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RINGS OF QUOTIENTS OF $C(X)$ INDUCED BY POINTS

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Abstract. We study two rings of quotients of $C(X)$, the ring of continuous functions on the Tychonoff space X. The first, $\mathfrak{F}(X)$, is the ring of quotients induced by the filter of ideals consisting of dense finite intersections of fixed maximal ideals. The second, $C[\mathcal{F}]$, is the ring of quotients induced by the filter of dense cofinite subspaces of X. After some preliminary information we explicitly describe in §2 and §3 the constructions of these rings of quotients. In the third section, we use $\mathfrak{F}(X)$ and $C[\mathcal{F}]$ to define and study the class of h-points and h-spaces. In particular, we show that C-spaces and P-spaces are h-spaces. In the last section we construct an ideal of $C(X)$ which will be used to give an ideal theoretic characterization of h-points.

1. Preliminaries

For a topological space X, the ring $C(X)$ consisting of all real-valued continuous functions with domain X has several natural extensions. In [5], the authors investigated the classical and maximal rings of quotients of $C(X)$. They obtained representation theorems for these two rings as direct limits induced by filters of dense open sets. Our aim is to continue the investigations begun in [12] on rings of quotients of $C(X)$ induced by the filter $\mathfrak{F}$ of finite intersections of dense fixed maximal ideals. We shall make these and other concepts precise. For those readers familiar with [6] and the construction of the maximal ring of quotients of a ring, skipping this section is an appropriate action. The reader should refer to [3] for any undefined topological terms.

If $f \in C(X)$, we write

$$Z(f) = \{x \in X : f(x) = 0\}, \qquad \operatorname{coz}(f) = \{x \in X : f(x) \neq 0\}$$

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and call these the *zeroset* and *cozeroset* of f respectively. If $I \leqq C(X)$ is an ideal then we use

$$Z(I) = \bigcap \{ Z(f) : f \in I \}, \qquad \operatorname{coz}(I) = \bigcup \{ \operatorname{coz}(f) : f \in I \}$$

for the *zeroset* and *cozeroset* of I respectively. Observe that each zeroset of an ideal is closed, and each cozeroset of an ideal is open. When we speak of a subset of X as being a cozeroset, we mean one of the form $\operatorname{coz}(f)$ for some $f \in C(X)$. In general, every open subset of X is the form $\operatorname{coz}(I)$ for some ideal, but not every open subset is a cozeroset. As is customary when discussing $C(X)$, we assume that all topological spaces are Tychonoff, that is, Hausdorff and completely regular.

The set of all maximal ideals of $C(X)$ is denoted by $\operatorname{Max}\big(C(X)\big)$, and when endowed with the hull-kernel topology it is naturally homeomorphic to βX, the Stone–Čech compactification of X. There are two kinds of maximal ideals: fixed and free. A *fixed* maximal ideal is of the form

$$M_p = \{ f \in C(X) : f(x) = 0 \}$$

for some $p \in X$. A *free* maximal ideal is of the form

$$M^p = \{ f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f) \},$$

for some $p \in \beta X - X$. Another ideal which we will make use of is

$$O^p = \{ f \in C(X) : \operatorname{cl}_{\beta X} Z(f) \text{ is a neighbourhood of } p \}$$

where $p \in \beta X$. As above if $p \in X$, then we write O_p. Finally, we denote the collection of all prime ideals of $C(X)$ by $\operatorname{Spec}\big(C(X)\big)$.

We review some basic facts on rings of quotients.

RINGS OF QUOTIENTS 1.1. Suppose A is a commutative ring with identity and R is a ring extension of A. R is called a *ring of quotients of* A if for every $s, t \in R$ with t nonzero there exists an $a \in A$ such that $sa \in A$ and $ta \neq 0$. An ideal I of A is said to be *dense* provided that $a = 0$ whenever $aI = 0$. It follows that R is a ring of quotients of A if and only if for every nonzero $r \in R$, the set

$$r^{-1}A = \{ a \in A : ra \in A \}$$

is a dense ideal of A and $r(r^{-1}A) \neq 0$. We let $\mathfrak{D}$ denote the collection of all dense ideals of A. Observe that the fixed maximal ideal M_p of $C(X)$ is dense precisely when p is a non-isolated point of X.

It is known that for each commutative ring A with identity there exists a largest ring of quotients of A called the *maximal (or complete) ring of quotients of* A. We denote this ring by $Q(A)$ and observe that rings of quotients of A are precisely those ring extensions of A satisfying $A \leqq B \leqq Q(A)$. We review the construction of $Q(A)$. First, here are some properties of dense ideals of A:

(1) A is dense.
(2) If D is dense and $D \leqq D'$, then D' is dense.
(3) If D and D' are dense ideals, then so are DD' and $D \cap D'$.

For an ideal D of A, Hom (D) denotes the set of all A-module homomorphisms from D into A. We let

$$T = \bigcup \left\{ \operatorname{Hom}(D) : D \in \mathfrak{D} \right\}.$$

Observe that T is a directed union. We define an equivalence relation on T by saying $f : D \to A$ is equivalent to $g : D' \to A$ where D and D' are dense ideals of A if the restrictions of f and g to $D \cap D'$ agree. $Q(A)$ is defined to be $T/\sim$. For a more detailed description of $Q(A)$ and a proof of the fact that $Q(A)$ is the maximal ring of quotients, we suggest the reader check §2.3 of [9].

