GENERALIZATIONS OF COMPLEMENTED RINGS WITH APPLICATIONS TO RINGS OF FUNCTIONS.

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ABSTRACT. It is well known that a commutative ring R is complemented (that is, given $a \in R$ there exists $b \in R$ such that $ab = 0$ and $a + b$ is a regular element) if an only if the total ring of quotients of R is von Neumann regular. We consider generalizations of the notion of a complemented ring and their implications for the total ring of quotients. We then look at the specific case when the ring is a ring of continuous real-valued functions on a topological space.

1. Introduction and Preliminaries

In this article we are interested in determining which characteristics of a commutative ring R (with identity) determine when the classical ring of quotients of R has certain prescribed properties. An important and well known example of this states that the total (or classical) ring of quotients of R, denoted $q(R)$, is von Neumann regular if and only if R is a complemented ring. By that we mean, given any $a \in R$, there exists $b \in R$ such that $a+b$ is a regular element of R and $ab = 0$. We consider several generalizations of the complemented property and determine what these properties imposed on R say about $q(R)$. We then apply our work to rings of functions, obtaining characterizations of X, so that the ring of functions $C(X)$ (and its classical ring of quotients) satisfies these properties.

In Section 2 we first develop the aforementioned generalizations of complemented ring (such as weakly and quasi complemented) and relate these to when $q(R)$ satisfy weakened versions of the von Neumann regular property such as weak Baer (i.e., the annihilator of an element is generated by idempotents) and what we call feebly Baer. Another notion that has received much attention of late is that of a clean ring. Recall that a ring R is called *clean* if any element of R can be written as the sum of an idempotent and a unit. The notion of a clean ring was first developed for noncommutative rings by Nicholson in [22]. More recently [19], [5] have examined this condition for commutative rings obtaining among other things, a number of equivalent properties that will prove useful for our work. Since any zero-dimensional ring is clean, it follows that the clean property is a generalization of von Neumann regular. We give a condition on R characterizing when $q(R)$ is

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clean. We also present an example to show that R clean does not necessarily imply that $q(R)$ is clean.

In the last section we apply these results to $C(X)$, the ring of real-valued continuous functions on a Tychonoff space X. We find topological characterizations of X so that either $C(X)$ or the total ring of quotients of $C(X)$, denoted $q(X)$, satisfy the ring-theoretic properties considered in the earlier sections. We also show that certain properties of $C(X)$ are inherited by $q(X)$, properties that do not always transfer from R to $q(R)$ for an arbitrary ring R. In particular, if $C(X)$ is clean, so is $q(X)$.

For the record, an element a of the ring R is called *regular* if it is not a zero-divisor, and R is *reduced* (or *semiprime*) if it has no nonzero nilpotent elements. We let $Max(R)$ and $Min(R)$ denote the collection of maximal ideals and minimal prime ideals of R, respectively. We denote the nilradical of R by $n(R)$, and for any subset $S \subseteq R$, $Ann(S)$ denotes the annihilator ideal of S. When $S = \{a\}$ (or $S = \{a, b\}$) we shall instead write $Ann(a)$ ($Ann(a, b)$). A subset $S \subseteq R$ for which $Ann(S) = 0$ is said to be *dense*. Whenever an ideal contains a regular element we say the ideal is *regular*. Observe that a regular ideal is dense, but the converse is not true in general. Our main reference for this subject is [13]. **We shall assume throughout that all rings are commutative with identity.**

We note that there is some overlap of this paper with a preprint by Burgess and Raphael [5] which recently came to our attention.

2. Weakly complemented rings

In this section we examine a number of generalizations of a complemented ring and their relation to known rings such as weak Baer and PF rings. As in the case of complemented rings, these conditions often imply properties of the total ring of quotients. We begin with a couple of standard lemmas that shall be very useful. Proofs of these results can be found in [13].

Lemma 2.1. *Suppose P is a prime ideal of R. Then P is a minimal prime ideal if and only if whenever $a \in P$ there is an $x \in R \setminus P$ such that $ax \in n(R)$.*

Lemma 2.2. *Suppose R is reduced and let I be a finitely generated ideal of R. Then I is contained in a minimal prime ideal of R if and only if $Ann(I) \neq 0$.*

Definition 2.3. The ring R is said to satisfy *Property A* if whenever I is a finitely generated ideal consisting of zero divisors, then $Ann(I) \neq 0$, i.e. I is not a dense ideal of R. Notice that this means that any finitely generated ideal is dense if and only if it is regular. It is known that R satisfies Property A precisely when $q(R)$ satisfies Property A. Rings satisfying Property (A) are also called *McCoy rings* by some authors.

We say R satisfies *the annihilator condition* (or *the a.c.* for short) if for any $a, b \in R$ there is a $c \in R$ such that $Ann(a, b) = Ann(c)$. As with Property A, R satisfies the a.c. precisely when $q(R)$ satisfies the a.c.

Topologies on $Min(R)$ will play a pivotal role in this article. We recall the Zariski topology (a.k.a the hull-kernel topology) on $Min(R)$. For an ideal $I \leq R$ we let

$$U(I) = \{P \in Min(R) : I \nsubseteq P\}.$$

When $I = aR$ for some $a \in R$, then we instead write $U(a)$ and note that $U(a) = \{P \in Min(R) : a \notin P\}$. The set-theoretic complement of $U(I)$ (resp. $U(a)$) will be denoted by $V(I)$ (resp. $V(a)$). Observe that $U(a) \cap U(b) = U(ab)$ and so the collection $\{U(a) : a \in R\}$ forms a base for the topology known as the *Zariski topology*. The Zariski topology on $Min(R)$ is Hausdorff and zero-dimensional (that is, it has a base of clopen sets). In fact, each set $U(a)$ is clopen in the Zariski topology. Since the Zariski topology on $Min(R)$ is homeomorphic to the Zariski topology on $Min(R/n(R))$ often times we will assume that our rings are reduced. For more information on the Zariski topology on $Min(R)$ the reader should consult [11]. We now recall an important and well-known type of rings.

Definition 2.4. The ring R is called a *von Neumann regular* ring if for each $a \in R$ there is an $x \in R$ such that $axa = a$. It is well-known that a commutative ring with identity is von Neumann regular if and only if every finitely generated ideal is generated by an idempotent. This means, in particular, that von Neumann regular rings are Bézout rings, that is, a ring in which every finitely generated ideal is principal. Notice that every Bézout ring satisfies both Property A and the a.c.

The ring R is called *complemented* if for every $a \in R$ there is a $b \in R$ such that $ab = 0$ and $a + b$ is a regular element of R. It is well known that R is complemented if and only if $q(R)$ is von Neumann regular. Hence a complemented ring is reduced. We record the following result which relates the various properties that we have defined so far.

Theorem 2.5. *Suppose R is a reduced ring. The following statements are equivalent.*

(i) *R is complemented.*
(ii) *$q(R)$ is complemented.*
(iii) *$q(R)$ is a von Neumann regular ring.*
(iv) *$Min(R)$ is compact and R satisfies property A.*
(v) *$Min(R)$ is compact and R satisfies the a.c.*

Proof. The equivalence of (ii), (iii), (iv) and (v) are from [13, Theorem 4.5]. The equivalence of (i) to the other conditions is well known and can be deduced from [13, Corollary 3.3]. ■

We next present one of the main definitions of this article which is a generalization of the notion of a complemented ring.

Definition 2.6. We call the ring R *weakly complemented* if whenever $ab = 0$ there exists finitely generated ideals of R, say I, J, such that $a \in I, b \in J$,

$IJ = 0$, and $I + J$ contains a regular element. A weaker notion is the following. R is called *quasi-complemented* if whenever $ab = 0$ there exists finitely generated ideals of R, say I, J, such that $a \in I, b \in J$, $IJ = 0$, and $I + J$ is a dense ideal of R.

Lemma 2.7. *If R is complemented, then R is weakly complemented. If R is weakly complemented, then R is quasi-complemented.*

Proof. The second statement is obvious since a regular ideal is dense.

Suppose R is complemented and $ab = 0$. Choose $x \in R$ such that $ax = 0$ and $a + x$ is regular. Let $I = aR$ and $J = bR + xR$. Clearly, $IJ = 0$. Since $a + x \in I + J$ it follows that R is weakly complemented. ■

Proposition 2.8. *Suppose R is quasi-complemented. Then R is reduced.*

Proof. Suppose a is nilpotent element. Without loss of generality we assume that $a^2 = 0$. Since R is quasi-complemented there are finitely generated ideals I and J such that $a \in I$, $a \in J$, $IJ = 0$ and $I + J$ is a dense ideal of R. Let $i \in I$ and $j \in J$ be arbitrary. Then $ai = 0 = aj$ and so $a(I + J) = 0$, whence $a = 0$. ■

Remark 2.9. It is obvious that if R satisfies Property A, then R is weakly complemented if and only if R is quasi-complemented. At this point we have been unable to determine whether this also holds for rings satisfying the a.c. We also do not know whether it necessarily follows a weakly complemented ring satisfies Property A, as is the case for complemented rings. Later, we will explain why Quentel's famous example provides us with an example of a quasi-complemented ring which is not weakly complemented. We will also supply some examples of weakly complemented rings which are not complemented.

