WHEN $C(X)$ IS AN h-LOCAL RING

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Abstract. An h-local domain is a domain for which each nonzero prime ideal is contained in a unique maximal ideal and each nonzero element has finite character. In [17], the author generalizes the notion to rings with zero-divisors by restricting the definition to regular ideals and elements. In this article, we give a characterization of when $C(X)$ the ring of continuous real-valued functions on a space X is an h-local ring. We are left with more questions than answers.

1. Introduction

In the theory of integral domains, the concept of an h-local domain is well-known. Originally defined by Matlis [15], later Olberding [16] supplied a list of equivalent conditions for a domain to be an h-local domain. Interestingly, P. Jaffard [11], prior to Matlis, studied domains he called Dedekind type. In [6, Theorem 2.1.5], the authors show that a domain is h-local if and only if it is of Dedekind type. One of the main goals of a recent dissertation [17] was to generalize the concept of an h-local domain to the case of rings with zero-divisors. Our interest in this article is to characterize the h-local condition for rings of continuous functions.

Recall that an integral domain R is said to be an *h-local domain* if each nonzero prime ideal of R is contained in a unique maximal ideal and each nonzero element has finite character. As has become customary in generalizing to rings with zero-divisors the focus is on regular elements (i.e. non-zero-divisors). Throughout by **ring** we mean a (non-trivial) commutative ring with identity. An ideal is said to be **regular** if it contains a regular element.

Definition 1.1. Let R be a ring. We say R is **h-local** if each regular prime ideal of R is contained in a unique maximal ideal and each regular element is contained in at most finitely many maximal ideals of R.

Remark 1.2. In [14], the authors study h-local rings but their definition is a bit stronger in the sense that they do not restrict to regular prime ideals nor regular elements.

Our interest in h-local rings was spurred on by attendance in a seminar at Florida Atlantic University covering material in the dissertation by A. Omairi ([17]). We were interested in understanding what the h-local condition means in the context of rings of real-valued continuous functions.

We shall employ the following notation. For a ring R, $\mathrm{Spec}(R)$ denotes its space of prime ideals and $\mathrm{Max}(R)$ the subspace of maximal ideals.

2. When $C(X)$ is h-local

Throughout, $C(X)$ denotes the ring of of real-valued continuous functions defined on the topological space X. We assume throughout that X is a Tychonoff space, that is, completely regular and Hausdorff. Rings of the form $C(X)$ are always *pm*-rings, that is, every prime ideal is contained in a unique maximal ideal and so half of the definition of h-local is already satisfied. We state this formally.

Lemma 2.1. *The ring $C(X)$ is h-local if and only if every regular element of $C(X)$ is contained in at most finitely many maximal ideals of X.*

We assume knowledge of the ring $C(X)$ and cite [7] as our main reference. In order to keep the material self-contained we recall that for $f \in C(X)$, the **zero-set** of f is the set

$$Z(f) = \{x \in X : f(x) = 0\}.$$

The set-theoretic complement of $Z(f)$ is denoted by $\mathrm{coz}(f)$ and is called the **cozero-set** of f. A subset of X is called a zero-set (resp. cozero-set) if it has the form $Z(f)$ (resp. $\mathrm{coz}(f)$) for some $f \in C(X)$. It is known that a Hausdorff space is Tychonoff if and only if the set of cozero-sets forms a base for the open sets. All zero-sets are closed but not conversely. We let the operators int $(\cdot)$ and cl $(\cdot)$ denote the interior and closure of a subset of X.

Observe that the invertible and regular elements of $C(X)$ can be characterized by their zero-sets. Namely, $f \in C(X)$ is invertible if and only if $Z(f) = \emptyset$, and $f \in C(X)$ is regular if and only if $\mathrm{int}\, Z(f) = \emptyset$.

