between the notions of lamron, stranded primes, \mathbf{W} -stranded primes, complemented, and weakly complemented ℓ -groups.

1. INTRODUCTION

In [2], the author investigated the hull-kernel topology and the inverse topology on the space of minimal prime elements of an algebraic frame satisfying the FIP. Several of the results there generalized the situation of what happens in the theory of lattice-ordered groups. In addition, the author gave a frame-theoretic characterization of when the inverse topology is Hausdorff. Recently, in [3], the authors studied the set of ultrafilters on the lattice of G -cozero-sets of (G, u) , an archimedean lattice-ordered group with distinguished unit; this space of ultrafilters is denoted there by $\text{Ult}(\text{coz}(G))$. The study of spaces of ultrafilters on lattices of sets has a long and well-developed history. It is well-known that when equipped with the Wallman topology such a space is compact and T_1 , but need not be Hausdorff. One of the major results of [3] states that $\text{Ult}(\text{coz}(G))$ and $\text{Min}(G)^{-1}$ are homeomorphic. Furthermore, an ℓ -group theoretic classification of when these topologies are Hausdorff was given. In this article, we prove some fundamental results regarding this class of ℓ -groups.

We assume the reader is familiar with the notion of a lattice-ordered group and the major results regarding them. We assume that $(G, +, 0, \wedge, \vee)$ denotes an ℓ -group. In general, it will not be assumed that the additive operation is commutative. However, it is a fact that an archimedean ℓ -group is abelian. Section 3 will be devoted to archimedean ℓ -groups. Excellent references for the theory of ℓ -groups are the following: [1], [8], [11], [22], and [29]. Recall that $G^+ = \{g \in G : 0 \leq g\}$ is the positive cone of G .

For an ℓ -group G , the collection of all convex ℓ -subgroups of G is denoted by $\mathfrak{C}(G)$. It is a well-known fact that $\mathfrak{C}(G)$ is a complete distributive lattice under inclusion, and, moreover, is an algebraic frame satisfying the FIP. The main point of this result is that the intersection of a family of convex ℓ -subgroups is again a convex ℓ -subgroup. For $H, K \in \mathfrak{C}(G)$, the join and meet of H and K are denoted by $H \vee K$ and $H \cap K$, respectively. In general, $H \vee K$ need not be the sum $H + K$. However, this is the case when G is abelian.

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For a given $a \in G$, we use $G(a)$ to denote the convex ℓ -subgroup generated by a . For any $A \subseteq G$, we denote the polar of A by A^\perp and recall that

$$A^\perp = \{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}.$$

When $A = \{a\}$, we instead write a^\perp . Polars are convex ℓ -subgroups. If $a^\perp = 0$, then a is called a *weak order unit*. On the other hand, if $G = G(a)$ for some $a \in G^+$, then a is called a *strong order unit*. When G possesses a strong order unit, G is called *unital*. It is worth noting that a strong order unit is also a weak order unit, but not conversely.

Recall that for $P \in \mathfrak{C}(G)$ is called a *prime subgroup* of G if $a \wedge b = 0$ implies that $a \in P$ or $b \in P$. The set of all prime subgroups is denoted by $\text{Spec}(G)$. It is known that for a prime subgroup P the complement, $G^+ \setminus P$, is a filter on the (bounded below) lattice G^+ . The intersection of a chain of prime subgroups is again a prime subgroup, and so it follows from Zorn's Lemma that minimal prime subgroups exist and every prime subgroup contains a minimal prime subgroup. The set of minimal prime subgroups of G is denoted by $\text{Min}(G)$. The classification of minimal prime subgroups is among the major achievements in the theory of ℓ -groups.

Lemma 1.1 (Lemma on Ultrafilters). *Suppose U is an ultrafilter on G^+ . Then $P = \bigcup\{x^\perp : x \in U\}$ is a minimal prime subgroup and $G^+ \setminus P = U$.*

Theorem 1.2. *For $P \in \text{Spec}(G)$, the following are equivalent.*

- (1) P is a minimal prime subgroup.
- (2) $G^+ \setminus P$ is an ultrafilter on G^+ .
- (3) For all $g \in P$, there is a $t \in G^+ \setminus P$ such that $g \wedge t = 0$.
- (4) $P = \bigcup\{g^\perp : g \in G^+ \setminus P\}$.

Observe that if P and Q are distinct minimal prime subgroups, then, first, we know we can choose $0 < p \in P^+ \setminus Q$. Theorem 1.2 states that there is some $0 < g \in G^+ \setminus P$ such that $g \wedge p = 0$. By primality of Q , $g \in Q$. Thus, there is a $p \in P^+ \setminus Q$ and a $g \in Q^+ \setminus P$ such that $p \wedge g = 0$. This actually holds for any pair of incomparable prime subgroups, a result that is useful in what follows.

Lemma 1.3. *Suppose P and Q are two incomparable prime subgroups of G . There exists $0 < p \in P^+ \setminus Q$ and $0 < q \in Q^+ \setminus P$, with $p \wedge q = 0$.*

Proof. Choose $0 \leq p \in P^+ \setminus Q$ and $0 < q \in Q^+ \setminus P$. Observe that by convexity p and q are incomparable elements. Set $p' = p - (p \wedge q)$ and $q' = q - (p \wedge q)$ and observe that $p' \wedge q' = 0$. Then $0 < p' \in P^+$ and $0 < q' \in Q^+$. Also, by convexity, $p \wedge q \in P \cap Q$. If $p' \in Q$, then so is p , by addition. Similarly, if $q' \in P$, then so is q . Therefore, $0 < p' \in P^+ \setminus Q$, $0 < q' \in Q^+ \setminus P$, and $p' \wedge q' = 0$. \square

Our last result of this section is a foreshadow of things to come. Its proof follows from Theorem 1.2.

Proposition 1.4. *Let G be an ℓ -group and $0 < u \in G$. Then u is a weak order unit if and only if u belongs to no minimal prime subgroup.*

2. SPACES OF MINIMAL PRIMES

The authors of [3] are interested in studying spaces of ultrafilters on G^+ and, via the Lemma on Ultrafilters, spaces of minimal prime subgroups. We shall recall the two well-known topologies constructed on $\text{Min}(G)$. Certain interesting questions about these topological structure spaces lead to some interesting classes of ℓ -groups.

For $g \in G$, define

$$\mathcal{S}(g) = \{P \in \text{Spec}(G) : g \notin P\}.$$

We view $\mathcal{S}(\cdot)$ as an operator which has the following properties. (The result was first proved for vector lattices in [26, Ch.5,§35]. In [8, Lemma 10.1.1.], the authors point out that the result is true for abelian ℓ -groups. A routine verification shows that the result is also true for arbitrary ℓ -groups. We include a proof of (iii) to illustrate that the commutativity of the operation $+$ is not needed.)

Theorem 2.1. *The following statements hold.*

- (i) $\mathcal{S}(0) = \emptyset$. In particular, $\mathcal{S}(g) = \emptyset$ if and only if $g = 0$.
- (ii) For all $g \in G$, $\mathcal{S}(g) = \mathcal{S}(|g|)$.
- (iii) For all $g, h \in G$, $\mathcal{S}(g) \cup \mathcal{S}(h) = \mathcal{S}(|g| \vee |h|)$.
- (iv) For all $g, h \in G$, $\mathcal{S}(g) \cap \mathcal{S}(h) = \mathcal{S}(|g| \wedge |h|)$. In particular, the sets $\mathcal{S}(g), \mathcal{S}(h)$ are disjoint if and only if g, h are orthogonal.
- (v) For all $g \in G$, $\mathcal{S}(g) = \mathcal{S}(g^+) \cup \mathcal{S}(g^-)$.