Recall that the classical ring of quotients of A, $q(A)$, is obtained in a similar fashion. $\mathfrak{D}_0$ denotes the subcollection of $\mathfrak{D}$ consisting of dense ideals which contain a regular element of A (that is, a non divisor of zero). $q(A)$ is the subring of $Q(A)$ consisting of those $f : D \to A$ where $D \in \mathfrak{D}_0$ (modulo the equivalence relation).

REMARK 1.2. In the case that $A = C(X)$, the authors of [5] obtained representations of the classical and maximal rings of quotients of $C(X)$ in terms of continuous functions defined on dense open subsets of X. In particular, they showed that for each dense ideal D of $C(X)$

$$\operatorname{Hom}(D) \subseteqq C\bigl(\operatorname{coz}(D)\bigr).$$

Let $f \in \operatorname{Hom}(D)$ be given. For $x \in \operatorname{coz}(D)$, choose $d \in D$ for which $d(x) \neq 0$, and define

$$g(x) = \frac{f(d)(x)}{d(x)}.$$

This definition is independent of which $d \in D$ is chosen and thus $g \in C\bigl(\operatorname{coz}(D)\bigr)$. Since $f(d) = g \cdot d$ for any $d \in D$, it follows that $f = g$.

Let $\mathcal{G}(X)$ denote the set of dense open subsets of X and

$$G = \bigcup \left\{ C(U) : U \in \mathcal{G}(X) \right\}.$$

An equivalence relation on G is obtained by defining $f \in C(U)$ and $g \in C(V)$ to be equivalent if they agree on $U \cap V$. Define $Q(X) = G/\sim$. It then follows that $Q\big(C(X)\big) \leqq Q(X)$. It is shown in [5] that $Q(X)$ is a ring of quotients of $C(X)$ so that in fact $Q\big(C(X)\big) = Q(X)$. In particular, they showed that each $f \in C(U)$ ($U \subseteqq X$ dense and open) may be viewed as a fraction for a suitable dense ideal.

In a similar fashion, $q\big(C(X)\big) = q(X)$ where $q(X)$ is the union of the collection $\big\{ C(U) : U \in \mathcal{C}(X) \big\}$ modulo said equivalence relation. Here $\mathcal{C}(X)$ denotes the collection of dense cozero sets of X. Due to these facts we shall denote the maximal and classical ring of quotients of $C(X)$ by $Q(X)$ and $q(X)$, respectively.

REMARK 1.3. It should be evident that whenever $F \subseteqq X$, then $\mathrm{Hom}\,(\cap_{x \in F} M_x)$ is the collection of those $f \in C(X \setminus F)$ for which fg may be continuously extended to each point of the subset F for all $g \in \cap_{x \in F} M_x$. For further elaboration on this point the reader should consult [5]. For a subset $F \subseteqq X$ we will use the following notation: $M_F = \cap_{x \in F} M_x$.

A natural question is to determine, in terms of the space X, the coincidence of the following three rings: $C(X)$, $q(X)$, $Q(X)$.

PROPOSITION 1.4 [5]. *Let X be a Tychonoff space.*

(i) $q(X) = Q(X)$ *if and only if every continuous function on a dense open set agrees with a continuous function on a dense cozeroset.*

(ii) $C(X) = q(X)$ *if and only if every zeroset of X has nonempty interior, i.e. X is an almost P-space.*

(iii) $C(X) = Q(X)$ *if and only if X is an extremally disconnected P-space.*

NOTES. The definition of a ring of quotients is due to Utumi and as mentioned [9] is an excellent resource on the theory of rings of quotients. Examples of P-spaces may be found in [6], in particular η_1-sets are P-spaces. Levy [10] investigated the class of almost P-spaces. Spaces which satisfy condition (i) of Proposition 1.4 are called *strongly fraction dense* and they include perfectly normal spaces as well as extremally disconnected spaces. We urge the reader to look at [8] for more information on this interesting class of spaces as well as the more general class of fraction dense spaces. If one assumes that measurable cardinals do not exist then the spaces that satisfy (iii) are the discrete ones (see 12H of [6]).

2. The filter $\mathfrak{F}$

As noted in Remark 1.2, the representation theorem of [5] shows that the two direct limit constructions associated to $\mathfrak{D}$ and $\mathcal{G}(X)$ both give rise to the maximal ring of quotients of $C(X)$. Likewise, the pair $\mathfrak{D}_0$ and $\mathcal{C}(X)$ induces the classical ring of quotients $q(X)$. We now define a third pair which will induce a pair of rings of quotients of $C(X)$, and their relationship will be investigated.

We let $\mathfrak{F}$ denote the filter of ideals of $C(X)$ generated by the dense, fixed maximal ideals. Observe that $\mathfrak{F}$ consists of finite intersections of dense, fixed maximal ideals and $\mathfrak{F} \subseteqq \mathfrak{D}$. Let

$$H = \bigcup \{ \operatorname{Hom}(D) : D \in \mathfrak{F} \}$$

and define $\mathfrak{F}(X) = H/\sim$ (where $\sim$ is the usual equivalence relation). Since $\mathfrak{F} \subseteqq \mathfrak{D}$, it follows that $\mathfrak{F}(X)$ is a ring of quotients of $C(X)$. We call $\mathfrak{F}(X)$ the *fixed ring of quotients of* $C(X)$.