To each $x \in X$ there is a corresponding maximal ideal of $C(X)$, namely

$$M_x = \{f \in C(X) : f(x) = 0\}.$$

One version of the Gelfand-Kolmogorov Theorem states that the space of maximal ideals of $C(X)$ equipped with the hull-kernel (aka Zariski) topology, denoted $\mathrm{Max}(C(X))$, is homeomorphic to the Čech-Stone compactification of X, denoted βX; we view βX as the space of zero-set ultrafilters on X. Furthermore, the Gelfand-Kolmogorov Theorem ([7, Theorem 7.3]) states that for $p \in \beta X$,

$$M^p = \{f \in C(X) : Z(f) \in p\} = \{f \in C(X) : p \in \mathrm{cl}\,_{\beta X} Z(f)\}$$

is the maximal ideal corresponding to p. When $p \in X$, then $M_p = M^p$ and the point p can be viewed as the principal zero-set ultrafilter generated by p. And this correspondence is a continuous dense embedding of X into βX. Since we are interested in considering the set of maximal ideals of $C(X)$ containing a given $f \in C(X)$, we shall, as is customary, let $V(f) = \{M \in \mathrm{Max}(C(X)) : f \in M\}$. Observe then that $V(f) \cap X = Z(f)$.

Remark 2.2. For those interested in understanding rings of the form $C(X, \mathbb{A})$ for some proper subring $\mathbb{A}$ of $\mathbb{R}$, say $\mathbb{A} = \mathbb{Z}$, let us dispel with any hopes that this might be interesting. Observe that for each $x \in X$, the ideal $M_x = \{f \in C(X, \mathbb{A}) : f(x) = 0\}$ is a prime ideal since $C(X, \mathbb{A})/M_x = \mathbb{A}$. If $\mathbb{A}$ is not a field, then let N

be a maximal ideal of $\mathbb{A}$ and choose a nonzero $a \in N$. For each $x \in X$, there is a maximal ideal N_x of $C(X, \mathbb{A})$ containing M_x such that $N_x/M_x = N$. The constant function $\mathbf{a} \in C(X, \mathbb{A})$ is regular and belongs to each N_x and therefore if X is infinite, then $C(X, \mathbb{A})$ is not h-local. It follows that either X is finite and $\mathbb{A}$ is h-local or else, $\mathbb{A}$ is a field in which case the theory follows the situation of $C(X)$ with some minor modifications.

The aim here is investigate when $C(X)$ is an h-local ring. Now, as is typical, if the only regular elements of a ring R are the invertible ones, then trivially R is an h-local ring. Such rings are often called *classical* (or *total*) since they equal their classical ring of quotients. This leads us to recall the following class of spaces.

Definition 2.3. A space X is said to be an **almost P-space** if every zero-set has non-empty interior. This is equivalent to saying that X has no proper dense cozero-sets.

Recall X is an almost P-space if and only if $C(X)$ is classical; see [5, 3.2]. Put another way, X is an almost P-space if and only if for every regular $f \in C(X)$, $Z(f) = \emptyset$. It is now obvious that the condition that $C(X)$ is h-local generalizes that X is an almost P-space. We state this formally. (For the reader not fluent in the language of rings of continuous functions, a space X is a **P-space** if the topology of X is closed under countable intersections. Furthermore, X is a P-space if and only if $C(X)$ is a von Neumann regular ring.)

Proposition 2.4. *If X is an almost P-space, then $C(X)$ is an h-local ring.*

We observe that the space X is an almost P-space if and only if the interior of any non-empty G_δ-set is non-empty. (Recall that a G_δ-set in a space is a countable intersection of open sets.) A classic example of an almost P-space is the one-point compactification of an uncountable discrete space. We shall denote this space by αD. When D is a countable discrete set we instead shall write $\alpha\mathbb{N}$; this space is not an almost P-space as it is metrizable and metrizable almost P-spaces are discrete.

Another classic example of an almost P-space is $\beta\mathbb{N} \smallsetminus \mathbb{N}$. In fact, any space of the form $\beta X \smallsetminus X$ for some locally compact, realcompact space X is an almost P-space. One can find these results and other information concerning almost P-spaces in [12].

It is common in the study of rings of continuous functions to generalize a notion about spaces to a notion about points. This shall be useful here. In this vein, a point $p \in X$ is called an *almost P-point* if whenever $f \in C(X)$ and $p \in Z(f)$, then $\operatorname{int} Z(f) \neq \emptyset$. Obviously, an isolated point is an almost P-point. The point $\alpha \in \alpha D$ is a non-isolated almost P-point. A space X is an almost P-space if and only if every point of X is an almost P-point.