Proof. (iii) Let $g, h \in G$. By (ii), we can assume, without loss of generality, that $0 \leq g, h$. If $P \in \mathcal{S}(g)$, then since P is convex, it cannot be the case that $0 \leq g \leq g \vee h \in P$. Therefore, both $\mathcal{S}(g) \subseteq \mathcal{S}(g \vee h)$ and $\mathcal{S}(h) \subseteq \mathcal{S}(g \vee h)$. Conversely, if $P \in \mathcal{S}(g \vee h)$, then either $g \notin P$ or $h \notin P$. If this is not the case, then $g \in P$ and $h \in P$, which implies $g \vee h \in P$, since P is an ℓ -subgroup. □

For $g \in G$, we define

$$U(g) = \text{Min}(G) \cap \mathcal{S}(g) = \{P \in \text{Min}(G) : g \notin P\}.$$

It is obvious that the analogous properties of Theorem 2.1 hold as applied to $\text{Min}(G)$. Formally,

Proposition 2.2. *The following hold.*

- (i) $U(0) = \emptyset$. In particular, $U(g) = \emptyset$ if and only if $g = 0$.
- (ii) For all $g \in G$, $U(g) = U(|g|)$.
- (iii) For all $g, h \in G$, $U(g) \cup U(h) = U(|g| \vee |h|)$.
- (iv) For all $g, h \in G$, $U(g) \cap U(h) = U(|g| \wedge |h|)$. In particular, the sets $U(g), U(h)$ are disjoint if and only if g, h are orthogonal.
- (v) For all $g \in G$, $U(g) = U(g^+) \cup U(g^-)$.

In $\text{Min}(G)$, the set-theoretic complement of $U(g)$ is denoted by $V(g)$. Notice that by (iii) and (iv) of Proposition 2.2 the collection $\mathcal{B}_U = \{U(g) : g \in G\}$ is closed under finite unions and intersections. Moreover, the collection $\mathcal{B}_V = \{V(g) : g \in G\}$ also is closed under finite unions and intersections. Thus, each of these forms a basis for a topology of open sets on $\text{Min}(G)$. The topology generated by \mathcal{B}_U is referred to as the *hull-kernel* topology. The

topology generated by \mathcal{B}_V has many names; here it will be called the *inverse* topology. We use $\text{Min}(G)$ (respectively, $\text{Min}(G)^{-1}$) to denote the set of minimal prime subgroups equipped with the hull-kernel (respectively, inverse) topology. Here are some of the fundamental results concerning these two important topologies. First, some useful definitions.

Definition 2.3. Recall that an element $0 \leq x \in G^+$ is called *complemented* if there is some $0 \leq y \in G^+$ such that $x \wedge y = 0$ and $x \vee y$ is a weak order unit of G . The pair x, y is called a complementary pair. G is called *complemented* if every positive element is complemented. G is called *weakly complemented* if whenever $a \wedge b = 0$, then there is a complementary pair $0 \leq x, y \in G^+$ such that $a \leq x$ and $b \leq y$.

By a boolean space we mean a compact zero-dimensional Hausdorff space.

Theorem 2.4. For an ℓ -group G , the following hold.

- (a) For each $g \in G$, $U(g)$ is a clopen subset of $\text{Min}(G)$. Consequently, $\text{Min}(G)$ is a zero-dimensional Hausdorff space.
- (b) $\text{Min}(G)^{-1}$ is a compact space satisfying the T_1 -separation axiom.
- (c) $\text{Min}(G)$ is finer than $\text{Min}(G)^{-1}$.
- (d) The following statements are equivalent.
 - (1) $\text{Min}(G) = \text{Min}(G)^{-1}$.
 - (2) $\text{Min}(G)$ is compact.
 - (3) G is a complemented ℓ -group.
- (e) $\text{Min}(G)^{-1}$ is a boolean space if and only if G is weakly complemented.

Proof. The statement (a) is [10, Proposition 2.1.iii], while (d) is [10, Theorem 2.2].

For (b), (c), and (e) the reader can check [28] for the abelian case. For the general case, the article [2] studies this topic from the point of view of algebraic frames satisfying the FIP. Then, once it is observed that $\mathfrak{C}(G)$ is such a frame and $\text{Min}(G) = \text{Min}(\mathfrak{C}(G))$, the result follows, see [2, Lemma 4.1, Theorem 4.6]. Also, the reader can check [24, Theorem 2.13] for the general case of (e). □

Remark 2.5. A projectable ℓ -group with a weak order unit is complemented while a feebly projectable ℓ -group with a weak order unit is weakly complemented. (See [24] for more information on feebly projectable ℓ -groups.) The ℓ -group $\text{Aut}(\Omega)$ of all order preserving permutations on the chain (Ω, \leq) is always a complemented ℓ -group, and rarely a projectable ℓ -group. Since a boolean space is indeed Hausdorff, a natural process is to characterize when $\text{Min}(G)^{-1}$ is simply a Hausdorff space. Our next theorem is a complete characterization of the general situation for ℓ -groups. First, the central definition of the article.

Definition 2.6. We call the ℓ -group G *lamron* if whenever $a, b \in G^+$ such that $a \wedge b = 0$, then there are $x, y \in G^+$ such that $a \leq x$, $b \leq y$, $a \wedge y = 0 = b \wedge x$, and $x \vee y$ is a weak order unit.

Clearly, all weakly complemented ℓ -groups are lamron. Furthermore, it will be shown in 2.10 that every ℓ -group with stranded primes is lamron.

Theorem 2.7. Let G be an ℓ -group. The following conditions are equivalent.

- (1) $\text{Min}(G)^{-1}$ is a Hausdorff space.
- (2) $\text{Min}(G)^{-1}$ is a normal space.

- (3) Given any $a, b \in G^+$ with $a \wedge b = 0$, there exist $g, h \in G^+$ such that $a \in g^\perp$, $b \in h^\perp$, $g^\perp \cap h^\perp = 0$.
- (4) G is a lamron ℓ -group.
- (5) $\text{Min}(G)^{-1}$ is a regular space.
- (6) Given any $g \in G^+$ and $P \in \text{Min}(G)^{-1}$ with $g \in P$, there exists $a, b \in G^+$ such that $g \in a^\perp \subseteq b^{\perp\perp} \subseteq P$.
- (7) Whenever $a \wedge b = 0$, then $a^\perp \vee b^\perp$ contains a weak order unit.
- (8) For any pair of incomparable prime subgroups, the convex ℓ -subgroup they generate contains a weak order unit.
- (9) For any pair of distinct minimal prime subgroups, the convex ℓ -subgroup they generate contains a weak order unit.

Proof. (1) is equivalent to (2). If $\text{Min}(G)^{-1}$ is Hausdorff, then since it is always compact and compact Hausdorff spaces are normal, then $\text{Min}(G)^{-1}$ is normal. The converse is clear since a normal T_1 -space is Hausdorff.