DEFINITION 2.1. Let $\mathfrak{G}$ be a filter of ideals of A. If $IJ \in \mathfrak{G}$ whenever $I, J \in \mathfrak{G}$, then $\mathfrak{G}$ is called a *multiplicative filter.* If $\mathfrak{G}$ satisfies

- whenever J is an ideal of A for which there is an $I \in \mathfrak{G}$ with $a^{-1}J \in \mathfrak{G}$ for each $a \in I$, then $J \in \mathfrak{G}$

then $\mathfrak{G}$ is called a *Gabriel filter.* Gabriel filters are necessarily multiplicative. Both $\mathfrak{D}$ and $\mathfrak{D}_0$ are Gabriel filters. The aim of the rest of this section is to show that $\mathfrak{F}$ is a Gabriel filter.

LEMMA 2.2. *$\mathfrak{F}$ is a Gabriel filter if and only if it satisfies the following property:*

(∗) *If I is an ideal for which there is a finite subset F of nonisolated points of X such that whenever $f \in C(X)$ vanishes on F, then $f^{-1}I = M_{F_f}$ for some finite subset of nonisolated points F_f, then $I \in \mathfrak{F}$.*

PROOF. This is a simple translation of the Gabriel condition • from the previous definition. □

NOTATION 2.3. Let I be an ideal of $C(X)$. We let

$$V_S(I) = \{P \in \operatorname{Spec}(C(X)) : I \leqq P\},$$

$$V_M(I) = \{M \in \operatorname{Max}(C(X)) : I \leqq M\},$$

and

$$V_I = \{ M \in V_M(I) : M = M_x \text{ for some non-isolated } x \in X \}.$$

Note that V_I is precisely the set of dense, fixed maximal ideals containing I.

LEMMA 2.4. *If I is an ideal satisfying $(*)$, then $V_M(I) = V_I$.*

PROOF. Suppose that I satisfies $(*)$ yet there is an $M \in V_M(I) - V_I$. Since I satisfies $(*)$ we choose a finite subset F of X which witnesses this fact. For any $f \in M_F \smallsetminus M$, there is a finite subset F_f of X (in fact a subset of V_I) such that $\{g : fg \in I\} = M_{F_f}$. By choice of M, if $fg \in I$, then $fg \in M$. Since $f \notin M$, a prime ideal, $M_{F_f} \leqq M$, a contradiction. $\square$

LEMMA 2.5. *If I satisfies $(*)$, then V_I is finite.*

PROOF. Let F be a finite subset of X which exhibits that I satisfies $(*)$. Suppose $|V_I| \geqq \omega$ and let $\{M_{s_i} : i \in \mathbf{N}\}$ be a countable subset of V_I which is disjoint from F. For each $i < \omega$ select $f_i \in C(X)$ for which $s_i \in \operatorname{coz}(f_i)$ and $F \subseteqq Z(f_i)$. Recall that the cozerosets of X are closed under countable unions (1.14,[6]), whence there is a cozeroset U containing each s_i yet $U \cap F = \emptyset$. Choose $f \in C(X)$ such that $\operatorname{coz}(f) = U$. By $(*)$, $f^{-1}I = M_G$ for some finite subset G. Now if $fg \in I$, then $fg \in M_{s_i}$ for every natural i. But by our choice of f, $g \in \cap_{i\in\mathbf{N}} M_{s_i}$, i.e.,

$$M_G \leqq \cap_{i\in\mathbf{N}} M_{s_i},$$

a contradiction. Therefore, V_I is finite. $\square$

The proof of the following lemma is straightforward.

LEMMA 2.6. *If I satisfies $(*)$ and $I \leqq J$, then J satisfies $(*)$.*

First, it should be noted that the only prime ideals which satisfy $(*)$ are the fixed maximal ideals. To see this observe that if the non-maximal prime P satisfies $(*)$ and F witnesses this, then without loss of generality we may assume that $p \in F$, where M_p is the unique maximal ideal containing P. Next, select an $f \in M_F \smallsetminus P$ and then, since $f^{-1}P = M_G$ for some finite G, it follows that we may choose a $g \in M_G \smallsetminus P$. But this means that $fg \in P$, even though neither f nor g is in P, contradicting that P is prime.

We are now ready for the main result of this section.

THEOREM 2.7. *Let X be a Tychonoff space. Then $\mathfrak{F}$ is a Gabriel filter.*

PROOF. Our goal is to show that if I is an ideal satisfying $(*)$, then $I \in \mathfrak{F}$, that is, I is an intersection of a finite number of dense fixed maximal ideals. We recall from [7], Proposition 5.2 that

$$I = \bigcap_{p\in\beta X} (I, O^p),$$

where if $I \nsubseteq M^p$, then $(I, O^p) = C(X)$. Thus, since I satisfies $(*)$,

$$I = \bigcap_{p \in V_I} (I, O_p).$$

Now if for each $p \in V_I$ we have $(I, O_p) = M_p$, then we are done. Without loss of generality, we assume that for some $p \in V_I$, $(I, O_p) < M_p$. By Lemma 2.6, (I, O_p) satisfies $(*)$. Observe that $O_p \leqq (I, O_p) \leqq M_p$. We have just simplified our task by reducing to the case where J is an ideal satisfying $(*)$ and $O_p \leqq J \leqq M_p$. We intend to show that $J = M_p$.