One might think that the concept of an almost P-space would characterize when $C(X)$ is h-local. However, a quick check shows that if $X = \alpha\mathbb{N}$, the one-point compactification of the naturals, then $C(X)$ is h-local since the regular elements are the ones which either vanish nowhere (i.e. invertible) or only vanish at α. The latter type belong to only one maximal ideal, M_α. This leads one to see that if there are at most a finite number of points which are not almost P-points, then $C(X)$ is h-local. The question then becomes does having only a finite number of non-almost P-points characterize h-local $C(X)$. Our next example demonstrates that the answer is still no.

Example 2.5. Let I be an uncountable indexing set. For each $i \in I$, let X_i be a distinct copy of the one-point compactification of the naturals, with unique non-isolated point α_i. Let Y be the disjoint union of the copies of the X_i. The space Y is locally compact and therefore we may consider the space X which is the one-point compactification of Y; $X = \alpha Y$. We claim that $C(X)$ is h-local even though there are uncountably many points which are not almost P-points.

There are two interesting features here. First, we claim that the point α is an almost P-point of X. Let $f \in C(X)$ and suppose that f vanishes at α. Choose a natural number n and let n_i be its copy in X_i. Let $T = \{n_i\}_{i\in I} \cup \{\alpha\}$. Since X is compact it follows that T is a copy of αD, an almost P-space since I is uncountable. Then, by considering the restriction of f to T, it follows that f must vanish at some n_i, an isolated point. Therefore, $Z(f)$ has non-empty interior.

Next, if $f \in C(X)$ is regular, then as we just pointed out, f cannot vanish at α. Thus, $Z(f) \subseteq \{\alpha_i : i \in I\}$. Now, if $Z(f)$ were infinite then it would be forced to contain α since α is in the closure of any infinite collection of the α_i. Consequently, $Z(f)$ is finite.

Remark 2.6. Not surprisingly, if we were to instead use a countable indexing set I, then the space X is not h-local. Since, in this case, the set $\{\alpha_i\} \cup \{\alpha\}$ is a nowhere dense zero-set of X.

In the literature there has been a need to study subsets of a space which behave like almost P-points.

Definition 2.7. First, we recall [9, Definition 3.9] that a closed subset S of X is called an **almost P-set** if whenever $S \subseteq Z$, for a zero-set Z of X, then $S \subseteq \text{cl int } Z$.

We have found that this notion of an almost P-set is not exactly what we need for our purposes. Therefore, we define the following related concept.

Definition 2.8. We shall call a closed set S with the property that any zero-set containing S has non-empty interior an ***ap*-set**.

Clearly, any almost P-set is an *ap*-set. Here is a method to construct *ap*-sets which are not almost P-sets, whence the notion of almost P-set is stronger than that of an *ap*-set. Take a space X that is not an almost P-space but contains an almost P-point, say $p \in X$. Let Z be a nonempty nowhere dense zero-set of X and set $S = Z \cup \{p\}$. Any zero-set containing S will have non-empty interior. Since $p \notin Z$, there is some zero-set Z_1 containing p and disjoint from Z. The zero-set $Z_2 = Z_1 \cup Z$ contains S, yet does not satisfy $S \subseteq \text{cl int } Z_2$.

Many authors have used the existence of almost P-sets in certain spaces to further their work. We have not systematically gone through the literature to see if their results hold more generally for *ap*-sets though we do recognize that it is possible. We did find that [9, Proposition 3.10] holds for *ap*-sets. What is for sure is that the notion of an *ap*-set is of use to us in characterizing h-local spaces.

Theorem 2.9. *Let X be a Tychonoff space. The following statements are equivalent.*

1. *$C(X)$ is h-local.*
2. *Every infinite closed subset of X is an ap-set.*
3. *Every closed subset of X which is not an ap-set is finite.*
4. *For every regular $f \in C(X)$, $Z(f)$ is finite.*
5. *X has no infinite nowhere dense zero-sets.*

Proof. 1. implies 2. Let S be an infinite closed subset of X and suppose that $S \subseteq Z(f)$ for some $f \in C(X)$. Since $Z(f)$ is infinite, then f is not regular, i.e. $\operatorname{int} Z(f) \neq \emptyset$. Thus, S is an *ap*-set.