Definition 2.10. Recall that a ring is called a *weak Baer ring* if the annihilator of every element is generated by an idempotent. A ring R is weak Baer ring if and only if R is complemented and R_P is a domain for each prime ideal P of R. Another equivalent condition is that every principal ideal of R is projective (hence such rings are also called *p.p. rings*). The class of weak Baer rings lies properly between the class of von Neumann regular rings and complemented rings. Therefore, we can add the following two conditions to Theorem 2.5:

(vi) $q(R)$ is a weak Baer ring.
(vii) $q(R)$ is a p.p. ring.

In a natural generalization of the p.p. property, which will be of direct interest to us, a ring R is called a *PF-ring* (or PIF by Matlis [18]) if every principal ideal of R is a flat R-module. Three conditions that are equivalent to being PF are:

(i) R is locally a domain.
(ii) Whenever $a, b \in R$ such that $ab = 0$, then $Ann(a) + Ann(b) = R$.

(iii) For each $a \in R$, $Ann(a)$ is a pure ideal.
For proofs see [18], [10, Theorem 4.2.2] and [3].

In our next result we show that condition (ii) can be modified so as to give a necessary and sufficient condition on R so that $q(R)$ is a PF ring.

Proposition 2.11. *Let R be a commutative ring. Then $q(R)$ is a PF-ring if and only if given $a, b \in R$ such that $ab = 0$, we have $Ann_R(a) + Ann_R(b)$ is a regular ideal of R.*

Proof. Observe that for any $\frac{a}{s}, \frac{b}{t} \in q(R)$, we have $\frac{a}{s}\frac{b}{t} = 0$ precisely when $ab = 0$. Furthermore, $Ann_{q(R)}(\frac{a}{s}) = Ann_{q(R)}(a)$ and $Ann_{q(R)} \cap R = Ann_R(a)$. Therefore, $Ann_{q(R)}(\frac{a}{s}) + Ann_{q(R)}(\frac{b}{t}) = q(R)$ exactly when $Ann_R(a) + Ann_R(b)$ contains a regular element. ■

Definition 2.12. Our definition of weakly complemented leads us to define a *feebly Baer ring* as a ring R for which whenever $a, b \in R$ and $ab = 0$, there is an idempotent $e \in R$ such that $a \in eR$ and $b \in (1-e)R$. Clearly, a feebly Baer ring is weakly complemented, and hence reduced. It is straightforward to check that a weak Baer ring is feebly Baer. Al-Ezeh [1] called a ring R an *almost p.p. ring* if for every $a \in R$, $Ann(a)$ is generated by idempotents. Since an ideal which is generated by idempotents is a pure ideal it follows from condition (iii) above that an almost p.p. ring is a PF-ring.

We are now in a position to state and prove a theorem in the same vein as Proposition 4.2.10 of [10].

Theorem 2.13. *Suppose R is a commutative ring with identity. The following statements are equivalent.*

(i) *R is a feebly Baer ring.*
(ii) *R is an almost p.p. ring.*
(iii) *$q(R)$ is an almost p.p. ring and every idempotent of $q(R)$ is in R.*
(iv) *R is a weakly complemented PF-ring.*
(v) *R is a quasi-complemented PF-ring.*

Proof. (i) ⇔ (ii). Suppose R is a feebly Baer ring. Let $a \in R$ and $b \in Ann(a)$. Since $ab = 0$, there is an idempotent $e = e^2 \in R$ such that $a \in eR$, $b \in (1-e)R$. Notice that $a(1-e) = 0$ and so $b \in (1-e)R \subseteq Ann(a)$. Therefore, $Ann(a)$ is generated by idempotents and so R is an almost p.p. ring.

Conversely, let $ab = 0$. Then $b \in Ann(a)$ which is generated by idempotents so that there is an $e = e^2 \in R$ such that $b \in eR \in Ann(a)$. Multiplying $1 = e + (1-e)$ by a yields $a = a(1-e)$ so that $a \in (1-e)R$. Therefore, R is a feebly Baer ring.

(ii) ⇔ (iii). This is Theorem 4 of [2].

(i) ⇒ (iv). It has already been pointed out that a feebly Baer ring is a weakly complemented PF-ring.

(iv) ⇒ (v). Trivial.

(v) $\Rightarrow$ (i). Let $a, b \in R$ satisfy $ab = 0$. By (v) there are finitely generated ideals I and J, such that $a \in I$, $b \in J$, $IJ = 0$ and $I + J$ is a dense ideal. For each pair of elements $i \in I$ and $j \in J$ we have $Ann(i) + Ann(j) = R$ since R is a PF-ring. Since I and J are finitely, generated a direct computation shows that $Ann(I) + Ann(J) = R$. Furthermore, $Ann(I) \cap Ann(J) = Ann(I+J) = 0$ since $I + J$ is a dense ideal. It follows that (since R is reduced) $Ann(I)$ and $Ann(J)$ are complementary ideals and hence each is generated by an idempotent. Say $Ann(I) = eR$ and $Ann(J) = (1-e)R$. Since $a \in Ann(J)$ and $b \in Ann(I)$ we have shown that R is a feebly Baer ring. ■

Next, we classify when $q(R)$ is a feebly Baer ring, which as we see, is equivalent to $q(R)$ being weakly complemented.

Theorem 2.14. *Suppose R is a commutative ring with identity. The following statements are equivalent.*

(i) *R is a weakly complemented ring.*
(ii) *$q(R)$ is a weakly complemented ring.*
(iii) *$q(R)$ is a feebly Baer ring.*
(iv) *$q(R)$ is an almost p.p. ring.*
(v) *$q(R)$ is a weakly complemented PF-ring.*
(vi) *$q(R)$ is a quasi-complemented PF-ring.*

Proof. (i) $\Rightarrow$ (iii). Suppose R is weakly complemented and let $as^{-1}, bt^{-1} \in q(R)$ satisfy $as^{-1}bt^{-1} = 0$. It follows that $ab = 0$. By hypothesis, there are finitely generated ideals, $I, J \subset R$ such that $a \in I$, $b \in J$, $IJ = 0$, $I + J$ contains a regular element of R, say $r \in I + J$. Let $I' = IT$ and $J' = JT$ and note that I', J' are finitely generated ideals of $q(R)$. Also, $I'J' = 0$ and $T = I' + J'$ (since r is a unit of $q(R)$). It follows that there is an idempotent $e^2 = e \in q(R)$ such that $I' = eq(R)$ and $J' = (1-e)q(R)$, whence (iii) is true.

(iii) $\Rightarrow$ (ii). Clear.

(iii) $\Leftrightarrow$ (iv). See the previous theorem.

(ii) $\Rightarrow$ (i). Suppose $q(R)$ is weakly complemented and let $a, b \in R$ satisfy $ab = 0$. Then this equation also holds in $q(R)$. By hypothesis, there are finitely generated ideals $I', J' \subset q(R)$ for which $a \in I'$, $b \in J'$, $I'J' = 0$, and $I' + J'$ contains a regular element of $q(R)$. This last condition actually implies that $I' + J' = q(R)$. Since $q(R)$ is reduced, it is the direct sum of I' and J'. Hence I', J' are principal ideals of $q(R)$. Thus there exists $x, y \in R$ such that $I' = xq(R)$, $J' = yq(R)$. Clearly $x + y$ is a regular element of R. Then $I = aR + xR$ and $J = bR + yR$ are the ideals of R needed to show that R is weakly complemented.

(iv) $\Leftrightarrow$ (v) $\Leftrightarrow$. This follows from Theorem 2.14. ■

Remark 2.15. During our investigations we considered the following property, which we had hoped would be the appropriate generalization of complementedness. Suppose whenever $a, b \in R$ with $ab = 0$, then there are $x, y \in R$ such that $a \in xR, b \in yR$, $xy = 0$ and $x + y$ is a regular element of R. Observe that a feebly Baer ring satisfies this property with x and y as idempotents. Interestingly, we are able to construct an example of a complemented ring not satisfying this property. It follows that our definition of weakly complemented is the best such generalization.