(2) implies (3). Suppose $\text{Min}(G)^{-1}$ is normal and let $a, b \in G^+$ satisfy $a \wedge b = 0$. Then $U(a)$ and $U(b)$ are disjoint closed subsets of $\text{Min}(G)^{-1}$. By normality and compactness, there are disjoint basic open subsets, say $V(g), V(h)$ for some $g, h \in G^+$, such that $U(a) \subseteq V(g)$ and $U(b) \subseteq V(h)$. In particular, this means that

$$U(a \wedge g) = U(a) \cap U(g) = \emptyset$$

and

$$U(b \wedge h) = U(b) \cap U(h) = \emptyset.$$

In other words, $a \wedge g = 0$ and $b \wedge h = 0$. Therefore, $a \in g^\perp$ and $b \in h^\perp$. Finally, since $V(g \vee h) = V(g) \cap V(h) = \emptyset$, we gather that $g \vee h$ is a weak order unit. In other words, $g^\perp \cap h^\perp = (g \vee h)^\perp = 0$.

(3) implies (4). Let $a \wedge b = 0$, then we have $g, h \in G^+$ as in (3). Set $x = a \vee h$ and $y = b \vee g$ and observe that $a \leq x$, $b \leq y$. Moreover, $a \wedge y = a \wedge (b \vee g) = (a \wedge b) \vee (a \wedge g) = 0$ and $b \wedge x = b \wedge (a \vee h) = (b \wedge a) \vee (b \wedge h) = 0$. It is clear that since $g \vee h$ is a weak order unit, $x \vee y$ is a weak order unit.

(4) implies (1). Let $P, Q \in \text{Min}(G)$ be distinct. Then, using Lemma 1.3, there exists $g, h \in G^+$ such that $g \wedge h = 0$, $g \in P \setminus Q$, and $h \in Q \setminus P$. By (4) we can find $x, y \in G^+$ such that $g \leq x$, $h \leq y$, $g \wedge y = 0 = h \wedge x$, and $x \vee y$ is a weak order unit. Since $g \wedge y = 0$ and $g \notin Q$, it follows that $y \in Q$; consequently $Q \in V(y)$. Similarly $P \in V(x)$. Finally, it is evident that $V(x) \cap V(y) = V(x \vee y) = \emptyset$, because $x \vee y$ is a unit.

(2) is equivalent to (5). It is clear that every normal space is regular and every T_1 regular space is Hausdorff.

(5) implies (6). Suppose that $\text{Min}(G)^{-1}$ is regular and that $P \in V(g)$. Now, $U(g)$ is a closed subset of $\text{Min}(G)^{-1}$, so by regularity, there exists disjoint basic open sets $V(a)$ and $V(b)$ such that $P \in V(b)$ and $U(g) \in V(a)$; assume that $0 \leq a, b \in G^+$. Note that $V(a \vee b) = V(a) \cap V(b) = \emptyset$, concluding that $a \vee b$ is a weak order unit. It follows that $a^\perp \cap b^\perp = (a \vee b)^\perp = 0$, and so $a^\perp \subseteq b^{\perp\perp}$. To show the remaining inclusions, first observe that $b^{\perp\perp} \subseteq P$, since $b \in P$. Finally, $U(g \wedge a) = U(g) \cap U(a) = \emptyset$, proving that $g \wedge a = 0$. Hence, $g \in a^\perp$. Combining all the above inclusions it follows that $g \in a^\perp \subseteq b^{\perp\perp} \subseteq P$.

(6) implies (5). Suppose $P \in \text{Min}(G)^{-1}$ and C is a closed subset with $P \notin C$. Then we can choose some $g \in G^+$ such that $P \in V(g) \subseteq \text{Min}(G)^{-1} \setminus C$, that is, $g \in P$. Using (6) there exists $a, b \in G^+$ such that $g \in a^\perp \subseteq b^{\perp\perp} \subseteq P$. Note that $b \in P$, so that $P \in V(b)$. Also $g \wedge a = 0$, showing that $U(g) \cap U(a) = \emptyset$; consequently $C \cap U(a) = \emptyset$ which means $C \subseteq V(a)$. To complete the proof, it can be easily verified that $V(a) \cap V(b) = \emptyset$ since $a^\perp \cap b^\perp = 0$.

We have now proved conditions (1) through (6) are equivalent.

(3) implies (7). Suppose $a \wedge b = 0$, then there exists $g, h \in G^+$ such that $g \vee h$ is a weak order unit and $a \in g^\perp$, $b \in h^\perp$. It follows that $g \in a^\perp$ and $h \in b^\perp$. Hence, $a^\perp \vee b^\perp$ contains a weak order unit $g \vee h$.

(7) implies (8). Let $P \neq Q$ be a pair of incomparable prime subgroups of G . Using Lemma 1.3, there exist $0 < a \in P^+ \setminus Q$ and $0 < b \in Q^+ \setminus P$ such that $a \wedge b = 0$. Using (7) it follows that $a^\perp \vee b^\perp$ contains a weak order unit u . Observe that $a^\perp \subseteq Q$ and $b^\perp \subseteq P$, since they are primes. Hence, $P \vee Q$ contains a weak order unit.

(8) implies (9). This direction is clear.

(9) implies (1). Let $P \neq Q$ be distinct minimal prime subgroups of G . By (9), $P \vee Q$ contains a weak order unit u . Then we can choose $p_i \in P^+$ and $q_i \in Q^+$ with

$$u = p_1 + q_1 + \cdots + p_n + q_n.$$

It follows that $P \in V(p_1) \cap \cdots \cap V(p_n) = V_1$, and similarly, $Q \in V(q_1) \cap \cdots \cap V(q_n) = V_2$. If $R \in V_1 \cap V_2$, then $u = p_1 + q_1 + \cdots + p_n + q_n \in R$, which is a contradiction, since R is a minimal prime subgroup. Therefore $V_1 \cap V_2 = \emptyset$, proving that $\text{Min}(G)^{-1}$ is a Hausdorff space. \square

Corollary 2.8. *If G is lamron, then G possesses a weak order unit.*

Remark 2.9. In [5], the authors characterize when the inverse topology on the space of minimal prime ideals of a commutative ring with identity is Hausdorff. The reader familiar with the theory of commutative rings with identity will notice the similarity between Theorem 2.7 and [5, Theorem 2.6]. In our view, multiplication in a ring behaves in a fashion similar to the behavior of the infimum in an ℓ -group. For example, being disjoint in an ℓ -group corresponds to being annihilated in a ring. A weak order unit corresponds to a regular element in a ring, while principal convex ℓ -subgroups correspond to finitely generated ideals of a ring. Through this lens, condition (3) of Theorem 2.7 relates to condition (5) of [5, Theorem 2.6], and condition (7) of Theorem 2.7 relates to condition (6) of [5, Theorem 2.6].

Furthermore, condition (7) of Theorem 2.7 ought to look familiar. Recall that G is said to have *stranded primes* if each prime subgroup contains a unique minimal prime subgroup. This statement is equivalent to the condition that $\text{Spec}(G)$ is a union of disjoint maximal chains. The class of ℓ -groups with stranded primes has a long history. Conrad [9] showed that this class forms a torsion class of ℓ -groups; he denoted the class by \mathcal{B} . In [8], the authors call an object in \mathcal{B} “semi-projectable”; other authors have used the term “weakly projectable”. One more statement that characterizes when G has stranded primes is that, for each $g, h \in G^+$, $G = (g \wedge h)^\perp = g^\perp \vee h^\perp$ (see [8, 7.5.1]). For a proof from the frame-theoretic point of view, the reader is urged to look at [27, Theorem 2.4]. In particular, it is demonstrated that G has stranded primes if and only if whenever $g \wedge h = 0$, then $G = g^\perp \vee h^\perp$. Comparing this last characterization to condition (7) of Theorem 2.7, the following result is true.