2. is clearly equivalent to 3.

2. implies 4. Let $f \in C(X)$ be regular. Then $Z(f)$ cannot be infinite since then it would be an *ap*-set, which it clearly is not.

4. implies 1. Let $f \in C(X)$ be regular and so by hypothesis $Z(f)$ is finite. It follows that for all $p \in \beta X \smallsetminus X$, $Z(f) \notin p$, whence $V(f) = Z(f)$. Therefore, $C(X)$ is h-local.

That 4. and 5. are equivalent is obvious. □

Definition 2.10. For lack of a better term, it will be convenient to call a space X **h-local** if $C(X)$ is h-local.

Corollary 2.11. *If X is h-local, then every non-almost P-point of X is a G_δ-point.*

Proof. Suppose X is h-local and that $x \in X$ is not an almost P-point. Then there is some nowhere dense zero-set of X, say $Z(f)$, containing x. But then f is regular and hence $Z(f)$ is finite. Using the Tychonoff property one can construct an $f' \in C(X)$ such that $Z(f') = \{x\}$. Since zero-sets are G_δ-sets the statement follows. □

Example 2.12. Observe that the space X in Example 2.5 is an h-local space as every infinite closed set is *ap*-set. However, note that not every infinite closed set is an almost P-set. For example, take any countable subset $I' = \{i_n\}_{n\in\mathbb{N}} \subseteq I$ and let $S = \{\alpha_{i_n}\} \cup \{\alpha\}$; S is closed. It is straightforward to check that Z, the union of S together $\bigcup_{i\in I\smallsetminus I'} X_i$, is a zero-set. Furthermore, for each $i \in I'$, $\alpha_i \notin \operatorname{cl} \operatorname{int} Z$. This demonstrates the condition 2. of Theorem 2.9 cannot be strengthened to almost P-sets.

We let $C^*(X)$ denote the subring of $C(X)$ consisting of bounded continuous functions on X. The map that takes an $f \in C(\beta X)$ to the restriction of f to X, is a ring homomorphism from $C(\beta X)$ into $C^*(X)$. Since X is C^*-embedded in βX, it follows that the homomorphism is an isomorphism. We address when $C^*(X)$ is an h-local ring, i.e. when βX is h-local.

Theorem 2.13. *Let X be a Tychonoff space. The ring $C^*(X)$ is h-local if and only if X is pseudo-compact and $C(X)$ is h-local. In other words, βX is h-local if and only if X is pseudo-compact and h-local.*

Proof. Clearly, the sufficiency is true. So suppose that $C^*(X)$ is h-local and let $f \in C(X)^+$ be an unbounded function; without loss of generality $f \geq \mathbf{1}$. Since $C^*(X)$ and $C(\beta X)$ are isomorphic, the latter is also h-local, by hypothesis. Let $g = f^{-1}$ which belongs to $C^*(X)$ and let g^β be the extension to all of βX. Now, $\emptyset \neq Z(g^\beta) \subseteq \beta X \smallsetminus X$, and hence nowhere dense. Since $C^*(\beta X)$ is h-local it follows that $Z(g^\beta)$ is finite contradicting [7, Theorem 9.5]. Consequently, X is pseudo-compact. □

Example 2.14. Theorem 2.13 is mirroring Levy's result about almost P-spaces [12, Proposition 2.1] which states that βX is an almost P-space if and only if X is a pseudo-compact almost P-space. Levy also pointed out that an open subset of an almost P-space is again an almost P-space, and the same is true for dense subsets. These two facts cannot be generalized to h-local spaces as witnessed by Example 2.5. Notice that Y, the disjoint union of the uncountable collection of copies of the one-point compactification of $\mathbb{N}$ is a dense open subset of the h-local (compact) space $X = \alpha Y$, while Y is not h-local. To see this, observe that the infinite set $\{\alpha_i : i \in I\}$ is not an almost P-set of Y. Obviously, clopen subsets of h-local spaces are again h-local.