Example 2.16. Let $T = K[x, y, z, w]$, where K is any field. Let $P_1 = (x, y)$, $P_2 = (y, z)$, and $P_3 = (z, w)$. Clearly these are prime ideals of T. Let $I = P_1 \cap P_2 \cap P_3$ and set $R = T/I$. Then, as R is a reduced noetherian ring, it is complemented (its classical ring of quotients is the direct product of the quotient fields of R/P, where P runs through the minimal primes of R). We claim that R does not satisfy the condition of Remark 2.15. Assume that it does. For any element $h \in T$ we let $\bar{h} = h + I$. Clearly, $yz \in I$ and so $\bar{y}\bar{z} = \bar{0}$. Thus, there should exist $\bar{f}, \bar{g} \in R$, such that $\bar{y} \in \bar{f}R$ and $\bar{z} \in \bar{g}R$, $\bar{f}\bar{g} = \bar{0}$ and $\bar{f} + \bar{g}$ a regular element of R. This means that there exists $r, s \in T$ and $i, j \in I$ such that

$$y = fr + i \text{ and } z = gs + j,$$

with $fg \in I$, and $\bar{f} + \bar{g}$ a regular element of R.

We will say that the indeterminate X *appears in some* $h \in T$, if when we view the canonical representation of h as a sum of monomials, then one of those terms is a nonzero constant times X. We claim that the indeterminate y cannot appear in i. Suppose otherwise, then when we make the substitution $x = z = w = 0$, i becomes a non-zero polynomial in y. However, an element in I (such as i) becomes zero under the substitution $z = w = 0$, which proves the claim. Similarly z cannot appear in j.

Since $y = fr + i$, it follows that y appears in fr. The only way this can happen is if a constant appears in one of f or r and y appears in the other. Similarly for z in g and s. Recall that the set of zero divisors of a reduced ring is the union of the minimal prime ideals (see for instance [13, Corollary 2,4], though when the set of minimal primes is finite, this is easily proved directly). Thus the set of elements of T which are zero-divisors modulo I is the union of the ideals P_1, P_2, and P_3. Furthermore, if a polynomial of T has a nonzero constant term, it cannot be in any P_i. Hence any such polynomial is a regular element modulo I. Since both $\bar{f}$ and $\bar{g}$ are zero divisors of R, it follows that neither f nor g contains a constant, whence both r and s contain constants. But $fr = y - i \in P_1 \cap P_2$. Since r is in neither P_1 or P_2, $f \in P_1 \cap P_2$. Similarly $g \in P_2 \cap P_3$. Thus $f + g \in P_2$, which contradicts the fact that $f + g$ is regular mod I.

We end this section with some questions we have yet been unable to answer.

Questions:

(1) If R is weakly complemented, does R satisfy the a.c.?
(2) If R is quasi complemented and satisfies the a.c., then is R necessarily weakly complemented?
(3) There is a notion of a weak Baer hull, i.e. a weak Baer ring T such that $R \leq T \leq Q(R)$ (where $Q(R)$ is the maximal ring of quotients of R in the sense of Utumi) and T is the smallest weak Baer ring inside of $Q(R)$ containing R. Is there such a thing as a feebly Baer hull?
(4) If R is a feebly Baer ring, then does R satisfy Property A?
(5) If R is a feebly Baer ring, is $R[x]$ a feebly Baer ring? This and its converse are true for weak Baer rings.

3. The inverse topology on $Min(R)$

As we previously mentioned there are other topologies on $Min(R)$ that play a pivotal role in our investigation. Since $V(a) \cap V(b) = V(aR + bR)$ it follows that the collection

$$\{V(I) : I \text{ is a finitely generated ideal of } R\}$$

forms a base for a topology on $Min(R)$ (with the collection $\{V(a) : a \in R\}$ forming a subbase). We call this topology the *inverse topology* on $Min(R)$ and denote it by $Min(R)^{-1}$. Since for any finitely generated ideal I of R we have that $V(I)$ is an open set in the Zariski topology, it follows that the Zariski topology is finer than the inverse topology. We next show that $Min(R)^{-1}$ is always T_1 and compact, whence it is not the case that the two topologies always coincide. Furthermore, it is not always the case that $Min(R)^{-1}$ is Hausdorff (let alone zero-dimensional).

Theorem 3.1. *For any commutative ring R, the inverse topology on $Min(R)$ is compact and T_1.*

Proof. Let $P, Q \in Min(R)$ be distinct minimal prime ideals and let $a \in P \setminus Q$. By Lemma 2.1 there is an $x \notin P$ such that $ax \in n(R)$. It follows that $ax \in Q$ and so $x \in Q \setminus P$. Notice that $P \in V(a) \setminus V(x)$ and $Q \in V(x)$, so $V(x) \not\subset V(a)$. Hence the inverse topology satisfies the T_1-separation axiom.

To show that $Min(R)^{-1}$ is compact let $\mathcal{S} = \{V(a)\}_{a \in S}$ be an arbitrary collection of subbasic open sets with the property that no finite union of elements covers $Min(R)$. We show that $\mathcal{S}$ is not an open cover of $Min(R)^{-1}$. Since $V(a) \cup V(b) = V(ab)$ and $V(a) = Min(R)$ precisely when $a \in n(R)$ it follows that for any finite subset $\{a_1, \cdots, a_n\} \subseteq S$

$$a_1 a_2 \cdots a_n \notin n(A).$$

Let $\hat{S}$ be totality of all finite products of elements belonging to S and observe that $\hat{S}$ is a multiplicative system not containing 0. The usual Zorn's Lemma argument yields the existence of some prime ideal, say P, for which $P \cap \hat{S} = \emptyset$.

Without loss of generality we assume that $P \in Min(R)$. The fact that P and S are disjoint implies that for all $a \in S$, $P \notin V(a)$. Therefore, the collection $\mathcal{S}$ is not an open cover of $Min(R)$. ■

Proposition 3.2. *Suppose R is a reduced ring. The following are equivalent.*

(i) $Min(R) = Min(R)^{-1}$
(ii) $Min(R)$ *is compact.*
(iii) *For every $a \in R$ there is a finitely generated ideal $I \leq Ann(a)$, such that $Ann(aR + I) = 0$.*

Proof. That (ii) and (iii) are equivalent is a result of Quentel ([24]) and its proof can be found in the proof of Theorem 4.3 [13]. Since $Min(R)^{-1}$ is compact so is $Min(R)$, clearly (i) implies (ii). Finally assume (iii). Since the Zariski topology is finer than the inverse topology it suffices to show that any (Zariski) basic open set, say $U(a)$, is of the form $V(I)$ for some finitely generated ideal of R. Given $a \in R$ let I be the ideal of (iii). Since $Ann(a) \subseteq I$, we have $U(a) \subseteq V(I)$. Conversely, let $P \in V(I)$ and suppose that $P \notin U(a)$. Then $aR + I \subseteq P$. Since $Ann(aR + I) = 0$, we have a contradiction to Lemma 2.2, which proves the result. ■

Observe that when $Min(R)$ is compact, $Min(R)^{-1}$ is zero-dimensional (in fact a *boolean space*, that is, a compact zero-dimensional Hausdorff space). The converse is not true. More importantly we can characterize when $Min(R)^{-1}$ is zero-dimensional by using concepts defined in the previous section. To that end we begin with a lemma that classifies the clopen subsets of $Min(R)^{-1}$ (assuming that R is reduced).

Lemma 3.3. *Suppose R is a reduced ring and let $K \subseteq Min(R)^{-1}$. Then K is clopen under the inverse topology if and only if there exist finitely generated ideals I, J such that $V(I) = K$, $IJ = 0$ and $I + J$ is a dense ideal of R.*

Proof. Suppose K is a clopen subset of $Min(R)^{-1}$. Since the inverse topology is compact it follows that K is compact as well. Furthermore, as K is a union of basic open sets we deduce that $K = V(I_1) \cup \cdots V(I_n) = V(I)$ where each I_k is finitely generated. Hence $I = I_1 + \cdots + I_n$ is a finitely generated ideal. Since the complement of K is also clopen we gather that $Min(R) \smallsetminus K = V(J)$ for some finitely generated ideal. Now, observe that $\emptyset = V(I) \cap V(J) = V(I + J)$ so that by Lemma 2.2 $I + J$ is a dense ideal. Finally, $V(IJ) = V(I) \cup V(J) = Min(R)$ which means that $IJ \leq n(R)$, whence $IJ = 0$ since R is reduced. The converse is straightforward. ■

Theorem 3.4. *Suppose R is a reduced ring. Then R is quasi-complemented if and only if $Min(R)^{-1}$ is a compact zero-dimensional Hausdorff space.*

Proof. First suppose that R is quasi-complemented. To show that $Min(R)^{-1}$ has a base of clopen sets it is sufficient to show that given $P \in V(a)$ there is some clopen subset $K \subseteq Min(R)$ for which $P \in K \subseteq V(a)$ (since the collection $\{V(a) : a \in R)\}$ forms a subbase for the inverse topology). For $P \in V(a)$ we have $a \in P$. Since P is a minimal prime and R is reduced, by Lemma 2.1 there is some $b \in R \setminus P$ such that $ab = 0$. Since R is quasi-complemented there are finitely generated ideals, $I, J \subseteq R$ such that $a \in I$, $b \in J$, $IJ = 0$ and $I + J$ is a dense ideal. By Lemma 3.3 we gather that $V(I)$ is a clopen subset of $Min(R)^{-1}$, and since $a \in I$, it follows that $V(I) \subseteq V(a)$. All that is left to show is that $P \in V(I)$. To that end, notice that if $P \notin V(I)$, then $P \in V(J)$, that is, $J \subseteq P$. But then $b \in P$, a contradiction. This forces $P \in V(I)$, and so the space has the desired propreties.