Corollary 2.10. *Suppose G possesses a weak order unit u . If G has stranded primes, then G is lamron.*

Remark 2.11. It is worth mentioning that in the case that G possesses a strong order unit, having stranded primes is equivalent to a seemingly stronger condition (which we will call *strongly lamron*): given $a, b \in G^+$ such that $a \wedge b = 0$, there are $x, y \in G^+$ such that $a \in x^\perp$, $b \in y^\perp$, and $x \vee y$ is a strong order unit. Consequently, an ℓ -group G is unital and has stranded primes if and only if G is strongly lamron.

Proof. (\Leftarrow) Suppose $a, b \in G^+$ are disjoint. Then, there are $x, y \in G^+$ such that

$$a \wedge x = 0 = b \wedge y,$$

and $x \vee y$ is a strong order unit. Indeed, $x \in a^\perp$, $y \in b^\perp$, and $x \vee y \in a^\perp \vee b^\perp$. Therefore, $a^\perp \vee b^\perp = G$.

(\Rightarrow) Let $a \wedge b = 0$, then $a^\perp \vee b^\perp = G$ and hence it contains a strong order unit u since G is unital. It is known that $a^\perp \vee b^\perp$ is the subgroup generated by them. Furthermore, by the Riesz Decomposition Theorem, we can choose appropriate positive $x_1, \dots, x_n \in a^\perp$ and $y_1, \dots, y_n \in b^\perp$ such that $u = x_1 + y_1 + \dots + x_n + y_n$. Let $x = x_1 \vee \dots \vee x_n$ and $y = y_1 \vee \dots \vee y_n$. Then,

$$a \wedge x = a \wedge (x_1 \vee \dots \vee x_n) = (a \wedge x_1) \vee \dots \vee (a \wedge x_n) = 0.$$

Similarly, $b \wedge y = 0$. Finally, note that

$$0 < u \leq x + y + \dots + x + y \in G(x) \bigvee G(y) = G(x \vee y).$$

Therefore, $G(x \vee y) = G$, whence $x \vee y$ is a strong order unit. \square

3. \mathbf{W}

In this section we focus on \mathbf{W} , the category of archimedean ℓ -groups with distinguished weak order unit. An object in \mathbf{W} has the form (G, u) where G is an archimedean ℓ -group and $0 < u$ is a weak order unit of G . Morphisms in \mathbf{W} are ℓ -group homomorphisms which preserve the distinguished units. The beauty of this category is seen through the Yosida Embedding Theorem which establishes a representation theorem for archimedean ℓ -groups with weak order units. The most prominent example is $(C(X), \mathbf{1})$, where $C(X)$ is the collection of real-valued continuous functions on a Tychonoff space X , and $\mathbf{1}$ is the constant function. When one speaks of $C(X)$ as a \mathbf{W} -object, it is usually understood as $(C(X), \mathbf{1})$.

For more information on $C(X)$, the text [16] is still the best source. As for the Yosida Embedding Theorem, a good place to start is [19] as well as all of the more recent papers which reference it, for example [17]. For a more detailed look at \mathbf{W} , we encourage the reader to peruse [6] and [8].

Some recollections are in order. Start with a \mathbf{W} -object (G, u) . Since $0 < u$, there are convex ℓ -subgroups of G which are maximal with respect to not containing u . Such an object is indeed a prime subgroup of G , known as a *value* of u . In the archimedean case, the set of values of u is known as the *Yosida space* of (G, u) and is denoted by YG . The restriction of the hull-kernel topology on $\text{Spec}(G)$ to YG results in a compact Hausdorff space. A basic open set has the form $\mathcal{S}(g) \cap YG$ for an arbitrary $g \in G$. This set is called the *cozero-set* of g and is instead written as $\text{coz}(g)$. Any subset of YG of this form is known as a *G -cozero-set*; the collection of all such sets is denoted by $\text{coz}(G)$ and is, obviously, a base for the topology

of open subsets of YG . The complement of a G -cozero-set is a G -zero-set and the set of these is denoted by $Z(G)$. In the few cases where a discussion of (G, u) and (G, v) takes place with $0 < u, v$ different weak order units, we shall use the symbol $\text{Yos}_G(u)$ to denote the Yosida space relative to u .

Next, $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ is the two-point compactification of the real numbers, with the obvious ordering. For a Tychonoff space X and a continuous function $f : X \rightarrow \bar{\mathbb{R}}$, set $\text{re}(f) = f^{-1}(\mathbb{R})$; this is known as the reality set of f ; such a set is a cozero-set of X .

$$D(X) = \{f : X \rightarrow \bar{\mathbb{R}} : \text{re}(f) \text{ is a dense subset of } X\}.$$

In general, $D(X)$ is a lattice under the pointwise operations but not a group under (almost) pointwise addition. However, by an ℓ -subgroup of $D(X)$ is meant a subcollection H of $D(X)$ that is a sublattice and is also closed under the addition defined as follows: for $f, g \in H$ there is an $h \in H$ for all $x \in \text{re}(f) \cap \text{re}(g)$, $f(x) + g(x) = h(x)$. Next, we state the major result in the theory of \mathbf{W} .

Theorem 3.1 (The Yosida Embedding Theorem). *Let (G, u) be a \mathbf{W} -object. There is an ℓ -isomorphism of G ($g \mapsto \hat{g}$) onto an ℓ -subgroup $\hat{G} \leq D(YG)$ such that $\hat{u} = \mathbf{1}$ and \hat{G} has the following separation property: for each $p \in YG$ and closed set $V \subseteq YG$ not containing p , there is some $g \in G$ for which $\hat{g}(p) = 1$ and $\hat{g}(q) = 0$ for all $q \in V$. Moreover, YG is the unique compact space, up to homeomorphism, satisfying these two properties.*

Example 3.2. For a Tychonoff space X , it is standard to call a subset Z of X a zero-set of X if $Z = \{x \in X : f(x) = 0\}$ for some $f \in C(X)$. The set of all zero-sets of X is denoted by $Z(X)$. The Gelfand-Kolmogorov Theorem states that space $\text{Max}(C(X))$ of maximal (ring) ideals of $C(X)$ (endowed with the hull-kernel topology) is homeomorphic to βX , the Stone-Ćech compactification of X . One construction of βX is as the space of $Z(X)$ -ultrafilters. In this way, the correspondence is as follows: to each $M \in \text{Max}(C(X))$ there is a unique $p \in \beta X$ such that $M = M^p = \{f \in C(X) : Z(f) \in p\}$.

In order to characterize the Yosida space of $(C(X), \mathbf{1})$, notice that each $M^p \in \text{Max}(C(X))$ is a convex ℓ -subgroup, so that since $1 \notin M^p$, there is a unique value $V^p \in YC(X)$ containing M^p . This correspondence is, in fact, a homeomorphism. In this way, we can say that the Yosida space of $C(X)$ is βX . Therefore, if X is not compact, then $Z(X)$ and $Z(C(X))$ are not the same, as the first is a collection of subsets of X , while the second is a collection of subsets of βX .