Suppose $O_p \leqq J \leqq M_p$ and J satisfies $(*)$. Let F be a finite subset witnessing this, and without loss of generality we assume that $p \in F$. We claim that if $g \in M_p$, then $g^{-1}J \geqq M_p$. Once this is shown, observe that for arbitrary $f, g \in M_p$, $fg \in J$. This implies that $M_p^2 \leqq J \leqq M_p$ and so $J = M_p$ since M_p is an idempotent ideal. Thus all we need show is that $g \in M_p$ implies $g^{-1}J \geqq M_p$.

Let $g \in M_p$. Enumerate $F = \{p, x_1, \dots, x_m\}$ and let $r_i = g(x_i)$ for each $1 \leqq i \leqq m$. By the Tychonoff property, it follows that there is a k_i in O_p (and hence in J) for which $k_i(x_j) = \delta_{ij} r_i$. Let

$$k = \sum_{1 \leqq i \leqq m} k_i.$$

Observe that $k \in J$ and that $g - k \in M_F$. Also we have $g^{-1}J = (g-k)^{-1}J$. But since $g - k$ vanishes on F, it follows that $g^{-1}J \in \mathfrak{F}$ for all $g \in M_p$. Since $J \leqq g^{-1}J$ and $V(J) = \{M_p\}$ it follows that $M_p \leqq g^{-1}J$. $\square$

NOTES. In the theory of integral domains, a Gabriel filter is often called a localizing system. The basic results on Gabriel filters may be found in [15], specifically the ones regarding $\mathcal{D}$ and $\mathcal{D}_0$ and the fact that a directed union of hom groups (modulo the given equivalence relation) associated to a Gabriel filter is a ring.

3. The fixed ring of quotients $\mathfrak{F}(X)$ and h-spaces

Let $\mathcal{F}$ be the filter of dense, cofinite subsets of X and let $C[\mathcal{F}]$ denote the set of equivalence classes of the collection $\{C(U) : U \in \mathcal{F}\}$ under the obvious relation. $C[\mathcal{F}]$ is called the *cofinite ring of quotients of* $C(X)$. Observe that $\mathcal{F} \subseteqq \mathcal{G}(X)$ and so $C[\mathcal{F}]$ is a ring of quotients of $C(X)$. By Remark 1.3

it follows that $\mathfrak{F}(X) \leqq C[\mathcal{F}]$. We will show by example that these two rings of quotients are generally not the same. In a later section, we examine when they are equal. We begin with a notational device which will make our descriptions and calculations easier.

NOTATION 3.1. If F is a finite subset of X, then we let $X_F = X - F$. If F is the singleton set $\{p\}$, then we will write X_p.

LEMMA 3.2. *Let $f \in C(X_F)$. If $f \in \mathfrak{F}(X)$, then $f \in \operatorname{Hom}(M_F)$.*

PROOF. To say that $f \in \mathfrak{F}(X)$ means that there is a finite set G of nonisolated points of X for which $f \in \operatorname{Hom}(M_G)$. If $G \subseteqq F$, then we are done. So suppose that this is not the case. We use the Tychonoff condition to obtain an $h \in C(X)^+$ for which $G \setminus F \subseteqq \operatorname{int} Z(h)$ and $F \setminus G \subseteqq \operatorname{int} Z(h-1)$. Next let $g \in M_F^+$ and $k = gh \in M_G$. It follows from our assumption that $fk \in C(X)$. By construction, we may now define $fg(t) = fk(t)$ for every $t \in F \setminus G$ and obtain a continuous extension. (We are using the fact that $f \in C(X_F)$ implies that fg is continuous on $G \setminus F$.) $\square$

The above lemma leads to the following theorem. The theorem gives a characterization of when $\mathfrak{F}(X) = C[\mathcal{F}]$. This will be explored in a later section.

THEOREM 3.3. *$\mathfrak{F}(X) = C[\mathcal{F}]$ if and only if $C(X_p) = \operatorname{Hom}(M_p)$ for every $p \in X$.*

EXAMPLE 3.4. Let $X = [0,1]$. The function $f(x) = \frac{1}{x^2}$ belongs to $C(X_0)$. The function $g(x) = x$ is in M_0 and $fg \notin C(X)$. Therefore, $\operatorname{Hom}(M_0) < C(X_0)$, whence Theorem 3.3 implies that $\mathfrak{F}(X) \neq C[\mathcal{F}]$.

DEFINITION 3.5. A point $p \in X$ is called an *h-point* if $\operatorname{Hom}(M_p) = C(X_p)$. If every point of X is an h-point then X is called an *h-space*. Equivalently, X is an h-space if and only if $\mathfrak{F}(X) = C[\mathcal{F}]$.

Example 3.4 has an obvious generalization.