Conversely suppose that $Min(R)^{-1}$ is zero-dimensional and let $a, b \in A$ satisfy $ab = 0$. Since $Min(R)^{-1}$ is T_1 and zero-dimensional, it is a Hausdorff space. This means that disjoint closed subsets of $Min(R)^{-1}$ can be separated by a clopen set. Thus, since $U(a) \cap U(b) = U(ab) = U(0) = \emptyset$, and both $U(a)$ and $U(b)$ are disjoint closed subsets of $Min(R)^{-1}$, there is some clopen subset $K \subseteq Min(R)^{-1}$ such that $U(a) \subseteq K$ and $K \cap U(b) = \emptyset$. By Lemma 3.3 $K = V(J)$ and $Min(R) \setminus K = V(I)$ for some finitely generated ideals I and J of R. Next, $V(I) \subseteq V(a)$ and $V(J) \subseteq V(b)$. A quick check gives us that $V(I) = V(I + aR)$ and $V(J) = V(J + bR)$ and so without loss of generality we assume that $a \in I$, $B \in J$. We leave it to the reader to verify that $IJ = 0$ and $I + J$ is a dense ideal of R. ■

Example 3.5. A well known example by Quentel [24] gives a commutative reduced ring R for which $Min(R)$ is compact, yet R satisfies neither Property A nor the a.c. Furthermore, Quentel's example satisfies $q(R) = R$, R is indecomposable, and (hence) R is not von Neumann regular. The compactness of $Min(R)$ implies that $Min(R) = Min(R)^{-1}$, thus R is quasi-complemented. Since R is indecomposable it cannot be a feebly Baer ring, and hence not a weakly complemented ring.

In our next example we use a minor modification of an example of Burgess & Raphael in [5] to construct a ring that is weakly complemented but not complemented. Later, we will see that rings of continuous functions give a number of additional examples. However, the example below has the advantage of having finite Krull dimension (dimension 2 to be exact).

Example 3.6. For now, consider an arbitrary natural number $n \in \mathbb{N}$. If n is even, we let $R_n = \mathbb{Z}_3[X]$, and if n is odd, say $n = 2k + 1$, then define $R_n = \mathbb{Z}_{p_k}[X]$, where p_k is the k-th prime number. Let $T = \prod_{n \in \mathbb{N}} R_n$. For each $n \in \mathbb{N}$ there is a canonical projection map from $\mathbb{Z}[X]$ to R_n. This family of maps defines a map $\phi : \mathbb{Z}[X] \to T$. Note that since the p_k run through all prime numbers, ϕ is an injection. We denote the image of this map by A. Let R be defined as the ring $R := A + B$, where $B = \sum R_n$. We note that this construction is similar to the $A + B$ construction described by J. Huckaba [13, Section 26] (and as noted earlier is a minor modification of the ring used in [5]).

We claim the ring R just defined is weakly complemented, but not complemented. We will first show that $q(R)$ is not zero dimensional (and so is not von Neumann regular) by showing that there is a prime ideal of R that consists entirely of zero divisors and yet is not a minimal prime ideal.

Observe that since R contains $\sum R_n$ and the rings R_n are all integral domains, an element of R is a zero divisor if and only if it is zero at some coordinate. By composing the map $\phi : \mathbb{Z}[X] \to R$ with the natural projection, we see that $\mathbb{Z}[X] \simeq R/B$. Hence B is a prime ideal of R. We will identify $\mathbb{Z}[X]$ with its image A, and hence view it as a subring of R. The ideal $(3)+B$ is a prime ideal of R, since R modulo the ideal is clearly isomorphic to $\mathbb{Z}_3[X]$. It is also clear that it is not a minimal prime ideal of R. However, since the image of any element of (3) is zero in infinitely many coordinates, it follows that every element of $(3) + B$ is a zero divisor. Thus, R is not complemented.

Next we show that R is weakly complemented. Let $a, b \in R$ such that $ab = 0$. We need to show that there exists finitely generated ideals I and J of R such that $IJ = 0$ and $I+J$ is a regular ideal. Write $a = f+s$ and $b = g+t$, where $f, g \in A$ and $s, t \in B$. We break the proof into two cases. First assume that $f = 0$. Let $e' \in B$ be defined as 1 on the elements of $\mathbb{N}$ where s is nonzero and zero elsewhere. Let $e = 1 - e' \in R$. Thus the ideals $I = (a)$, $J = (b, e)$ are the required ones to show weakly complemented. Clearly a symmetric argument works if $g = 0$. We may now safely assume that both f and g are nonzero. Thus we have $0 = ab = (f+s)(g+t) = fg+ft+sg+st$. However, when viewed as elements of the subring $Z[X]$, we see that $fg \neq 0$. Hence when viewed as an element of R, fg is nonzero at infinitely many points of $\mathbb{N}$. On the other hand $ft + sg + st \in B$ has finite support. Thus the sum $fg+(ft+sg+st)$ cannot be zero, a contradiction which proves the last case and hence the example.

For any ring R, the fundamentals of ring theory tell us that there is a bijection between $Min(R)$ and $Min(q(R))$. This bijection is in fact a homeomorphism with respect to the Zariski topology. Moreover, it is the case that for any $a \in R \subseteq q(R)$, the set of minimal primes of R not containing a maps in bijective correspondence to the minimal primes of $q(R)$ not containing a. It then follows that this bijection is a homeomorphism with respect to the corresponding inverse topologies, which leads us to our last two results of this section.

Proposition 3.7. *Let R be any commutative ring with identity. The standard bijection $\Psi : Min(R) \to Min(q(R))$ is a homeomorphism with respect to the corresponding inverse topologies.*

Corollary 3.8. *Let R be a reduced ring. Then R is quasi-complemented if and only if $q(R)$ is quasi-complemented.*

4. When is $q(R)$ clean?

We have thus far looked at when $q(R)$ satisfies several interesting properties that are shared by all von Neumann regular rings. In particular, we have looked at the notions of weakly complemented, quasi-complemented, weak Baer (p.p.), and feebly Baer (almost p.p.), as well as the connections of these to PF-rings. In this section we investigate when $q(R)$ is one of the following types of rings: a clean ring, a PF-ring, and a *pm*-ring. We begin with the necessary definitions.

Definition 4.1. An element of a ring is said to be *clean* if it is the sum of a unit and an idempotent of the ring. A *clean ring* is a ring in which every element is clean. These rings are also known as exchange rings (in the commutative context). Every clean ring is a *pm*-ring, that is, every prime ideal is contained in a unique maximal ideal. Equivalently, a ring R is a *pm*-ring if whenever $a+b=1$, there are $r,s\in R$ such that $(1+ra)(1+sb)=0$. (Such rings are known as Gelfand rings.) Below we recite the main characterization of clean rings that will be useful for our purposes. For more information as well as a brief history of clean rings we urge the reader to peruse [20].

When studying clean rings the maximal ideal space plays a more important role than the minimal prime ideal space. Recall that for a ring R and $a\in R$, $U_M(a)$ is the set of maximal ideals that do not contain a, and the collection of all of these form a base for the Zariski topology on $Max(R)$. Its set-theoretic complement is denoted $V_M(a)$.

Theorem 4.2. *Let R be a commutative ring with identity. The following statements are equivalent.*

(i) *R is a clean ring.*
(ii) *R is a pm-ring and $Max(R)$ is a zero-dimensional space with respect to the Zariski topology.*
(iii) *For every $a,b\in R$ for which $a+b=1$ there is an idempotent $e^2=e$ such that $e\in aR$ and $1-e\in bR$.*
(iv) *For each $a\in R$ there is an idempotent $e^2=e\in R$ for which $V_M(a)\subseteq U_M(e)$ and $V_M(1-a)\subseteq V_M(e)$.*

Proof. For the equivalency of (i) and (ii) see [14], for (i) and (iii) [22], and for (i) and (iv) [20]. ■

Proposition 4.3. *Let R be a commutative ring. Then $q(R)$ is a pm-ring if and only if for every $a,b\in R$ such that $t=a+b$ is a regular element of R, there are $r,s,u\in R$ with u regular such that $(ut+ra)(ut+sb)=0$.*

Proof. Suppose $q(R)$ is a *pm* ring and let $t=a+b$ be a regular element of R. Then $1=\frac{a}{t}+\frac{b}{t}$ and so there are $\frac{r}{u},\frac{s}{u}\in q(R)$ such that

$$(1+\frac{a}{t}\frac{r}{u})(1+\frac{b}{t}\frac{s}{u})=0.$$

Then in R we have $(ut + ra)(ut + sb) = 0$.