Proposition 2.15. *Every dense cozero-set of an h-local space is again h-local.*

Proof. Let X be an h-local space and Y a dense cozero-set of X. Suppose $f \in C(Y)$ is a regular element. Recall that cozero-sets are z-embedded. Thus, there is a zero-set of X, say Z, such that $Z \cap Y = Z(f)$. Notice that $Z \smallsetminus Z(f) \subseteq X \smallsetminus Y$ and therefore, Z must be nowhere dense, and hence finite. It follows that $Z(f)$ is also finite. □

Proposition 2.16. *Suppose X has the property that every dense open subset of X is a cozero-set. Then the following statements are equivalent.*

1. *X is h-local.*
2. *X has only a finite number of non-almost P-points.*
3. *X is almost discrete.*

Proof. Recall that X is **almost discrete** if it has only a finite number of non-isolated points.

Clearly, 3. implies 2., and 2. implies 1.

1. implies 3. Suppose X is an h-local space. First of all, the hypothesis implies that every non-isolated point is a G_δ-point, and hence not an almost P-point. Now, in any infinite (Hausdorff) space, one can construct a discrete subset. Therefore, if the set of non-almost P-points is infinite, then there is a discrete collection of such points, say S. Then cl S is nowhere dense. In our case, cl S is an infinite nowhere dense zero-set, a contradiction. Thus, there are only a finite number of non-isolated points. □

Example 2.17. As we have seen, the metrizable space $\alpha\mathbb{N}$ is an h-local space. Since a metric space, and more generally, a perfectly normal space satisfies the hypothesis of Proposition 2.16, it follows that a perfectly normal space is an h-local space if and only if it is almost discrete. Interestingly, this includes all countable spaces.

Definition 2.18. Recall from [1] that a function $f \in C(X)$ is said to be **nowhere constant** if for every nonempty open subset of X, say O, there are $x, y \in O$ such that $f(x) \neq f(y)$. This is equivalent to saying that for each $r \in \mathbb{R}$, $f^{-1}(\{r\})$ is nowhere dense.

If X has a nowhere constant function, then X is does not possess any almost P-points. For more information on nowhere constant functions the reader is referred to [2].

Proposition 2.19. *Suppose X is h-local. Then no $f \in C(X)$ is nowhere constant.*

Proof. Suppose $f : X \longrightarrow \mathbb{R}$ is nowhere constant. For any $r \in f(X)$, the function $f - r$ is also nowhere constant. So we may assume that $Z(f) \neq \emptyset$. Since for any $s \in \mathbb{R}$ both $f^{-1}(s)$ and $f^{-1}(-s)$ are nowhere dense it follows that $|f|$ is also nowhere constant. So, we may assume further that $f \geq 0$. There must exist a decreasing sequence, say $\{r_n\}$, of positive real numbers belonging to $f(X)$ converging to 0, (otherwise, $Z(f)$ is clopen). Let $h \in C(\mathbb{R})$ such that $Z(h) = \{0\} \cup \{r_i\}$. Set $r_0 = 0$ and consider $h \circ f$.

$$Z(h \circ f) = \bigcup_{n \in \mathbb{N}} Z(f - r_n).$$

Set $Z = Z(h \circ f)$ and observe that for each $n \neq 0$ there is an open subset of X, say O_n, containing $Z(f - r_n)$ and so that $O_n \cap Z(f - r_m) = \emptyset$ for all $m \neq n$. We claim that Z is nowhere dense. If not, then there is some non-empty open set, say O, such that $O \subseteq Z$. If there is some $z \in O \cap Z(f - r_n)$ (with $n \neq 0$) then $\emptyset \neq O \cap O_n \subseteq Z(f - r_n)$, contradicting that $Z(f - r_n)$ is nowhere dense. Therefore, $O \subseteq Z(f - r_0) = Z(f)$. But this contradicts that $Z(f)$ is nowhere dense.