PROPOSITION 3.6. *If p is a nonisolated G_δ-point of X, then it is not an h-point.*

LEMMA 3.7. *For any point $p \in X$ we have*

$$C^*(X_p) \leqq \operatorname{Hom}(M_p) \leqq C(X_p).$$

PROPOSITION 3.8. *Let X be any space and $p \in X$. Then the following are equivalent.*

(i) *p is an h-point of X.*

(ii) *If $f \in C(X_p)$, then $fg \in C(X)$ for every $g \in M_p$.*

(iii) *Every $C(X)$-module homomorphism $\psi : O_p \to C(X)$ may be extended to M_p.*

(iv) $\operatorname{Hom}(M_p) = \operatorname{Hom}(O_p)$.

Furthermore, the next condition implies the first four and if X is pseudocompact, then all five are equivalent:

(v) *X_p is pseudocompact.*

REMARK 3.9. A space for which $C(X) = C[\mathcal{F}]$ is called a *C-space.* Obviously, X is C-space if and only if every cofinite subset of X is C-embedded. It clearly follows that a C-space is an h-space. For example, compact extremally disconnected spaces are C-spaces. There are examples of C-spaces which are not extremally disconnected. It follows from Proposition 3.8 (iv) that a P-space is an h-space. Finally, there are h-spaces which are neither a P-space nor a C-space.

DEFINITION 3.10. We say $p \in X$ is a *strongly Fréchet–Urysohn* point if whenever $p \in \operatorname{cl} A - A$ there is a sequence $\{x_i\}_{i\in\mathbf{N}} \subseteqq A$ converging to p and $\{x_i\}$ is a C-embedded subset of X_p.

Through slight modification of Example 3.4 we obtain the following result.

PROPOSITION 3.11. *If p is a strongly Fréchet–Urysohn point, then it is an h-point if and only if it is isolated.*

COROLLARY 3.12. *Let X be a totally ordered space. The point $p \in X$ is an h-point if and only if it is a P-point.*

PROOF. Recall that totally ordered spaces are normal. Suppose p is not a P-point. By Exercise 5O.1 of [6] we may assume without loss of generality that there is a sequence of points, say $\{x_i\}_{i\in\mathbf{N}}$ such that $x_i < x_{i+1}$ and the x_i converge to p. It follows that in the subspace $(-\infty, p]$, p is a G_δ-point. Any continuous function which witnesses that p is not an h-point of $(-\infty, p]$ will have a continuous extension to X which witnesses that p is not an h-point of X. □

PROPOSITION 3.13. *Suppose $p \in X$ is point of X with countable character. Then* $\operatorname{Hom}(M_p)$ *is precisely the set of those $f \in C(X_p)$ for which there exists a deleted neighbourhood of p on which f is bounded.*

PROOF. We assume that p is a non-isolated point of X. The set in question is always a ring of quotients of $C(X)$ lying inside of $\operatorname{Hom}(M_p)$. Therefore, let $f \in \operatorname{Hom}(M_p)$ and suppose that $f \in C(X_p)$ is positive and unbounded on every neighbourhood of p. Since p has countable character, we can construct a sequence of points $\{x_i\}_{i\in\mathbf{N}}$ converging to p so that $\{f(x_i)\}$ is increasing and converges to $+\infty$. We can also choose a function $g \in M_p$ so that $g(x_i) = 1/\sqrt{f(x_i)}$. It then follows that fg cannot be extended to p, so that $f \notin \operatorname{Hom}(M_p)$. □

Example 3.14. The space $X = \beta\mathbf{N} - \mathbf{N}$ is not a P-space. (In fact, it is consistent that X has no P-points.) It is known that X_p is countably compact for every $p \in X$ (see [13]), and hence pseudocompact. Thus, (v) of Proposition 3.8 implies that $\beta\mathbf{N} - \mathbf{N}$ is an h-space. Whether X is a C-space is independent of the axioms of ZFC.

We now end this section with some applications of our definitions. In particular the next theorem shows that being an h-space is a ring theoretic property of $C(X)$.

Proposition 3.15. *X is an h-space if and only if υX is an h-space.*

Proof. Suppose X is an h-space and let $p \in \upsilon X$. We wish to show that if $f \in C\big((\upsilon X)_p\big)$ and $g \in M_p \leqq C(\upsilon X)$, then fg may be continuously extended to all of υX.

If $p \in \upsilon X - X$, then $X \subseteqq (\upsilon X)_p \subseteqq \upsilon X$ and so any $f \in C\big((\upsilon X)_p\big)$ may be extended to all of υX, let alone fg. So we assume that $p \in X$.

Let f' be the restriction of f to X_p and g' be the restriction of g to X. Then since X is an h-space it follows that $f'g'$ may be extended to X and then further to υX. This final extension extends fg to all of υX.

Conversely, assume that υX is an h-space, $p \in X$, $f \in C(X_p)$, and $g \in M_p$. Let g^{υ} be the extension of g to all of υX. By Exercise 8G.2 [6], X_p is C-embedded in $(\upsilon X)_p$, so we may extend f to $(\upsilon X)_p$; call the extension f^{υ}. Then by hypothesis $f^{\upsilon}g^{\upsilon}$ is extendable to all of υX. Restricting back to X, we obtain our desired extension. $\square$

A similar proof shows that if X is an h-space, then so is βX. Recall that Σ (4M[6]) is the space obtained by adjoining a point $p \in \beta\mathbf{N} - \mathbf{N}$ to $\mathbf{N}$ with the subspace topology inherited from $\beta\mathbf{N}$. $\beta\Sigma$ is a C-space, yet Σ is not an h-space.

