# Free Meets and Atomic Assemblies of Frames

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Abstract. In J. T. Wilson's doctoral dissertation, the author takes up the subject of *free meets*, that is, subsets $S$ of a frame $L$ whose meet $\bigwedge S$ is preserved by a designated class of frame homomorphisms out of $L$. Wilson proves that all subsets of $L$ have free meets if and only if the dual frame law holds in $L$ and the assembly $\mathbb{N}L$ is a boolean frame.

In this paper it is shown that a frame $L$ has free meets with an atomic assembly if and only if it is freely generated by its subset of meet-irreducible elements and that poset satisfies the descending chain condition.

## 1 Introduction

The assembly of a frame $L$ is the frame $\mathbb{N}L$ of nuclei defined on $L$, ordered pointwise. There is a natural embedding of $L$ in its assembly; it assigns $x \in L$ to the closed nucleus $c_x$ determined by $x$. The assembly has been studied by several authors, notably in [Wi94], and most recently in [SS04, Si06, MMZ08]. Some aspects of the way $L$ lies embedded in its assembly are well understood.

In this paper the free meets of a frame are considered as a relative feature: freeness is to be construed *vis-à-vis* a designated class of morphisms out of the frame of discourse. In his thesis ([Wi94]) J. T. Wilson proves a remarkable and intriguing theorem: *A frame $L$ has free meets – relative to all frame maps out of $L$ – if and only if (a) the dual frame law is satisfied by $L$ and (b) $\mathbb{N}L$ is boolean.* Using Conrad's Theorem on complete lattices which are freely generated by their meet-irreducible elements, it is shown here (Theorem 3.6) that that $L$ has free meets and $\mathbb{N}L$ is atomic if and only if $L$ is freely generated by the subset $\mathcal{M}(L)$ of meet irreducible elements, and $\mathcal{M}(L)$ satisfies the descending chain condition (henceforth abbreviated *DCC*). Proposition 4.5 and Corollary 5.3 then characterize the frames with free meets and atomic assembly, for which the canonical embedding into the assembly is coherent.

We assume general familiarity with elementary frame theory. We refer the reader to [J82] and [PT01] for any unexplained concepts and terms.

Let $L$ be a frame. A function $p : L \longrightarrow L$ is a *prenucleus* if it is inflationary, order preserving and satisfies

$$p(x \wedge y) \geq p(x) \wedge y.$$

The set $\mathbb{P}L$ of prenuclei is closed under composition of functions, arbitrary joins, and finite meets, so it is a subframe of the frame of all functions, with respect to pointwise operations. An idempotent – $p(p(x)) = p(x)$ – prenucleus is a *nucleus*. For each prenucleus $p$, there is a nucleus $\bar{p}$ with the same fixed points; $\bar{p}$ is the smallest idempotent prenucleus above $p$. In fact, $\bar{p}$ may be obtained from $p$ by transfinite iteraton of composites of $p$, but that is not relevant to our purposes.

What is needed in the proof of Theorem 3.6 is this. A moment's reflection will make the claim transparent.

**Lemma 1.1.** *Suppose $L$ is a frame and $p$ is a prenucleus on $L$. Then $\bar{p} = 1$ – that is $\bar{p}(x) = 1$, for each $x \in L$ – if and only if $p(x) > x$ for every $x < 1$.*

The map $p \mapsto \bar{p}$ is itself a prenucleus on $\mathbb{P}L$. Therefore, the set $\mathbb{N}L$ of all nuclei is a frame. However, joins in the frame of nuclei cannot generally be computed by taking pointwise joins of functions. The frame $\mathbb{N}L$ is called the *assembly* of $L$.

**Definition & Remarks 1.2.** Throughout these remarks, $L$ denotes a frame. For the record, the bottom 0 in $\mathbb{N}L$ is the identity function, whereas the top 1 is the nucleus which inflates $L$ to 1.

Recall first that $x \to y$ denotes the *Heyting operation*, defined by

$$z \leq x \to y \iff x \wedge z \leq y.$$

Given $a \in L$, the functions

$$c_a(x) = a \vee x \quad \text{and} \quad o_a(x) = a \to x$$

are nuclei. These nuclei satisfy ([J82, II, Lemma 2.6])

$$c_a \overset{\mathbb{N}L}{\vee} o_a = 1 \quad \text{and} \quad c_a \wedge o_a = 0;$$

that is, they are complementary nuclei. Motivated by topology, $c_a$ is referred to as *the closed quotient of $L$ by $a$* and $o_a$ as *the open quotient of $L$ by $a$*. Note that

- the map $c : L \longrightarrow \mathbb{N}L$ by $a \mapsto c_a$ is a frame embedding, while
- $o : L \longrightarrow \mathbb{N}L$ by $a \mapsto o_a$ is one-to-one and satisfies

$$o_{(a\wedge b)} = o_a \vee o_b, \quad \text{and} \quad o_{(\bigvee S)} = \bigwedge \Big\{ o_s \, : \, s \in S \Big\}.$$

(See [Wi94], Proposition 15.1,(e) and (f).)

The assembly is generated by the complemented open and closed nuclei.

**Proposition 1.3.** ([J82, II, Proposition 2.7]) *Let $L$ be any frame. Then $\mathbb{N}L$ is generated (by suprema) of nuclei of the form $c_a \wedge o_b$.*

Finally, we recall (from [MMZ08]) the notion of a *smooth* filter: a filter $S$ of the form $S = j^{-1}\{1\}$, for some nucleus $j$ on $L$. In [NR88] smooth filters are said to be *closed*, while Simmons calls them *admissible* in [Si06].

## 2 Free Meets.

Throughout this section, $L$ stands for a frame, unless the contrary is specified.