Conversely, suppose $1 = \frac{a}{t} + \frac{b}{t}$ in $q(R)$. Then in R we get $t = a + b$ is a regular and so by hypothesis there are $r, s, u \in R$ with u regular such that $(ut + ra)(ut + sb) = 0$. Dividing by the regular element ut yields

$$(1 + \frac{r}{u}\frac{a}{t})(1 + \frac{s}{u}\frac{b}{t}) = 0$$

showing that $q(R)$ is a *pm* ring. ■

We do not know if the existence of the $u \in R$ in the previous proposition is absolutely necessary. Next is the main result of this section, which characterizes in terms of R as to when $q(R)$ is a clean ring.

Proposition 4.4. *Suppose R is a commutative ring with identity. The following statements are equivalent.*

(i) *$q(R)$ is a clean ring.*
(ii) *For each $a, s \in R$ with s regular, there exist $u, t, e \in R$ with u, t regular so that $e^2 = et$, and $at = s(u + e)$.*
(iii) *For $a, b \in R$ such that $a + b$ is a regular element, there exist $x \in aR$ and $y \in bR$ such that $xy = 0$ and $x + y$ is a regular element of R.*

Proof. (i) $\Leftrightarrow$ (ii). Suppose $q(R)$ is a clean ring and let $a, s \in R$ with s regular. Then $\frac{a}{s} \in q(R)$ and so there exist $f, g \in q(R)$ such that f is invertible, $g^2 = g$, and $\frac{a}{s} = f + g$. Hence without loss of generality there are regular elements $t, u \in R$ as well as $e \in R$ such that $f = \frac{u}{t}$, $g = \frac{e}{t}$, and so

$$\frac{a}{s} = \frac{u}{t} + \frac{e}{t}.$$

The invertibility of f means that u is a regular element of R. That g is idempotent means that $e^2t = et^2$. Since t is regular it follows that $e^2 = et$. Furthermore, we know that $at = s(u + e)$. The converse is similar. Since the argument is completely reversible, we have both directions of this proof.

(i) $\Rightarrow$ (iii). Let $T = q(R)$ and assume that T is clean. Let $a, b \in R$ with $a+b$ regular. Then there exists a unit $s \in T$ such that $as+bs = 1$ (where this takes place in T). Since T is clean, there exists an idempotent $e \in T$ with $e \in bsT = bT$ and $1 - e \in asT = aT$. Since $e \in bT$, we can write $e = w/u$, with $w \in bR$, and $u \in R$ a regular element of R. Similarly, we can write $1 - e = z/t$, with $z \in aR$ and $t \in R$ a regular element of R. Thus, we have $z/t+w/u = 1$. Multiply this equation by ut. Hence $x = (1-e)tu = zu \in aR$ and $y = etu = wt \in bR$ satisfy the desired properties since $x+y = ut$ which is the product of two regular elements and therefore regular.

(iii) $\Rightarrow$ (i). Assume R has the stated property. Let $a', b' \in T$ satisfy the equation $a' + b' = 1$. We can write $a' = \frac{a}{s}$ and $b' = \frac{b}{s}$ for $a, b \in R$ and $s \in R$ a regular element. It follows that $a + b = s$ is a regular element of R. By hypothesis there is an $x' \in aR$ and $y' \in bR$ such that $x'y' = 0$ and $x' + y'$ is regular. Let $x = \frac{x'}{x'+y'}$ and $y = \frac{y'}{x'+y'}$; both elements of T. Then

$x + y = 1$ and $xy = 0$, whence x and y are idempotents of T. Furthermore, $x \in aT = a'T$ and $y \in bT = b'T$. ■

Example 4.5. We now give an example of a Noetherian commutative clean ring R for which $q(R)$ is not clean. Given a ring T and a T-module M, recall the construction of the idealization of T by M denoted $T(+)M$. The underlying set is the cartesian product with addition defined in the usual fashion and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. These two binary operations make $T(+)M$ into a commutative ring with nilpotent elements. In particular, the set $0(+)M$ is a nilpotent ideal of nilpotency class 2. A good reference for this construction is Chapter 27 of [13].

Now let T be a local Noetherian domain of Krull dimension 2 (e.g., let $T = K[x, y]_{(x,y)}$). Then T has at least two prime ideals of height one, say P_1, P_2 (of course, it must have infinitely many such primes). Let $M = T/P_1 \oplus T/P_2$ and let $R = T(+)M$. Since $0(+)M$ is nilpotent, the prime ideals of T are in 1-1 order-preserving correspondence with those of R; in particular the primes of R are of the form $Q(+)M$, where Q is a prime ideal of T. It follows that R is a local ring and whence clean. (It is also the case that T is clean if and only if $T(+)M$ is clean for any T-module M.) We claim that $q(R)$ is not a clean ring.

Since T is a domain and P_1 and P_2 are prime ideals, the element $(t, m) \in R$ is a zero divisor if and only if $t \in P_1 \cup P_2$. Denote the maximal ideal of T by N. It is well known that N is not contained in $P_1 \cup P_2$ (one could use the prime avoidance theorem, though that canon is not necessary, since it is trivial to prove the case for the union of two prime ideals). Thus the unique maximal ideal $N(+)M$ of R is a regular ideal. In particular $(N(+)M)q(R) = q(R)$. On the other hand the height one prime ideals $P_1(+)M$ and $P_2(+)M$ consist entirely of zero divisors. Hence their images in $q(R)$ are maximal ideals. Thus the image of the prime ideal $0(+)M$ in $q(R)$ is contained in two maximal ideals. Since a clean ring is a *pm*-ring, $q(R)$ cannot be clean. ■

Remark 4.6. Burgess and Raphael [5] have constructed a reduced ring R, so that R is clean but $q(R)$ is not. Of course their ring is not Noetherian, since the total ring of quotients of a reduced Notherian ring is a finite product of fields and so von Neumann regular.

5. Applications to $C(X)$

In this section we concern ourselves with rings of real-valued continuous functions on a topological space X. The set of all real-valued continuous functions with pointwise operations is denoted $C(X)$. $C(X)$ is a ring with zero-divisors that possesses special ring-theoretic properties due not only to its ring structure but also to its order structure. $C^*(X)$ is the subring of $C(X)$ consisting of all bounded continuous real-valued functions on X. Our main references for rings of continuous functions are [9] and [8], and for topological spaces [7] and [23].

We first discuss some notation and terminology used in the theory of rings of continuous functions. For $f \in C(X)$, the *zeroset of* f is the set $Z(f) = \{x \in X : f(x) = 0\}$. Its set theoretic complement is denoted $\text{coz}(f)$ and is called the *cozeroset* of f. A set $U \subseteq X$ is a zeroset (resp. cozero set) if there is some $f \in C(X)$ such that $U = Z(f)$ (resp. $\text{coz}(f)$). Observe that every zeroset (resp. cozero set) is a closed (resp. open) subset of X. We shall assume that **all spaces are Tychonoff spaces**, that is, completely regular and Hausdorff. In particular, the collection of cozero sets of X forms a base for the topology of X.

The units of $C(X)$ are precisely those $f \in C(X)$ for which $Z(f) = \emptyset$. The idempotents are the characteristic functions of clopen subsets of X. Consequently, $C(X)$ is an indecomposable ring exactly when X is a connected space. The element $f \in C(X)$ is a regular element of $C(X)$ precisely when $Z(f)$ ($\text{coz}(f)$) is a nowhere dense (dense) subset of X. For any $f \in C(X)$,

$$Ann(f) = \{g \in C(X) : \text{coz}(f) \cap \text{coz}(g) = \emptyset\}.$$

Moreover, for any $f_1 \cdots, f_n \in C(X)$, $Ann(f_1, \cdots, f_n) = Ann(f_1^2 + \cdots + f_n^2)$. Therefore, $C(X)$ is always a McCoy ring which satisfies the a.c. property. (In fact, it is known that $C(X)$ satisfies the countable annihilator condition see [11].)

The prime ideals of $C(X)$ form a root system, yielding that both $C(X)$ and $q(X)$ are always *pm*-rings. (It is customary to write $q(X)$ instead of $q(C(X))$.) Since we are interested in $q(X)$ we take the time to recall the representation theorem of Fine, Gillman, and Lambek [8]. Their theorem states that $q(X)$ can be obtained as the direct limit of partial continuous functions. Specifically,

$$q(X) = \bigcup_{T \in \mathcal{C}_0[X]} C(T)/\sim;$$

the union of rings of the form $C(T)$, where T is an arbitrary dense cozero set of X, modulo the equivalence relation defined by $f \in C(U)$, $g \in C(V)$ are equivalent if and only if the restrictions of f and g to $U \cap V$ are equal.