Finally, since Z is clearly an infinite set, we gather that X is not h-local. □

Corollary 2.20. *If X is dense in itself and an h-local space, then by* [1, Theorem 2]*, X cannot be separable.*

Our aim is to determine whether there exists an h-local space with no almost P-points. We do not have a complete answer but our next result rules out a big class of spaces.

Theorem 2.21. *Suppose X is not pseudo-compact. If X is h-local, then X has an almost P-point.*

Proof. Suppose X is an h-local space which is not pseudo-compact. If X contains no almost P-point, then every point of X is a non-isolated G_δ point. Since X is not pseudo-compact there is some C-embedded copy of $\mathbb{N}$, say $\{x_n\} \subseteq X$ is C-embedded (see [7, Corollary 1.2]). Then there is some $f \in C(X)$ such that $f(x_n) = n$. Considering $f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))$ it follows that there is a sequence of pairwise disjoint zero-sets, say $\{Z_n\}$, such that $x_n \in C_n \subseteq Z_n$ where $C_n = \text{coz}(g_n)$ for $g_n \in C(X)$.

We can assume that $0 \leq g_n \leq 1$ and that $g_n(x_n) = 1$. Now since each x_n is a G_δ point we can find $t_n \in C(X)$ such that $0 \leq t_n \leq 1$ and $t_n^{-1}(1) = x_n$. Then the function $h_n = g_n \cdot t_n$ has the property that $0 \leq h_n \leq 1$, $x \in \text{coz}(h_n) \subseteq Z_n$ and $h_n(y) = 1$ implies $y = x_n$.

Then the infinite sum of the h_n, call it h, is continuous and $Z(h-1) = \{x_n\}$ which means that $\{x_n\}$ is an infinite nowhere dense zero-set, a contradiction. □

Our next aim to rule out another significant class of spaces: compact spaces, see Corollary 2.24. We get a more general result.

Theorem 2.22. *Suppose X is normal and each non-almost P-point has countable character. Then X is h-local if and only if there is no countably infinite closed set consisting of non-almost P-points. Moreover, a normal first countable space is h-local if and only if it is almost discrete.*

Proof. Suppose there is a countably infinite closed set consisting of non-almost P-points, say T. Choose a discrete subset D of T. Notice that D is nowhere dense and thus so is the countable set cl D. If $D = \text{cl } D$, then D is a closed discrete set and hence a G_δ-set. Otherwise, choose $x \in \text{cl } D \setminus D$ and then select a a sequence in D, say $\{d_n\}$, such that $d_n \to x$. Then $\{x\} \cup \{d_n\}$ is also a closed G_δ-set. In either case, there is an infinite nowhere dense closed G_δ-set. Since X is normal this set is a zero-set, whence X is not h-local.

Conversely, if X is not h-local, then there is an infinite nowhere dense zero-set, say Z. Since each point of Z is not an almost P-point, each point of Z is of countable character. Select a discrete subset of Z, and either it is closed or we can choose a convergent sequence in Z. Either way, there is an infinite closed sequence in Z consisting of non-almost P-points.

Finally, suppose that X is a normal first countable space that is also h-local. Notice that every point in a first countable space is a G_δ-point and therefore an almost P-point must be isolated. The collection of non-isolated points is a closed subset of X and therefore we can find either a closed discrete set or a non-constant sequence that converges in it. Either way we obtain a countably infinite closed set of non-almost P-points, contradicting the hypothesis.

□

Remark 2.23. Observe that the last statement of Theorem 2.22 does not hold for Frechet-Urysohn spaces. Let Y be a countable collection of copies of αD, and then let $X = \alpha Y$, the one-point compactification of Y. Both X and Y are Frechet-Urysohn spaces, and both are h-local spaces. Neither is almost discrete.

Corollary 2.24. *Suppose X is compact. Then X is h-local if and only if there is no countably infinite closed set consisting of non-isolated G_δ-points. Moreover, a compact h-local space must contain an almost P-point.*

Proof. In a compact space, the pseudo-character of a point equals the character of said point. Therefore, the first statement follows from Theorem 2.22.