For a given \mathbf{W} -object (G, u) , it is useful to discuss the set of “bounded” elements. This is simply the convex ℓ -subgroup generated by u . In terms of the Yosida Embedding Theorem, it is the set of elements whose Yosida representation is a bounded continuous function. This set is denoted by G^* , resulting in a \mathbf{W} -object (G^*, u) . Observe that $G = G^*$ if and only if u is a strong order unit. In the unital case, the Yosida Embedding Theorem embeds G into $C(YG)$. Now, the uniqueness established in the Yosida Embedding Theorem leads to the statement that $YG = YG^*$. Furthermore, the embedding of G^* into G is an example of *rigid* embedding: for each $g \in G$ there is an $h \in G^*$ such that $g^{\perp\perp} = h^{\perp\perp}$. In this case,

the contraction mapping $\text{Min}(G) \longrightarrow \text{Min}(G^*)$ is a homeomorphism with regards to the hull-kernel topology as well as the inverse topology (see [10, Proposition 2.3] and [28, Proposition 6.5]).

An intermission is in order. In particular, the topics discussed in the previous section have interesting applications to \mathbf{W} . The main point here is that in the previous section each interesting class of ℓ -groups was described in terms of spaces of minimal prime subgroups. In \mathbf{W} , these classes can be described by their Yosida spaces. The following lemma is vital and well-known.

Lemma 3.3. *Let (G, u) be a \mathbf{W} -object and $g \in G^+$. Then*

$$g^\perp = \{h \in G : \text{coz}(g) \cap \text{coz}(h) = \emptyset\}$$

and

$$g^{\perp\perp} = \{f \in G : \text{coz}(f) \subseteq \text{cl}_{YG} \text{coz}(g)\}.$$

Observe that $0 \leq g \in G$ is a weak order unit if and only if $\text{coz}(g)$ is a dense subset of YG .

Theorem 3.4. *The following are equivalent for the \mathbf{W} -object (G, u) .*

- (1) G is a complemented ℓ -group.
- (2) YG has the following property: for each $C \in \text{coz}(G)$ there is a $D \in \text{coz}(G)$ such that $C \cap D = \emptyset$ and $C \cup D$ is a dense subset of YG .
- (3) G^* is a complemented ℓ -group.
- (4) YG^* has the following property: for each $C \in \text{coz}(G^*)$ there is a $D \in \text{coz}(G^*)$ such that $C \cap D = \emptyset$ and $C \cup D$ is a dense subset of YG^* .

Proof. Since the complemented property is characterized by a topological property on $\text{Min}(G)$ and both $\text{Min}(G)$ and $\text{Min}(G^*)$ are homeomorphic, it suffices to prove (1) and (2) are equivalent.

Suppose G is complemented and let $C \in \text{coz}(G)$. This means there is some $g \in G^+$ such that $C = \text{coz}(g)$. Let $h \in G^+$ be a complement of g . Then $g \wedge h = 0$ implies that $\text{coz}(g) \cap \text{coz}(h) = \emptyset$. That $g \vee h$ is a weak order unit means that $\text{coz}(g) \cup \text{coz}(h) = \text{coz}(g \vee h)$ is a dense subset.

The proof of the converse is clear. □

Theorem 3.5. *The following are equivalent for the \mathbf{W} -object (G, u) .*

- (1) G is a weakly complemented ℓ -group.
- (2) YG satisfies: for each disjoint pair $C, D \in \text{coz}(G)$ there are $C', D' \in \text{coz}(G)$ such that $C \subseteq C', D \subseteq D', C' \cap D' = \emptyset$, and $C' \cup D'$ is a dense subset of YG .
- (3) G^* is a weakly complemented ℓ -group.
- (4) YG^* satisfies: for each disjoint pair $C, D \in \text{coz}(G^*)$ there are $C', D' \in \text{coz}(G^*)$ such that $C \subseteq C', D \subseteq D', C' \cap D' = \emptyset$, and $C' \cup D'$ is a dense subset of YG^* .

Proof. As in the proof of Theorem 3.4, it suffices to prove that (1) and (2) are equivalent.

Suppose that G is weakly complemented and let $C, D \in \text{coz}(G)$ be disjoint G -cozero-sets and let $0 \leq g, h \in G$ satisfy $C = \text{coz}(g)$ and $D = \text{coz}(h)$. By hypothesis, there is a complementary pair $0 \leq x, y \in G$ such that $g \leq x$ and $h \leq y$. Clearly, $\text{coz}(g) \subseteq \text{coz}(x)$ and $\text{coz}(h) \subseteq \text{coz}(y)$, and the G -cozero-sets $\text{coz}(x)$ and $\text{coz}(y)$ have the desired property in (2).

Once again, the proof of the converse is clear. □

Remark 3.6. A Tychonoff space X for which $C(X)$ is a complemented ℓ -group is known as a *cozero complemented space*, while X is *weakly cozero complemented* when $C(X)$ is weakly complemented. The interesting result for Tychonoff spaces is that X is a (weakly) cozero complemented space if and only if βX is a (weakly) cozero complemented space. References for more information on these classes of spaces are [21], [25], and [28].

Recall that a Tychonoff space X is an F -space whenever disjoint cozero-sets of X are completely separated. Moreover, $C(X)$ has stranded primes if and only if X is an F -space. The next result generalizes the situation for a \mathbf{W} -object. First, recall that for an ℓ -group G and $P \in \text{Spec}(G)$,

$$O(P) = \{m \in G : \text{there exists an } x \in G^+ \setminus P \text{ such that } |m| \wedge x = 0\}.$$

It is known that $O(P) = \bigcap \{Q \in \text{Spec}(G) : Q \leq P\}$. The condition that $O(P)$ is prime is the same as saying there is a unique minimal prime subgroup beneath P .

Theorem 3.7. *The following are equivalent for the \mathbf{W} -object (G, u) .*

- (1) *Each pair of disjoint G -cozero-sets are completely G -separated.*
- (2) *For each $P \in YG$, $O(P)$ is a prime subgroup.*
- (3) *If P and Q are distinct minimal prime subgroups, then $u \in P + Q$.*
- (4) *G^* has stranded primes.*
- (5) *Whenever $a \wedge b = 0$, then $u \in a^\perp + b^\perp$.*
- (6) *Disjoint G -cozero-sets have disjoint closures.*
- (7) *Disjoint G^* -cozero-sets have disjoint closures.*

Proof. (1) implies (5). Let $a, b \in G$ satisfy $a \wedge b = 0$. Then, $\text{coz}(a), \text{coz}(b)$ are disjoint G -cozero-sets of YG . By hypothesis, there are $0 \leq g_1, g_2 \in G$ such that $\text{coz}(a) \subseteq Z(g_1)$, $\text{coz}(b) \subseteq Z(g_2)$, and $Z(g_1) \cap Z(g_2) = \emptyset$. It follows that $g_1 \in a^\perp$, $g_2 \in b^\perp$, and $Z(g_1 + g_2) = Z(g_2) \cap Z(g_2) = \emptyset$. Thus, $g_1 + g_2 \in a^\perp + b^\perp$ is nowhere 0, and so by compactness, there is some $\epsilon > 0$ such that for all $p \in YG$, $\epsilon < (g_1 + g_2)(p)$. Thus, some multiple of $g_1 + g_2$ surpasses u . Hence, by convexity, $u \in a^\perp + b^\perp$.

(4) is equivalent to (5). If (5) holds, then it holds for all $a, b \in G^*$ and so G^* has stranded primes. Conversely, if G^* has stranded primes, then for any $a, b \in G^+$ satisfying $a \wedge b = 0$, it is the case that $(a \wedge u) \wedge (b \wedge u) = 0$. Therefore, $u \in (a \wedge u)^\perp + (b \wedge u)^\perp$. Now, since u is a weak order unit, $(a \wedge u)^\perp = a^\perp$ and $(b \wedge u)^\perp = b^\perp$.