Proposition 3.16. *The topological sum of the spaces X_λ, $\oplus_\lambda X_\lambda$, is an h-space if and only if each X_λ is an h-space.*

Notes. If X is an infinite P-space then condition (iv) of Proposition 3.8 is satisfied, however condition (iv) is not. Also observe that $\alpha\mathbf{D}$, the one-point compactification of a discrete space, is an example of an almost P-space which is not an h-space.

4. E_p, the ideal of extendable functions

In this section, we find a characterization of h-points which is internal to $C(X)$.

DEFINITION 4.1. Let $p \in X$. Define

$$E_p = \{ g \in C(X) : \text{for every } f \in C(X_p), fg \in C(X) \} .$$

It is an easy exercise to show that E_p is an ideal of $C(X)$ containing O_p. We stress that E_p may be all of $C(X)$. Observe that E_p is the largest ideal of $C(X)$ containing O_p so that $\mathrm{Hom}\,(E_p) = \mathrm{Hom}\,(O_p) = C(X_p)$.

THEOREM 4.2. *The following are equivalent for a Tychonoff space X:*
(i) *For every $p \in X$, $E_p = C(X)$.*
(ii) *X is a C-space.*
(iii) *$C(X) = C[\mathcal{F}]$.*
(iv) *X is a C^*-space and $\psi(x) \neq \aleph_0$ for every nonisolated point $x \in X$.*

PROOF. That (i) and (ii) are equivalent follows from the fact that whenever $E_p = C(X)$ and $f \in C(X_p)$, then $f = f \cdot 1 \in C(X)$. The equivalence of (ii) and (iii) is clear. So is the implication (ii) $\Rightarrow$ (iv).

Suppose X is a C^*-space and every nonisolated point is not a G_δ-point. Let p be a nonisolated point of X and suppose $f \in C(X_p)$ is unbounded and has no continuous extension to p. Without loss of generality we may assume that $f \geqq 1$. Let $g = \frac{1}{f}$, then $g \in C^*(X_p)$. By hypothesis g has a continuous extension to p. There are two cases to consider, both leading to a contradiction. The first is that $g(p) \neq 0$. Then $f(p) = \frac{1}{g(p)}$ would continuously extend f. Otherwise, if $g(p) = 0$, then $Z(g) = \{p\}$ and hence p is a G_δ-point. □

The following proposition should not be surprising.

PROPOSITION 4.3. *Let $p \in X$. Then p is an h-point of X if and only if $M_p \leqq E_p$.*

THEOREM 4.4. *A space X is a C^*-space if and only if $C(X) = \mathfrak{F}(X)$.*

PROOF. First recall from Lemma 3.7 that $C^*(X_p) \leqq \mathfrak{F}(X)$. Thus if $C(X) = \mathfrak{F}(X)$, it follows that p is a C^*-point for every nonisolated point p.

Conversely, let X be a C^*-space and $p \in X$ (nonisolated). Suppose by way of contradiction that $C(X) < \mathrm{Hom}\,(M_p)$. Choose $f \geqq 0$ that witnesses this inequality and, without loss of generality, we suppose that $f \geqq 1$. By design $1/f \in C^*(X_p)$ and therefore our hypothesis forces $1/f \in C(X)$. Again by design we know that $(1/f)(p) \neq 0$ whence $1/f$ is a unit of $C(X)$, a contradiction. □

COROLLARY 4.5. *If X is a C^*-space, then X is an h-space if and only if X is a C-space.*

COROLLARY 4.6. *A point $p \in X$ is a C^*-point if and only if $C(X) = \mathrm{Hom}\,(M_p)$.*

Our approach for the rest of this section is to develop a basic algebraic and lattice theory of E_p which will demonstrate how E_p resembles the ideals O_p and M_p.

PROPOSITION 4.7. *E_p is closed under the lattice operations of $C(X)$.*

PROOF. Since E_p is a subgroup and every element of $C(X)$ can be written as a difference of positive elements, it is sufficient to demonstrate that $g \vee 0 \in E_p$ whenever $g \in E_p$. Let $f \in C(X_p)$. Note that the following equality allows us to assume without loss of generality that $f \geqq 0$:

$$f(g \vee 0) = (f^+ - f^-)(g \vee 0) = \bigl(f^+ \cdot (g \vee 0)\bigr) - \bigl(f^- \cdot (g \vee 0)\bigr).$$

Let $r = (fg)(p)$ and suppose that that $r > 0$. Choose ε so that $r > \varepsilon > 0$. There is a neighbourhood U about p so that if $x \in U$ then $(fg)(x) \in (r - \varepsilon, r + \varepsilon)$. Since $f \geqq 0$ it follows that $g \geqq 0$ on U. Thus on U, $g = g \vee 0$, whence we may extend $f(g \vee 0)$ to p.

If $r = 0$, then simply setting $f(g \vee 0)(p) = 0$ defines a continuous extension. □

We now show that the first case in the preceding proof is the only case that may occur.