Remark 5.1. Throughout this remark let $A = C(X)$ or $A = q(X)$. In general, it is not true that A has the property that every maximal ideal contains a unique minimal prime ideal. When this does happen then the set of prime ideals of A forms a disjoint union of maximal chains, and we say that A has *stranded primes*. Moreover, it is known that A has stranded primes if and only if A is a PF-ring. There are examples of Tychonoff spaces X for which $C(X)$ has stranded primes yet $C(X)$ is not a weak Baer ring. Before continuing on we need to recall some topological definitions.

Definition 5.2. Let X be a space and Y a subspace. We say Y is *C-embedded in* X if for every $f \in C(Y)$ there is a $g \in C(X)$ whose restriction to Y is f. If whenever $f \in C^*(Y)$ there is a $g \in C^*(X)$ whose restriction to Y is g, then Y is said to be *C^*-embedded* subspace of X. We let βX denote the Stone-Cech compactification of X, and recall that X is a dense C^*-embedded subspace of βX. Furthermore, $C(\beta X)$ and $C^*(X)$ are naturally isomorphic.

The space X is a *P-space* if every cozero set of X is closed. Since X is Tychonoff this is equivalent to saying that the topology of open subsets of X is closed under countable intersection. A *basically disconnected* space is a space X for which the closure of every cozero set is clopen. A space is called an *F-space* if every cozero set is C^*-embedded. Notice that every P-space is basically disconnected, and every basically disconnected space is an F-space. These spaces are useful in characterizing nice ring theoretic properties of $C(X)$.

Conditions equivalent to a space being a P-space, and to a space being an F-space, are given in [9]. Conditions equivalent to a space being basically disconnected are given in [9] and [10]. Characterizations of when $C(X)$ is complemented are given in [12].

We now give the appropriate topological definition of a topological space X for $C(X)$ to be a weakly complemented ring. We note that since $C(X)$ satisfies Property A, it is weakly complemented precisely when it is quasi-complemented.

Definition 5.3. The space X is called *weakly cozero complemented* if whenever C_1, C_2 are disjoint cozero sets of X there exist disjoint cozero sets D_1, D_2 of X such that $C_1 \subseteq D_1$, $C_2 \subseteq D_2$, and $D_1 \cup D_2$ is a dense subset of X. It follows that X is weakly cozero complemented if and only if whenever Z_1, Z_2 are zerosets such that $X = Z_1 \cup Z_2$, then there are zerosets Y_1, Y_2 such that $Y_1 \subseteq Z_1$, $Y_2 \subseteq Z_2$, $Y_1 \cup Y_2 = X$ and $Y_1 \cap Y_2$ is a nowhere dense set.

Theorem 5.4. *Suppose X is a Tychonoff space. The following statements are equivalent.*

(i) *X is a weakly cozero complemented space.*
(ii) *$C(X)$ is a weakly complemented ring.*
(iii) *$q(X)$ is a weakly complemented ring.*
(iv) *$q(X)$ is a feebly Baer ring.*
(v) *For each $f \in q(X)$, there exists a unit $u \in q(X)$ such that $f = u|f|$.*
(vi) *$q(X)$ is a clean ring and for each $f \in q(X)$ there is an $r \in q(X)$ such that $f = r|f|$.*
(vii) *$q(X)$ is a clean PF-ring and the natural bijection between $Max(q(X))$ and $Min(q(X))^{-1}$ is a homeomorphism.*
(viii) *$q(X)$ is a clean PF-ring.*

Remark 5.5. The proof of this can be found in [15], but the reader should be aware that $C(X)$ is treated as a lattice-ordered group. The main point is that for a semiprime f-ring with bounded inversion, being a weakly complemented (resp. feebly Baer) ring is equivalent to being a weakly complemented (resp. feebly projectable) ℓ-group.

Definition 5.6. To answer when $C(X)$ is a feebly Baer ring we need to recall a definition. The space X is a *strongly zero-dimensional space* if any two disjoint zerosets of X can be separated by a clopen subset. A strongly zero-dimensional space is zero-dimensional, but not conversely. For other equivalent conditions the reader is urged to look at [23].

Theorem 5.7. *Let X be a Tychonoff space. The following statements are equivalent.*

(i) *$C(X)$ is a feebly Baer ring.*
(ii) *$C(X)$ is an almost p.p. ring.*
(iii) *$C(X)$ is a weakly complemented PF-ring.*
(iv) *X is a strongly zero-dimensional F-space.*
(v) *βX is a strongly zero-dimensional F-space.*

Example 5.8. Set $X = \beta\mathbb{N} \setminus \mathbb{N}$. It is known that X is a compact strongly zero-dimensional F-space which is not basically disconnected. It follows that $C(X)$ is a feebly Baer ring, and hence a weakly complemented ring. Since an F-space is cozero complemented if and only if it is basically disconnected we conclude that $C(\beta\mathbb{N} \setminus \mathbb{N})$ is an example of a feebly Baer ring which is not a weak Baer ring. Furthermore, if we let Z be the topological sum of $\beta\mathbb{N} \setminus \mathbb{N}$ and $\mathbb{R}$ then we obtain a weakly complemented ring, $C(Z)$, which is neither complemented nor feebly Baer.

Example 5.9. Let $\mathbb{H} = [0, \infty)$. Then $X = \beta\mathbb{H} \setminus \mathbb{H}$ is an example of a compact connected F-space. It follows that $C(X)$ is a PF-ring which is not weakly complemented.

The articles [12] and [17] gather a collection of results regarding cozero complemented spaces. Our goal is to prove the analogous results for weakly cozero complemented spaces. We include the following definition and two lemmas from [12] for the sake of completeness.

Definition 5.10. Suppose $X \subseteq T$ satisfies the property that for each $f \in C(X)$ there is a $g \in C(T)$ such that $cl_X(int_X Z(f)) = X \cap cl_T(int_T Z(g))$, then X is said to be *$\mathcal{Z}^\sharp$-embedded in T*. It is easy to see that a C^*-embedded subspace is $\mathcal{Z}^\sharp$-embedded.

Lemma 5.11. *(*[12, Lemma 2.2]*) If X is dense in T and V is open in T, then $cl_T(V \cap X) = cl_T V$.*

Lemma 5.12. *(*[12, Lemma 2.3]*) Let X be a subspace of T that is either dense or open. Then the following are equivalent:*

(i) *X is $\mathcal{Z}^\sharp$-embedded in T.*
(ii) *For each $C \in \operatorname{coz}(X)$ there is a $V \in \operatorname{coz}(T)$ such that $cl_X C = X \cap cl_T V$.*

Theorem 5.13. *Suppose X is a dense $\mathcal{Z}^\sharp$-embedded subspace of T. X is weakly cozero-complemented if and only if T is weakly cozero-complemented.*

Proof. First, suppose T is weakly cozero-complemented and let C_1 and C_2 be disjoint cozerosets of X. By Lemma 5.12 there exist $V_1, V_2 \in Coz(T)$ such that $cl_X C_i = X \cap cl_T V_i, i = 1, 2$. We claim that $V_1 \cap V_2 = \emptyset$. Since X is dense in T, if $V_1 \cap V_2 \neq \emptyset$, we have $V_1 \cap V_2 \cap X$ is nonempty and hence $(V_1 \cap X) \cap (cl_T V_2 \cap X) = (V_1 \cap X) \cap cl_X C_2$ is nonempty. Now because $V_1 \cap X$ is open in X, we have $(V_1 \cap X) \cap C_2 \neq \emptyset$ and so $(cl_T V_1 \cap X) \cap C_2 = cl_X C_1 \cap C_2 \neq \emptyset$. It follows that $C_1 \cap C_2$ is nonempty, which is a contradiction. Hence $V_1 \cap V_2 = \emptyset$.

By hypothesis there exist disjoint cozerosets O_1, O_2 of T whose union is dense in T with $V_i \subseteq O_i, i = 1, 2$. Then $O_1 \cap X$ and $O_2 \cap X$ are disjoint cozerosets of X. For $i = 1, 2$ let $W_i = C_i \cup (O_i \cap X)$, whence $W_1, W_2 \in \text{coz}(X)$. We claim that $W_1 \cap W_2 = [C_1 \cup (O_1 \cap X)] \cap [C_2 \cup (O_2 \cap X)]$ is empty. To see this,

$$\emptyset = O_2 \cap cl_T O_1 \cap X \supseteq O_2 \cap cl_T V_1 \cap X = O_2 \cap cl_X C_1 \supseteq O_2 \cap C_1$$

and so $O_2 \cap C_1 = \emptyset$. Similarly, $O_1 \cap C_2 = \emptyset$, whence $W_1 \cap W_2 = \emptyset$.