(4) implies (7). Let C_1, C_2 be disjoint G^* -cozero-sets. Choose $0 \leq g_1, g_2 \in G^*$ such that $\text{coz}(g_1) = C_1$ and $\text{coz}(g_2) = C_2$. Then $g_1 \wedge g_2 = 0$, and so, $u \in g_1^\perp + g_2^\perp$. Choose $0 \leq f \in g_1^\perp$ and $0 \leq g \in g_2^\perp$ such that $u = f + g$. A simple calculation shows that $\text{coz}(g_1) \subseteq Z(f)$, $\text{coz}(g_2) \subseteq Z(g)$ and $Z(f) \cap Z(g) = \emptyset$. Therefore, C_1 and C_2 have disjoint closures.

(6) and (7) are equivalent since $YG = YG^*$ and $\text{coz}(G) = \text{coz}(G^*)$.

(1) and (6) are equivalent since YG is a compact Hausdorff space.

(2) is equivalent to (3). In particular, (2) is equivalent to the statement that any two distinct minimal primes are contained in distinct values. In particular, if $u \notin P + Q$, then they are contained in some value of u .

That (2) and (4) are equivalent is patent. □

Definition 3.8. If the \mathbf{W} -object (G, u) satisfies the equivalent conditions of Theorem 3.7, then G will be said to *have \mathbf{W} -stranded primes*. So, having \mathbf{W} -stranded primes means that each prime in $\mathcal{S}(u)$ contains a unique minimal prime subgroup.

Corollary 3.9. *The \mathbf{W} -object (G, u) has \mathbf{W} -stranded primes if and only if (G^*, u) has \mathbf{W} -stranded primes. Moreover, if (G, u) has \mathbf{W} -stranded primes, then G is a lamron ℓ -group.*

Proof. The first statement is clear. As to the second statement, if $a \wedge b = 0$, then $u \in a^\perp + b^\perp$, and so G is lamron by condition (7) of Theorem 2.7. \square

Next, we investigate the map $\lambda : \text{Min}(G) \rightarrow YG$ defined by taking a $P \in \text{Min}(G)$ and outputting $\lambda(P)$, the unique value of u containing P . This map is certainly a surjection, as every value of u contains a minimal prime subgroup. It is known that this map is continuous between $\text{Min}(G)$ and YG (see e.g. [8] and [18]). We can actually say more.

Lemma 3.10. *The map $\lambda : \text{Min}(G)^{-1} \rightarrow YG$ is continuous.*

Proof. The proof that we are about to give is a restricted proof from a more general result concerning the map between the space of all prime subgroups not containing u and the values of u .

Let $P \in \text{Min}(G)$ so that $\lambda(P) \in \text{coz}(h)$ for $0 < h \in G^+$. For each $Q \in Z(h)$ there are disjoint G -cozero-sets, say $\text{coz}(t_Q)$ and $\text{coz}(s_Q)$ with $0 \leq t_Q, s_Q$, such that $Q \in \text{coz}(t_Q)$ and $\lambda(P) \in \text{coz}(s_Q)$. The disjointness implies that $t_Q \wedge s_Q = 0$. The collection $\{\text{coz}(t_Q)\}_{Q \in Z(h)}$ is an open cover of $Z(h)$ which happens to be compact. Therefore, there is a finite subcover, say $Z(h) \subseteq \text{coz}(t_{Q_1}) \cup \dots \cup \text{coz}(t_{Q_n})$. Set $t = t_{Q_1} \vee \dots \vee t_{Q_n}$ and $s = s_{Q_1} \wedge \dots \wedge s_{Q_n}$. By distributivity, $s \wedge t = 0$.

Now, by design, for $Q \in Z(h)$, $\lambda(P) \in \text{coz}(s_Q)$ implies that $s_Q \notin P$, whence $P \in U(s_Q)$. It follows that $P \in U(s)$ and so $t \in P$, i.e. $P \in V(t)$. We claim that $\lambda(P) \in \lambda(V(t)) \subseteq \text{coz}(h)$.

Let $R \in V(t)$. This means that $t \in R$ and thus $t \in \lambda(R)$, i.e. $\lambda(R) \in Z(t)$. Now, if $\lambda(R) \in Z(h)$, then it is in one of the G -cozero-sets that belong to the finite subcover from above, say $\lambda(R) \in \text{coz}(t_{Q_i})$. But then $\lambda(R) \in \text{coz}(t)$, the desired contradiction. \square

A natural question is: when is λ injective? The injectivity of λ is equivalent to the condition that each value contains precisely one minimal prime subgroup, which as we just saw is equivalent to (G, u) having \mathbf{W} -stranded primes. This leads to another characterization of \mathbf{W} -stranded primes. We omit its proof.

Theorem 3.11. *Let (G, u) be a \mathbf{W} -object. The following are equivalent.*

- (1) (G, u) has \mathbf{W} -stranded primes.
- (2) $\lambda : \text{Min}(G) \rightarrow YG$ is injective.
- (3) $\lambda : \text{Min}(G)^{-1} \rightarrow YG$ is a homeomorphism.

Example 3.12. In Proposition 7.3 of [28], the author proves that when X is a compact F -space, then X and $\text{Min}(C(X))^{-1}$ are homeomorphic. The author attributes this to [14]. However, it appears that an appropriation citation would be Lemma 2 of [13]. Theorem 3.11 is a generalization of this result to \mathbf{W} -objects with \mathbf{W} -stranded primes. When G is lamron, the map $\lambda : \text{Min}(G)^{-1} \rightarrow YG$ is a cover of YG . This topic will be taken up in [7].

Lest the reader believe that the above is equivalent to G having stranded primes we supply an interesting example.

Example 3.13. Here is an example of an archimedean ℓ -group with two different weak order units u and v such that (G, u) has \mathbf{W} -stranded primes but (G, v) does not have \mathbf{W} -stranded primes. It follows then that G , as an ℓ -group, does not have stranded primes.

Let H be the ℓ -group of all bounded real-valued sequences, and let G be the ℓ -group generated by H and the sequence $i(n) = n$. Notice that $H = C^*(\mathbb{N})$. Now, for $(G, \mathbf{1})$, $YG = \beta\mathbb{N}$ and $G^* = H$. In this case, disjoint G -cozero-sets have disjoint closures and so $(G, \mathbf{1})$ satisfies the equivalent conditions of Theorem 3.7.

On the other hand, consider the \mathbf{W} -object (G, i) . In this case, besides the values corresponding to $n \in \mathbb{N}$, there is only one more value: namely, H . It follows that YG in this case is a copy of the one-point compactification of the naturals, $\alpha\mathbb{N}$. The Yosida Embedding Theorem represents G with i being sent to $\mathbf{1}$ and therefore $\mathbf{1}$ maps to the functions $n \mapsto \frac{1}{n}$. Observe that there are disjoint G -cozero-sets of $\alpha\mathbb{N}$ which do not have disjoint closures. Clearly, (G, v) does not have \mathbf{W} -stranded primes.

Even though it is possible to have an archimedean ℓ -group with different weak order units, one of which induces a \mathbf{W} -object with \mathbf{W} -stranded primes, while the other does not, the ℓ -group in question is a lamron ℓ -group. On the other hand, a natural question is whether there is a way to characterize the (global) stranded primes condition in \mathbf{W} .