LEMMA 4.8. *Assume that p is a non-isolated point of X. Suppose $g \in E_p$, $f \in C(X_p)$. Then $fg(p) = 0$.*

PROOF. First, due to the previous proposition we may assume that $g \geqq 0$. Let $r = fg(p)$ and suppose that $r \neq 0$. Then by continuity of fg, p is not an accumulation point of $Z(g)$. Choose a zeroset neighbourhood $Z(k)$ of p such that $Z(k) \cap Z(g) = \emptyset$ and $k \geqq 0$. Then $Z(g + k) = Z(k) \cap Z(g) = \{p\}$. Since $k \in O_p$, we obtain that $g + k \in E_p$. Let $h = (g + k)^{-1} \in C(X_p)$ and observe that $h^2 g$ must be extendable to p, which it is not; this is our desired contradiction. Thus $r = 0$. □

PROPOSITION 4.9. *E_p is a convex subset of $C(X)$. Moreover, if E_p is proper then*

$$O_p \leqq E_p \leqq M_p.$$

PROOF. Suppose that $0 \leqq h \leqq g$ where $g \in E_p$. Let $f \in C(X_p)$. Then $fg \in C(X)$ and $fg(p) = 0$. A simple argument involving continuity shows that if we define $fh(p) = 0$, then $fh \in C(X)$. □

PROPOSITION 4.10. *E_p is square-root closed, hence E_p is an idempotent ideal. Moreover, E_p is semiprime and so if it is properly contained in M_p then it is contained in a prime ideal that is not maximal.*

PROOF. Let $g \in E_p$ and suppose that $g \geqq 0$. To show that $\sqrt{g} \in E_p$ it is enough to observe that every positive $f \in C(X_p)$ is a square root, say $f = \sqrt{h}$, and thus $\sqrt{g}f = \sqrt{g}\sqrt{h} = \sqrt{gh}$ and $\sqrt{gh} \in C(X)$.

To see that E_p is semiprime suppose that $g^2 \in E_p$. Since E_p is a convex ℓ-ideal, we may assume that $g \geqq 0$. Let $f \in C(X_p)$. Next, f is the difference of positive elements so that we further assume that $f \geqq 0$. Since $fg = \sqrt{f^2g^2}$ and $f^2g^2 \in C(X)$ it follows that $fg \in C(X)$. □

PROPOSITION 4.11. *Suppose p is a strongly Fréchet–Urysohn point of X. Then $E_p = O_p$.*

PROOF. Suppose $g \in M_p$. If g does not vanish on a neighbourhood of p, then $p \in \mathrm{cl}_X \operatorname{coz}(g)$. Choose a sequence in $\operatorname{coz}(g)$ which converges to p, say $\{x_n\}$. Define $f(x_n) = \frac{1}{g(x_n)}$ and by hypothesis extend f to all of X_p. It follows from Lemma 4.8 that fg may not be extended to p, whence $E_p = O_p$. □

EXAMPLE 4.12. We may not generalize Proposition 4.11 to points of countable tightness nor to points of countable pseudo-character. Let X be the space defined in Example 1.6.19 in [3]. To remind the reader, recall that $X = \{0\} \cup bigcup_{i=1}^{\infty} X_i$, where $X_i = \{1/i\} \cup \{1/i + 1/i^2, 1/i + 1/(i^2+1), \dots\}$. Each point not of the form $1/i$ is isolated in X and for $x = 1/i$, sets of the form $\{1/i\} \cup \{1/i + 1/k, 1/i + 1/(k+1), \dots\}$, for $k = i^2, i^2+1, \dots$, form a neighbourhood base around x. A neighbourhood of 0 is any subset of X obtained by removing a finite number of X_i's and a finite number of points of the form $1/i + 1/j$ from each of the remaining X_i's. Since X is countable it is a countably tight space and $\psi(0) = \aleph_0$. But 0 is not a Fréchet–Urysohn point of X. Observe that each X_i is homeomorphic to the one-point compactification of the naturals.

Letting $p = 0$ we shall now define a function $g \in E_p - O_p$. The restriction of g to each X_i is defined to be $g_i(x) = x - 1/i$. We then define $g(0) = 0$. g is continuous and $g \notin O_p$.

Next, let $f \in C(X_p)$. Then $fg(1/i) = 0$ for every i. If $\varepsilon > 0$ then for each i choose a cofinite set N_i of X_i so that $|fg(x)| < \varepsilon$ for every $x \in N_i$. It then follows that $\{0\} \cup \bigcup_{i=1}^{\infty} N_i$ is a neighbourhood of 0 in X, which demonstrates that $fg(0) = 0$ defines a continuous extension.

A similar argument shows that E_p is in fact the set of all $g \in C(X)$ for which $g(1/i) = 0$ for all but a finite number of i. It follows then that $O_p < E_p < M_p$. Finally, it is true that if s is a nonisolated point of X and $\psi(s) = \aleph_0$, then every element $g \in E_s$ has an infinite zeroset. Moreover, $E_s < M_s$; more on this later.

PROPOSITION 4.13. *The following are equivalent for the point $p \in X$.*
(i) *The point p is a G_δ-point of X.*
(ii) *O_p is contained in a (proper) principal ideal of $C(X)$.*
(iii) *E_p is contained in a (proper) principal ideal of $C(X)$.*

PROOF. The fact that (i) and (ii) are equivalent may be found in 4I.8 [6]. Since $O_p \subseteqq E_p$, (iii) implies (ii).