It remains to be seen that $W_1 \cup W_2$ is dense in X. Suppose $p \in X$ and U is a neighbourhood of p in T. Because $O_1 \cup O_2$ is dense in T, we have $U \cap (O_1 \cup O_2)$ is nonempty. Then $U \cap (O_1 \cup O_2) \cap X$ is nonempty since X is dense in T, so

$$(U \cap X) \cap [(O_1 \cup O_2) \cap X] = (U \cap X) \cap [(O_1 \cap X) \cup (O_2 \cap X)] \neq \emptyset.$$

Since $O_i \cap X \subseteq C_i \cup (O_i \cap X) = W_i, i = 1, 2$, we have

$$\emptyset \neq (U \cap X) \cap [(O_1 \cap X) \cup (O_2 \cap X)] \subseteq (U \cap X) \cap (W_1 \cup W_2),$$

therefore $W_1 \cup W_2$ is dense in X.

Conversely, assume X is weakly cozero-complemented, and let C_1, C_2 be disjoint cozerosets of T. Then $C_1 \cap X$ and $C_2 \cap X$ are disjoint cozerosets of the weakly cozero-complemented space X, so there exist $V_1, V_2 \in \text{coz}(X)$ with $V_1 \cap V_2 = \emptyset, C_1 \cap X \subseteq V_1, C_2 \cap X \subseteq V_2$ and $V_1 \cup V_2$ dense in X. By Lemma 5.12 there exist $O_1, O_2 \in \text{coz}(T)$ such that for $i = 1, 2$ we have $cl_X V_i = X \cap cl_T O_i$. Let $W_i = O_i \cup C_i, i = 1, 2$. We claim that O_1 and O_2 are disjoint. If not, then $O_1 \cap O_2 \cap X \neq \emptyset$ since X is dense in T. Now, $(cl_T \cap X) \cap (O_2 \cap X) = cl_X V_1 \cap (O_2 \cap X) \neq \emptyset$, hence $V_1 \cap (O_2 \cap X) \neq \emptyset$. It follows that $V_1 \cap (cl_T O_2 \cap X) = V_1 \cap cl_X V_2 \neq \emptyset$ and so $V_1 \cap V_2 \neq \emptyset$, a contradiction. Thus O_1 and O_2 are disjoint.

Next we will show that $W_1 \cap W_2 = (O_1 \cup C_1) \cap (O_2 \cup C_2) = \emptyset$. We just showed $O_1 \cap O_2 = \emptyset$, so we need to show that $O_1 \cap C_2$ and $O_2 \cap C_1$ are empty. If $O_1 \cap C_2$ is nonempty, then $O_1 \cap C_2 \cap X \neq \emptyset$. So

$$(O_1 \cap X) \cap cl_X V_2 = (O_1 \cap X) \cap (X \cap cl_T O_2) \neq \emptyset$$

implies that $O_1 \cap cl_T O_2 \cap X \neq \emptyset$. By Lemma 5.11 we see that

$$O_1 \cap cl_T O_2 \cap X = O_1 \cap cl_T (O_2 \cap X) \cap X \neq \emptyset,$$

and hence $O_1 \cap O_2 \cap X \neq \emptyset$, a contradiction. Thus, $W_1 \cap W_2 = \emptyset$.

Finally, we need to show that $W_1 \cup W_2$ is dense in T. To do this, we will show that $O_1 \cup O_2$ is dense in T. By Lemma 5.11 we have $cl_X(O_i \cap X) = X \cap cl_T O_i = cl_X V_i$ for $i = 1, 2$. So

$$X \cap cl_T(O_1 \cup O_2) = X \cap [cl_T O_1 \cup cl_T O_2] = cl_X V_1 \cup cl_X V_2 = cl_X(V_1 \cup V_2) = X.$$

Hence,

$$cl_T(W_1 \cup W_2) \supseteq cl_T(O_1 \cup O_2) \supseteq cl_T X = T$$

implies that $W_1 \cup W_2$ is dense in T. ■

Corollary 5.14. *A space X is weakly cozero-complemented if and only if βX is weakly cozero-complemented.*

Notice that if X is a P-space, then $C(X)$ is von-Neumann regular and so, in this case, $C(X) = q(X)$. A space for which $C(X) = q(X)$ is called an *almost P-space.*

Theorem 5.15 ([16])**.** *Let X be a Tychonoff space. The following statements are equivalent.*

(i) *X is an almost P-space.*
(ii) *Every nonempty zeroset of X has nonempty interior.*
(iii) *Every dense cozero set of X is C-embedded.*
(iv) *X has no proper dense cozero sets.*

Next, we look at when $q(X)$ is a clean ring. When this happens we call X a *q-clean space.* We begin by showing that $C(X)$ forms a nice example where $C(X)$ being a clean ring is enough to force $q(X)$ to be a clean ring. (See Example 4.5.) First, we give a topological characterization of when $C(X)$ is a clean ring. (Recall that a ring is called *almost clean* if every element can be written as the sum of a regular element and an idempotent.)

Theorem 5.16. *(*[19, Theorem 13] *and* [4, Theorem 2.5]*) The following statements are equivalent for the Tychonoff space X.*

(i) *$C(X)$ is a clean ring.*
(ii) *$C^*(X)$ is a clean ring.*
(iii) *X is strongly zero-dimensional.*
(iv) *βX is zero-dimensional.*
(v) *$C(X)$ is almost clean.*
(vi) *$C^*(X)$ is almost clean.*

Proposition 5.17. *For a Tychonoff space X, the following are equivalent.*

(i) *$q(X)$ is a clean ring.*
(ii) *Given two zero sets U and V of X such that $U \cap V$ is nowhere dense, there exist zero sets U', V' such that $U \subseteq U'$, $V \subseteq V'$ and $U' \cap V'$ is nowhere dense and $U' \cup V' = X$.*
(iii) *For any pair of cozero sets $C_1, C_2 \subseteq X$ whose union is dense, there is a pair of disjoint cozero sets, say D_1, D_2 for which $D_i \subseteq C_i$ $(i = 1, 2)$ and whose union is dense.*

Proof. Since (ii) are (iii) are obviously equivalent it is enough to show that (i) implies (ii) and vice-versa. To that end, suppose that (i) is true, that is, suppose that $q(X)$ is clean. Let U and V be zero sets such that $U \cap V$ is nowhere dense. Let $0 \leq f, g \in C(X)$ satisfy $U = Z(f)$ and $V = Z(g)$. Since $Z(f+g) = U \cap V$ is nowhere dense it follows that $f+g$ is a regular element of $C(X)$. By Proposition 4.4, there exists $h \in fC(X)$ and $k \in gC(X)$ such that $h+k$ is regular and $hk = 0$. Let $U' = Z(h)$ and $V' = Z(k)$. Then U' and V' have the desired properties.

For the converse assume that X has the condition on any pair of zero sets whose intersection is nowhere dense. We show that $C(X)$ satisfies the conditions of Proposition 4.4. Let $f, g \in C(X)$ with $f+g$ regular. Thus $Z(f) \cap Z(g) \subseteq Z(f+g)$ is nowhere dense. Let $U = Z(f)$ and $V = Z(g)$. Let U' and V' be the zero sets given in the hypothesis of this direction. Say $U' = Z(h)$ and $V' = Z(k)$. Then $fh \in fC(X)$ and $gk \in gC(X)$. Furthermore, $Z(fh) = U'$ and $Z(gk) = V'$ (since $Z(fh) = U \cap U' = U'$). Thus $Z((fh)^2 + (gk)^2) = U' \cap V'$ is nowhere dense and so $(fh)^2 + (gk)^2$ is regular. Moreover, $Z[(fh)^2(gk)^2] = U' \cup V' = X$. Thus $(fh)^2(gk)^2 = 0$. Hence by Proposition 4.4, the total ring of quotients of $C(X)$ is clean. ■

It is not difficult to see, and is well-known (see, for example, [21, 4.9]), that in a strongly zero-dimensional space, every finite cozero-cover has a refinement consisting of a partition by clopen sets. Combining this fact with Proposition 5.16 and 5.17 immediately gives the following.

Corollary 5.18. *If $C(X)$ is clean, then $q(X)$ is clean.*

Of course the converse of Corollary 5.18 is not true. For eaxample given any infinite connected metric space X (for example, $X = \mathbb{R}$), $q(X)$ is clean but $C(X)$ is not, since X is not strongly zero-dimensional.

We now give another characterization of when $q(X)$ is a clean ring. This characterization models (iv). of Theorem 4.2.