Theorem 3.14. *Let G be an archimedean ℓ -group with a weak order unit. The following are equivalent.*

- (1) G has stranded primes.
- (2) For each weak order unit $0 < u \in G$, (G, u) has \mathbf{W} -stranded primes.
- (3) For each weak order unit $0 < u \in G$, disjoint G -cozero-sets of $\text{Yos}_G(u)$ have disjoint closures.
- (4) For each weak order unit $0 < u \in G$, (G^*, u) has stranded primes.
- (5) For all $a, b \in G^+$, if $a \wedge b = 0$, then $a^\perp + b^\perp$ contains every weak order unit.

Proof. (1) implies (2). If G has stranded primes, then for each weak order unit $0 < u \in G$, (G, u) has \mathbf{W} -stranded primes, and so (2) holds.

(2), (3), and (4) are equivalent by Theorem 3.7 and since (G, u) has \mathbf{W} -stranded primes if and only if (G^*, u) has \mathbf{W} -stranded primes.

(2) implies (5). Let $0 < u \in G$ be a weak order unit, and let $a, b \in G$ satisfy $a \wedge b = 0$. By hypothesis, (G, u) has \mathbf{W} -stranded primes and so $u \in a^\perp + b^\perp$.

(5) implies (1). Let $0 < u \in G$ be a weak order unit. To show that G has stranded primes let $a, b \in G$ satisfy $a \wedge b = 0$, and let $0 \leq g \in G$. Now, $u \vee g$ is a weak order unit and so, by (5), $u \vee g \in a^\perp + b^\perp$. By convexity, $g \in a^\perp + b^\perp$, whence $G = a^\perp + b^\perp$. □

Observe that knowing that $(C(X), \mathbf{1})$ satisfies the conditions of Theorem 3.7 forces the stranded primes condition. This actually occurs in any \mathbf{W} -object $(A, \mathbf{1})$ where A is an archimedean f -ring, and so each polar subgroup is a ring ideal. Consequently, if condition (5) of Theorem 3.7 is true, then A , as an ℓ -group, has stranded primes. This means that $C(X)$ has stranded primes if and only if $(C(X), \mathbf{1})$ has \mathbf{W} -stranded primes.

With regards to the lamron condition we ought not to expect that whether it holds for a given ℓ -group is dependent on the weak order unit, since the condition is based on whether $\text{Min}(G)^{-1}$ is Hausdorff. However, there is a nice characterization of when an archimedean ℓ -group is lamron, which is in the same vein as the previous conditions.

Theorem 3.15. *Suppose (G, u) is a \mathbf{W} -object. Then G is lamron if and only if for each pair of disjoint G -cozero-sets C_1, C_2 , there exists G -zero-sets Z_1, Z_2 such that $C_1 \subseteq Z_1, C_2 \subseteq Z_2$, and $\text{int}_{YG}Z_1 \cap \text{int}_{YG}Z_2 = \emptyset$.*

Proof. Suppose that G is lamron, and that C_1, C_2 are disjoint G -cozero-sets. This means there are $a, b \in G^+$ such that $\text{coz}(a) = C_1, \text{coz}(b) = C_2$, and $a \wedge b = 0$. Then $v \in a^\perp + b^\perp$ for some weak order unit $0 \leq v \in G$ (Theorem 2.7 (7)). So, $v = v_1 + v_2$ for appropriate $0 \leq v_1 \in a^\perp$ and $0 \leq v_2 \in b^\perp$ (Riesz Decomposition). Set $Z_1 = Z(v_1)$ and $Z_2 = Z(v_2)$.

Now, $v_1 \wedge a = 0$ implies that $\text{coz}(v_1) \cap C_1 = \emptyset$ and so $C_1 \subseteq Z_1$, whence $C_1 \subseteq \text{int}_{YG}Z_1$. Similarly, $C_2 \subseteq \text{int}_{YG}Z_2$. Clearly, $Z_1 \cap Z_2 = Z(v)$ is a nowhere dense set since v is a weak order unit, whence $\text{int}_{YG}Z_1 \cap \text{int}_{YG}Z_2 = \emptyset$.

Conversely, assume the sufficiency. To show that G is lamron let $a, b \in G^+$ satisfy $a \wedge b = 0$ and set $C_1 = \text{coz}(a)$ and $C_2 = \text{coz}(b)$. Then $C_1 \cap C_2 = \emptyset$ and so there are $g, h \in G$ such that $C_1 \subseteq \text{int}_{YG}Z(g), C_2 \subseteq \text{int}_{YG}Z(h)$ and $\text{int}_{YG}Z(g) \cap \text{int}_{YG}Z(h) = \emptyset$. Without loss of generality, we assume that $0 \leq g, h$. Clearly, $a \in g^\perp$ and $b \in h^\perp$. Let $0 \leq t \in g^\perp \cap h^\perp$. Then $t \wedge g = 0 = t \wedge h$ which means that $\text{coz}(t) \subseteq Z(g)$ and $\text{coz}(t) \subseteq Z(h)$. Furthermore, $\text{coz}(t) \subseteq \text{int}_{YG}Z(g) \cap \text{int}_{YG}Z(h)$, whence $\text{coz}(t) = \emptyset$. This means that $t = 0$ and so $g^\perp \cap h^\perp = \{0\}$. Consequently, G is lamron. □

Remark 3.16. The above condition has appeared in the context of $C(X)$ with regards to the σ -interpolation property: Theorem 12.13 of [12]. Our proof is modeled after theirs with the appropriate generalization to \mathbf{W} .

4. EXAMPLES

The main conclusion regarding the class of lamron ℓ -groups is that it contains the well-known class of ℓ -groups with stranded primes (and possessing a weak order unit) and the class of complemented ℓ -groups. Moreover, being lamron is even more general than being weakly complemented. In this section, the aim is to show how far apart these four classes can be.

Example 4.1. Let $H = \oplus \mathbb{Z}$ be the set of eventually-zero sequences. This ℓ -group has stranded primes. In particular, every prime subgroup is a minimal prime subgroup. However, H does not possess a weak order unit, and therefore is not lamron. A close inspection yields that $\text{Min}(H)$ is countable and that $\text{Min}(H)^{-1}$ is homeomorphic to the space of naturals with the co-finite topology (a non Hausdorff topology).

Taking $G = \mathbb{Z} \times \overline{H}$, the lexicographic extension of a copy of the integers over H , produces an ℓ -group with a weak order unit (e.g. $(1, \mathbf{0})$) so that $\text{Min}(G)^{-1}$ is homeomorphic to $\text{Min}(H)^{-1}$. Therefore, G is not lamron even though it has a weak order unit. (Clearly, G is not archimedean.)

Example 4.2. Any ℓ -group with a finite number of minimal prime subgroups is lamron. Simply, a finite T_1 -space is discrete, and hence compact and Hausdorff. Of course, such an ℓ -group need not have stranded primes, although it is a complemented ℓ -group. For example, the lexicographic extension of $G = \overline{Z} \times (Z \times Z)$ has the following prime structure:



Remark: the ℓ -groups with a finite number of minimal prime subgroups are known as the ℓ -groups with finite basis.

Example 4.3. If a \mathbf{W} -object is weakly complemented or has \mathbf{W} -stranded primes, then it is lamron. The weakly complemented condition and having \mathbf{W} -stranded primes are incomparable, which this example now explores. We show there is a lamron ℓ -group that does not have \mathbf{W} -stranded primes (for any positive weak order unit), and that there is an example of a lamron ℓ -group which is not weakly complemented.