If X is compact h-local and does not contain any almost P-points, then there is certainly a convergent sequence in X, contradicting that X is h-local. □

Corollary 2.25. *Suppose $(X, \leq)$ is a totally ordered space. Then X is h-local if and only if there is no countably infinite closed set consisting of non-isolated G_δ-points. Moreover, a totally ordered h-local space must contain an almost P-point.*

Proof. Every totally ordered space is normal and so Theorem 2.22 applies. Furthermore, in a totally ordered space the pseudo-character of a point equals its character. If $p \in X$ is not an almost P-point, then it is a G_δ-point and hence has a countable base of neighborhoods. Thus, X must contain an almost P-point otherwise there is a sequence of non-almost P-points converging to a non-almost P-point, a contradiction. □

Remark 2.26. The proof in Corollary 2.25 works for perfect images of GO-spaces since such spaces are normal and the pseudo-character equals the character, see [8].

We also point out that if X is a Suslin line, then X has no nowhere constant function by [2, Theorem 5]. Since X is a totally ordered space with no almost P-points, it is not h-local. Therefore, the class of h-local spaces is a proper subclass of the class of spaces with no nowhere constant functions.

At this point we are left with the following question.

Question. Is there a pseudo-compact h-local space with no almost P-points?

If there is such a space X it cannot be compact. Furthermore, we know that βX is a compact h-local space and so it must contain an almost P-point. Any such point must live inside of $\beta X \setminus X$. In fact, any point of $\beta X \setminus X$ must be an almost P-point of βX. Otherwise, it would be a G_δ-point of βX contradicting the fact that no point of $\beta X \setminus X$ is a G_δ-point.

In [4], the authors investigated almost P-spaces as a subclass of quasi F-spaces. In particular, they show that a product of two spaces is an almost P-space if and only if each space is an almost P-space. We now classify when a product is an h-local space.

Theorem 2.27. *The product space $X \times Y$ is h-local if and only if either both X and Y are almost P-spaces or one of them is h-local while the other is finite.*

Proof. Necessity. Suppose $X \times Y$ is h-local and let $Z(f)$ be a nowhere dense zero-set of X. Then $S = Z(f) \times Y$ is a zero-set of $X \times Y$. If $Z(f) \times Y$ has non-empty interior, say $(x, y) \in \operatorname{int} S$, then there is an open set of the form $O_1 \times O_2$ such that $(x, y) \in O_1 \times O_2 \subseteq Z(f) \times Y$. It follows that $x \in O_1 \subseteq Z(f)$, which cannot happen. Therefore, S is a nowhere dense zero-set of $X \times Y$. By hypothesis, S is finite, and therefore $Z(f)$ is finite. Consequently, X is h-local. Similarly, Y is h-local.

Suppose that one of X or Y is not an almost P-space, say X. Then there is some non-isolated point of X, say $p \in X$, which is a G_δ-point. Let $f \in C(X)$ satisfy $Z(f) = \{p\}$. Define $\phi : X \times Y$ by $\phi((x, y)) = f(x)$ and observe that $\phi \in C(X \times Y)$ and $Z(\phi) = \{p\} \times Y$. We claim that $Z(\phi)$ is nowhere dense, whence Y must be finite.

Suppose $(p, t) \in \text{int } Z(\phi)$. Then there are open sets $O_1 \subseteq X$ and $O_2 \subseteq Y$ such that

$$(p, t) \in O_1 \times O_2 \subseteq \{p\} \times Y.$$

It follows that p is an isolated point, a contradiction. Therefore, $Z(\phi)$ is nowhere dense.

Sufficiency. If both X and Y are almost P-spaces, then so is $X \times Y$ by [4, Proposition 5.10]. If X is h-local and Y is finite, then $X \times Y$ is homeomorphic to a finite sum of copies of X which is again h-local. □

Remark 2.28. In [3], the authors investigate what they call DC-spaces (short for densely constant). The space X is a DC-space if for each $f \in C(X)$ there exist open sets $\{U_i : i \in I\}$ which are pairwise disjoint, the union of which is dense in X, and such that f is constant when restricted to each of the U_i. Any space with a dense set of almost P-points is a DC-space. Therefore, not every DC-space is h-local. Furthermore, a DC-space does not have any nowhere constant functions.

We do not know if an h-local space must be a DC-space.

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