Next suppose (ii), and let f be a nonunit for which $Z(f) = \{p\}$, and $O_p \subseteqq (f)$. Let $g \in E_p$. Since $1/f \in C(X_p)$, we let $k = g(1/f) \in C(X)$. It is then clear that $g = fk$, whence $E_p \subseteqq (f)$. $\square$

PROPOSITION 4.14. *Let X be a space and $p \in X$. Suppose that there exists a subspace $S \subseteqq X$ such that S is a discrete, C-embedded subspace of X_p and S satisfies the following property: U is an open neighbourhood of p if and only if there exists a subspace $T \subseteqq S$ and a collection of sets $\{O_t\}_{t\in T}$ for which O_t is a neighbourhood of $t \in T$ and*

$$U = \bigcup_{t\in T} O_t \cup \{p\}.$$

Set $Y = S \cup \{p\}$. Then E_p is precisely those $g \in C(X)$ whose restriction to Y vanishes on a deleted neighbourhood of p.

PROOF. Let $g \in E_p$. The function defined by

$$f(s) = \begin{cases} \dfrac{1}{g(s)^2}, & \text{if } s \in S - Z(g); \\ 0, & \text{if } s \in S \cap Z(g) \end{cases}$$

can be continuously extended to X_p. Since $fg(s) = \frac{1}{g(s)}$ for all $s \in S - Z(g)$ and both g, fg are continuous at p it follows that g restricted to Y vanishes on a neighbourhood of p.

Conversely, let $\varepsilon > 0$. If g has said property then for any $f \in C(X_p)$, $Z(fg)$ contains a $T \subseteqq S$ such that $T \cup \{p\}$ is a clopen subset of Y. For each $t \in T$ choose a neighbourhood O_t such that $\left|fg(x)\right| \in [0, \varepsilon)$ for all $x \in O_t$. Then the union of each O_t plus p is an open neighbourhood of p satisfying continuity at p. $\square$

Note that any space with a unique non-isolated point satisfies the conditions of Proposition 4.14 with S as the set of isolated points. It then follows that in these cases $E_p = O_p$, where p is the unique non-isolated point. But Proposition 4.14 is much more useful than this. It gives an easy way of constructing spaces for which $O_p < E_p < M_p$. In the two examples that follow it is also true that E_p is a z-ideal. We do not know how to prove this in general nor do we have a counterexample.

In the following two examples Y_n will denote a copy of the one-point compactification of $\mathbf{N}$. The point at infinity will be represented by α_n. Analogously, Z_n will denote a copy of Σ. Its one non-isolated point will be σ_n. Set $Y = \bigcup_{n\in\mathbf{N}} Y_n$ and $Z = \bigcup_{n\in\mathbf{N}} Z_n$. Finally, let A and S denote the set of non-isolated points of Y and Z, respectively. It is obvious that A and S are copies of $\mathbf{N}$.

EXAMPLE 4.15. Let τ be a non-principal ultrafilter on S and let X_1 be the space obtained by adjoining τ to Z and defining U to be a neighbourhood of τ if its intersection with S is an element of τ and for each $\sigma_n \in U$, $U \cap Z_n$ lies in σ_n. This is a well-defined topology on X_1 and makes into a Tychonoff space. We leave the verification of this to the reader.

Observe that $E_\tau = \left\{ g \in M_\tau : Z(g) \cap S \in \tau \right\}$. In this case both E_τ and O_τ are prime ideals of $C(X_1)$ and $E_\tau \neq O_\tau$.

EXAMPLE 4.16. Let τ be a non-principal ultrafilter on A and let X_2 be the space obtained by adjoining τ to Y and defining U to be a neighbourhood of τ if its intersection with A is an element of τ and for each $\alpha_n \in U$, $U \cap Y_n$ is cofinite. This is a well-defined topology on X_2.

Observe that $E_\tau = \left\{ g \in M_\tau : Z(g) \cap A \in \tau \right\}$. Again E_τ is a prime ideal of $C(X_2)$ but now O_τ is not, i.e., τ is not an F-point of X_2.

NOTES. It is known that every dense open subset of an extremally disconnected space is C^*-embedded and so every extremally disconnected space is a C^*-space. Compact ones are in fact C-spaces since in this case no non-isolated point is a G_δ-point. In [12] the reader will find examples of C-spaces which are not extremally disconnected.

QUESTIONS. 1. Is there a topological characterization for when a filter of dense ideals of $C(X)$ is Gabriel?

2. Is every compact almost P, F-space an h-space? Specifically, if X is a locally compact, σ-compact space, is $\beta X - X$ an h-space? Is $\beta\mathbf{H} - \mathbf{H}$ an h-space, where $\mathbf{H} = [0, \infty)$?

3. Is there an example, in ZFC, of a space for which $\mathfrak{F}(X) \leqq q(X)$ yet $\mathfrak{F} \nsubseteq \mathfrak{D}_0$? Observe that $\mathfrak{F} \subseteqq \mathfrak{D}_0$ precisely if each almost P-point of X is isolated. It is known that the statement $\beta\mathbf{N} - \mathbf{N}$ is a C^*-space is independent of ZFC (see [2].)

4. Is Proposition 4.11 true in general for Fréchet–Urysohn points?

5. Is E_p always a z-ideal? (An ideal I of $C(X)$ is said to be a z-ideal if whenever $f \in I$ and $Z(f) = Z(g)$ then $g \in I$.)

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