As has been mentioned a few times now, $C(X)$ has stranded primes if and only if X is an F -space. On the other hand, $C(X)$ is complemented if and only if X is a cozero-complemented space. It is further the case that $C(X)$ is both complemented and has stranded primes precisely when X is basically disconnected. Therefore, if X is an F -space which is not basically disconnected, then $C(X)$ has stranded primes but is not complemented. Moreover, if X is such a space that is not strongly zero-dimensional, then $C(X)$ is not even weakly complemented. An example of such a space is $\beta\mathbb{R} \setminus \mathbb{R}$; $C(\beta\mathbb{R} \setminus \mathbb{R})$ is lamron but not weakly complemented. (In general, it makes sense to call a space X *lamron* if $C(X)$ is a lamron ℓ -group.)

Along the other vein, take $C([0, 1])$. This is a lamron ℓ -group (as it is complemented) with the property that for each weak order unit $0 < f \in C([0, 1])$, $(C([0, 1]), f)$ does not have \mathbf{W} -stranded primes, as follows. Since f is a weak order unit, $\text{coz}(f) \subseteq [0, 1]$ is a dense cozero-set of $[0, 1]$. For any $p \in \text{coz}(f)$, the ideal M_p is a value of f ; M_p is in fact a maximal convex ℓ -subgroup of $C(X)$. However, it is known that there are an infinite number of minimal prime subgroups beneath M_p . Thus, $C([0, 1])$ is an example of a lamron ℓ -group that has no \mathbf{W} -stranded primes representation.

Observe that the appropriate generalization for $C(X)$, with X compact, is that if there is a weak order unit $f \in C(X)$ such that $(C(X), f)$ has \mathbf{W} -stranded primes, then $\text{coz}(f)$ is a dense cozero-set of X consisting of F -points, i.e. O_p is a prime ideal. For more results along this direction we suggest the reader consult [7].

Example 4.4. Here is an example of a $C(X)$ with X metric (and hence cozero-complemented) and a weak order unit $f \in C(X)$ so that $(C(X), f)$ has \mathbf{W} -stranded primes, but $C(X)$ does not have stranded primes.

Let $X = \alpha\mathbb{N}$, the one-point compactification of the naturals. Since X is a metric space, it is cozero-complemented and hence $C(X)$ is a complemented ℓ -group. Since X is not an F -space, $C(X)$ does not have stranded primes. Let $f \in C(X)$ be defined by $f(n) = \frac{1}{n}$ and $f(\alpha) = 0$. Since $\text{coz}(f) = \mathbb{N}$ is a dense subset of $\alpha\mathbb{N}$, f is a weak order unit of $C(X)$. Consider the \mathbf{W} -object $(C(X), f)$. We claim that it has \mathbf{W} -stranded primes.

Recall [16, Exercise 14.G]. The minimal prime subgroups of $C(\alpha\mathbb{N})$ are known to have one of two forms. First, for each $n \in \mathbb{N}$, the ideal $M_n = \{f \in C(\alpha\mathbb{N}) : f(n) = 0\}$ is simultaneously a maximal ideal and a minimal prime subgroup. The remaining minimal prime subgroups are each contained in M_α and are constructed as follows. Let $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$, a free ultrafilter on the naturals. Then

$$P_{\mathcal{U}} = \{f \in C(\alpha\mathbb{N}) : Z(f) \setminus \{\alpha\} \in \mathcal{U}\}$$

is a minimal prime subgroup. Distinct free ultrafilters produce distinct minimal prime subgroups, and this indexes all of the minimal prime subgroups beneath M_α . Let \mathcal{U}, \mathcal{V} be distinct free ultrafilters on \mathbb{N} . There is a subset $S \subseteq \mathbb{N}$ such that $S \in \mathcal{U} \setminus \mathcal{V}$ and $T = \mathbb{N} \setminus S \in \mathcal{V} \setminus \mathcal{U}$. Observe that $\chi_S f \in P_{\mathcal{V}}$, $\chi_T f \in P_{\mathcal{U}}$, and $\chi_S f + \chi_T f = f$. It follows that

$$f \in P_{\mathcal{V}} + P_{\mathcal{U}}.$$

Consequently, by condition (3) of Theorem 3.7, $(C(X), f)$ has **W**-stranded primes.

Remark 4.5. A natural question is whether an ℓ -group with finite rank is lamron. Recall that the rank of a prime $P \in \text{Spec}(G)$ is the cardinality of the set

$$\{Q \in \text{Min}(G) : Q \leq P\}.$$

This cardinality is denoted by $\text{rank}_G(P)$. The prime P is said to have *finite rank* if $\text{rank}_G(P)$ is finite. The supremum of ranks indexed over all prime subgroups is denoted by $\text{rank}(G)$, and the ℓ -group G is said to have *finite rank* if $\text{rank}(G)$ is finite. Clearly, if G has stranded primes, then it is of finite rank and $\text{rank}(G) = 1$.

In the case of **W**, one can speak of the **W**-rank of (G, u) . This would mean the supremum of $\text{rank}_G(V)$ indexed over the Yosida space of u . Interestingly, $C(X)$ has the property that if each value of **1** has finite rank, then $C(X)$ has finite **W**-rank. This result is proved in the setting of uniformly complete f -algebras with identity; this includes objects of the form $C(X)$ (Theorem 4.2 [20]). (Note that the set of minimal prime subgroups beneath M^p is the same as the set of minimal prime subgroups beneath V^p ; see Example 3.2.)

Example 4.6. Here we produce an example of a **W**-object with finite **W**-rank which is not lamron.

Consider Example 5.2 of [20]. Let X be the space obtained by first taking two copies of $\beta\mathbb{N} \setminus \mathbb{N}$ and then gluing these at a non- P -point p . X is an example of a (compact) space with the property that every point of X has rank 1 except for p . The rank of p is 2 and thus X has finite rank but is not an F -space. Moreover, this space is a quasi F -space, that is, every dense cozero-set is C^* -embedded. So, by the next theorem, $C(X)$ is not lamron.

Theorem 4.7. *For a space X , $C(X)$ is an F -space if and only if it is a quasi F -space and lamron.*

Proof. Clearly, an F -space is a quasi F -space and lamron. Conversely, let C_1, C_2 be disjoint cozero-sets. Then by Theorem 3.15, there are zero-sets Z_1, Z_2 such that $C_1 \subseteq \text{int}_X Z_1$, $C_2 \subseteq \text{int}_X Z_2$, and $\text{int}_X Z_1 \cap \text{int}_X Z_2 = \emptyset$. Since X is a quasi F -space, [12, Lemma 11.9] states that $\text{int}_X Z_1$ and $\text{int}_X Z_2$ are completely separated, and so the same holds for C_1, C_2 . It follows that disjoint cozero-sets of X are completely separated, i.e. X is an F -space. \square

In an upcoming paper [7], this theorem will be explored in the context of \mathbf{W} -objects which will avoid any discussion about order completeness and (almost) σ -interpolation properties on uniformly complete vector lattices, the driving topics behind the results in [12].

Example 4.8. Notice then that $C(\alpha D)$ is not a lamron ℓ -group. Here D is an uncountable discrete space and αD is its one-point compactification. The space αD is an example of an almost P -space (hence a quasi F -space) which is not an F -space. Hence αD is not lamron by Theorem 4.7.

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