Definition 5.19. *We say the space X satisfies (q) if for every $f \in C(X)$ there are disjoint cozero sets $C_1, C_2 \subseteq X$ such that $U = C_1 \cup C_2$ is a dense subset of X, $U \cap Z(f) \subseteq C_1$, and $Z(1-f) \cap U \subseteq C_2$*

Lemma 5.20. *X satisfies (q) if and only if for each $f \in C(X)$, f is a clean element of $q(X)$.*

Proof. First suppose X satisfies (q), and let $f \in C(X)$. For $A \subseteq X$, let χ_A be the characteristic function of A. There are disjoint cozero sets C_1, C_2 of X such that $U = C_1 \cup C_2$ is dense in X, $U \cap Z(f) \subseteq C_1$, and $U \cap Z(1-f) \subseteq C_2$. Now let $u = (f-1)\chi_{C_1} + f\chi_{C_2}$ and $e = \chi_{C_1}$. Then $u, e \in C(U)$ are defined on the dense cozero set U of X and hence are elements of $q(X)$. It is straightforward to check that $f = u + e$ on U, and clearly e is an idempotent of $q(X)$. We need only show that u is a unit of $q(X)$. Observe that $U \cap Z(f) \subseteq C_1$ implies that $f\chi_{C_2}$ is nonzero on C_2 and $U \cap Z(1-f) \subseteq C_2$

implies that $f\chi_{C_2}$ is nonzero on C_1. It follows that $u(x) \neq 0$ for any $x \in U$, i.e. u is a unit of $q(X)$. Therefore f is a clean element of $q(X)$.

Conversely, suppose $f = u + e$ on a dense cozero set C of X where u is a unit of $q(X)$ and $e = e^2 \in q(X)$. Let $C_1 = C \cap coz(e) \cap coz(u^-)$, where $u^-(x) = \min\{u(x), 0\}$, and let $C_2 = C \smallsetminus C_1$. Observe that since e is an idempotent and u is a unit, $coz(e)$ and $coz(u^-)$ are both clopen subsets of C. Hence C_1 and C_2 are disjoint cozero sets of X with $C_1 \cup C_2 = C$ dense in X, $C \cap Z(f) \subseteq C_1$, and $C \cap Z(1 - f) \subseteq C_2$. ■

If Y is a subset of X, then the restriction map $f \to f \restriction Y$ maps $C(X)$ to $C(Y)$. If Y is dense in X, this map is a monomorphism, and if, in addition, Y is a cozero set of X, this map induces an isomorphism from $q(X)$ to $q(Y)$, as described in the following lemma.

Lemma 5.21. *Suppose Y is a dense cozero set of X. Then $q(X)$ is naturally isomorphic to $q(Y)$.*

Proof. Let $r\colon C(X) \to C(Y)$ be the restriction map. Then r extends to an isomorphism from $q(X)$ to $q(Y)$, it is enough to show that if f is a regular element of $C(X)$, then $r(f)$ is a regular element of $C(Y)$. But f is a regular element of $C(X)$ if and only if the zero-set of f is nowhere dense in X, and if f has nowhere dense zero set in X, then $Z(f \restriction Y)$ is nowhere dense in Y. ■

Theorem 5.22. *For the space X, $q(X)$ is a clean ring if and only if each dense cozero set of X satisfies property (q).*

Proof. Suppose $q(X)$ is clean, and let Y be a dense cozero set of X. By the previous lemma, it suffices to show that for each $f \in C(Y)$, f is a clean element of $q(Y)$. Since f is defined on a dense cozeroset Y of X, we can think of f is an element of the clean ring $q(X)$. So there exist $u, e \in q(X)$ such that u is a unit of $q(X)$, $e^2 = e$, and $f = u+e$ on some dense cozero set Z of X. Note that $e = \chi_K$ for some clopen subset K of Z. Hence $e' = \chi(K \cap Y)$ is an idempotent element of $q(Y)$. Since $f \in C(Y)$, we have

$$u|_{Z\cap Y}(x) = \begin{cases} f(x) - 1, & \text{if } x \in K \cap Z \cap Y; \\ f(x), & \text{if } x \notin K \cap Z \cap Y. \end{cases}$$

is an element of $C(Z \cap Y)$ and hence $u|_{Z\cap Y}$ is a unit of $q(Y)$. It follows that $f = u + e'$ on the dense cozeroset $Z \cap Y$ of X, and so f is a clean element of $q(X)$.

Now suppose each dense cozero set of X satisfies (q). To show $q(X)$ is clean let $f \in q(X)$. Then by the representation theorem for q(X) we can assume that $f \in C(Y)$ for some dense cozero set $Y \subseteq x$. By hypothesis, Y satisfies property (I) and so by Lemma 5.20 $f \in C(Y)$ is a clean element of $q(Y)$. But $q(Y) = q(X)$ and so f is a clean element of $q(X)$. ■

We use our results to construct a space which is q-clean yet not weakly complemented.

Proposition 5.23. *Let M be any compact metric space and let $\alpha D = D \cup \{\infty\}$ be the one-point compactification of an uncountable discrete space D. The space $X = M \times \alpha D$ is q-clean.*

Proof. Let $Z_1 = Z(f)$ and $Z_2 = Z(g)$ be zero-sets of X such that $Z_1 \cap Z_2$ is nowhere dense. It is not too hard to show the following:

($*$) There is a countable subset C of D such that if $y_1, y_2 \notin C$, then $f(x, y_1) = f(x, y_2)$ and $g(x, y_1) = g(x, y_2)$ for all $x \in M$.

(In other words, the function f restricted to horizontal strips corresponding to y-coordinates not in C is essentially always the same, and the same is true for g.)

Now consider the set $H = C \cup \{\infty\}$. Then because of ($*$), f and g can be viewed as well-defined functions on the metric space $Y = M \times H$, and so Z_1 and Z_2 can be viewed as being zero-sets of Y with nowhere dense intersection. Since a metric space is complemented, there exist zero-sets $Z_1^{\#}$ and $Z_2^{\#}$ of Y containing Z_1 and Z_2 respectively whose union is Y and whose intersection is nowhere dense. Now extend $Z_1^{\#}$ and $Z_2^{\#}$ to zero-sets Z_1^* and Z_2^* of X by making them constant on each of the sets $\{x\} \times (K \setminus C)$ for $x \in M$. ■

Proposition 5.24. $[0, 1] \times \alpha D$ *is not weakly cozero-complemented.*

Proof. Select $f, g \in C(\alpha D)^+$ such that $f \wedge g = 0$, $\alpha \notin coz(f) \cup coz(g)$, and both $coz(f)$ and $coz(g)$ are countably infinite. Define $F, G \in C([0, 1] \times \alpha D)^+$ by $F((r, x)) = f(x)$ and $G((r, x)) = g(x)$ for each $(r, x) \in [0, 1] \times \alpha D$. Assume, to get a contradiction, that $[0, 1] \times \alpha D$ is weakly cozero-complemented. Then there exist disjoint cozerosets C_1, C_2 of $[0, 1] \times \alpha D$ with $coz(F) \subseteq C_1$, $coz(G) \subseteq C_2$, and such that $C_1 \cup C_2$ is dense in $[0, 1] \times \alpha D$.

First we will show that $(r, \alpha) \notin (C_1 \cup C_2)$ for any $r \in [0, 1]$. If $(r, \alpha) \in C_1$, for instance, then $C_1 \cap (\{r\} \times \alpha D)$ must be a cofinite subset of $\{r\} \times \alpha D$ since it is a cozeroset which contains α. But $C_2 \cap (\{r\} \times \alpha D)$ is an infinite subset of $\{r\} \times \alpha D$ disjoint from $C_1 \cap (\{r\} \times \alpha D)$. This is a contradiction. Hence $(r, \alpha) \notin (C_1 \cup C_2)$ for any $r \in [0, 1]$.

Now let $\{q_n\}_{n \in \mathbb{N}}$ denote the set of rational numbers in $[0, 1]$, and let $\pi : [0, 1] \times \alpha D \to \alpha D$ be the continuous projection map. For each $n \in \mathbb{N}$ let $A_n = \pi[(\{q_n\} \times \alpha D) \cap C_1]$ and $B_n = \pi[(\{q_n\} \times \alpha D) \cap C_2]$, and finally let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. By the first proposition, A_n and B_n are countable for all n, so A and B are also countable subsets of αD. Then $C_1 \cup C_2 \subseteq [0, 1] \times (A \cup B)$ where $[0, 1] \times (A \cup B)$ is not dense in $[0, 1] \times \alpha D$, which is a contradiction. Therefore, $[0, 1] \times \alpha D$ is not weakly cozero-complemented.■

We end the article with a conjecture. We have been unable to prove it but it seems likely that it is so.

Conjecture. For any space X, $q(X)$ is a PF-ring if and only if it is a Bézout ring.

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