In [Wi94], Wilson defines a subset $S$ of a frame $L$ to *have a free meet* if for each frame map $h : L \longrightarrow M$, $h(\bigwedge S) = \bigwedge h(S)$. He gives several characterizations in §25. We highlight [Wi94, Theorem 25.5]: *A filter $S$ is smooth if and only if $S$ is closed under meets of subsets $T \subseteq S$ having free meets.* Wilson's principal theorem on free meets follows.

**Theorem 2.1.** ([Wi94, Theorem 26.1]) *$L$ has free meets if and only if the following two conditions hold:*

(i) *$L$ satisfies the dual frame law;*

(ii) *$\mathbb{N}L$ is a boolean frame.*

Theorem 2.1 should be put in a broader context. First, the *dual frame law* is the distribution of finite joins over arbitrary infima; to say that $L$ satisfies the dual frame law means that the join-meet dual of $L$ is a frame.

Next, we sharpen the definitions associated with free meets.

**Definition 2.2.** It is assumed that $S \subseteq L$, and $\mathcal{H}$ is a class of frame homomorphisms whose domain is $L$.

We say that $S$ has *$\mathcal{H}$-free meets* (or *free meets for $\mathcal{H}$*) if $h(\bigwedge S) = \bigwedge h(S)$, for each $h \in \mathcal{H}$. If $\mathcal{H}$ is the class of all frame maps out of $L$, we drop mention of $\mathcal{H}$, and adopt Wilson's usage: $L$ has *free meets* if every subset of $L$ has free meets.

The reader is reminded of some fairly standard terminology pertaining to lattices – *distributive* lattices, in this context.

**Remark 2.3.** In a frame $L$

- $1 > m \in L$ is *meet-irreducible* if $m = \bigwedge S$, with $S \subseteq L$, implies that $m \in S$. Evidently, every meet-irreducible element is prime.

- The dual of a prime element is a *finitely join-irreducible* element: for $a > 0$, $a = x \vee y$ implies that either $a = x$ or $a = y$. Likewise, the dual of a meet-irreducible element is a *join-irreducible* element. Observe for later use that a compact finitely join-irreducible element is necessarily join-irreducible.
- We say that $c \in L$ is *indecomposable* if $c = a \vee b$, with $a \wedge b = 0$, implies that $a$ or $b$ is $c$.

Let us put together some observations about free meets. First, regarding the dual frame law, there is the following result, proved by Conrad, in [C65, §2]. Let $\mathcal{M}(L)$ denote the set of meet-irreducible elements of the latttice $L$. Following Conrad's usage, say that $\mathcal{M}(L)$ *generates* $L$ if each $x \in L$ is a meet of members of $\mathcal{M}(L)$. If, in addition,

$$\bigwedge \mathcal{S}_1 = \bigwedge \mathcal{S}_2 \implies \mathcal{S}_1 = \mathcal{S}_2,$$

for any upsets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{M}(L)$, then $\mathcal{M}(L)$ *freely generates* $L$. The latter means that $L$ is isomorphic to the frame of all upsets of $\mathcal{M}(L)$.

**Proposition 2.4.** *Suppose $L$ is a lattice that is generated by $\mathcal{M}(L)$. Then the following are equivalent:*

(i) *$\mathcal{M}(L)$ freely generates $L$.*

(ii) *$L$ is completely distributive; that is, for all $x_{ij} \in L$,*

$$\bigwedge_{i\in I} \bigvee_{j\in J} x_{ij} = \bigvee_{f\in J^I} \bigwedge_{i\in I} x_{if(i)}.$$

(iii) *The dual frame law holds.*

Next, an assortment of facts; they are fairly obvious, or else they may be found in [Wi94]. Recall that $L$ is *joinfit* if for each $0 < a \in L$, there is a $z < 1$ in $L$ such that $a \vee z = 1$. This concept, obviously a kin to the better-known fitness conditions, was introduced in [M08]. It is the frame-theoretic counterpart of semisimplicity in commutative rings, and of archimedeaneity in lattice-ordered groups.

**Proposition 2.5.** *Suppose that $L$ is a frame.*

(a) *If $L$ has free meets, then each smooth filter $S$ of $L$ is principal – i.e., $S = \uparrow a$, for some $a \in L$.*

(b) *If $L$ has free meets, then for each $a > 0$ there is a $b \in L$ such that $a \vee b = 1$ and*

$$(a \wedge b) \vee z = 1 \implies z = 1.$$

*In particular, if $L$ is also joinfit, then it is a boolean frame.*

(c) *Suppose $L$ has free meets for all frame surjections of $L$. Then one has the following:*

(i) *eveery prime is meet-irreducible, and, in particular,* $\mathrm{Spec}(L)$ *satisfies the DCC;*

(ii) *the assembly of $L$ is boolean.*

**Proof.** (a) According to [Wi94, Theorem 25.5], a smooth filter $S$ is closed under every infimum of a subset having free meets. In particular, $\bigwedge S$ is in $S$.

(b) By (a), the filter $\Gamma_a \equiv \{\, x \in L \,:\, x \vee a = 1 \,\}$ has a least element $b$. Now, if $(a \wedge b) \vee z = 1$, then, simultaneously, $b \leq z$ and $b \vee z = 1$, whence $z = 1$, as claimed.

(c) As to (i), suppose $p \in L$ is prime. Let $j_p$ be the associated character: $j_p(x) = 0$, if $x \leq p$, while $j_p(x) = 1$, otherwise. Now, as $j_p$ preserves all meets, $\bigwedge S = p$ implies that $\bigwedge j_p(S) = 0$; this can be only if $p \in S$.

Since the meet of a chain of primes is prime, the remaining assertion is clear.

(ii) of (c) is implicit in Wilson's own proof of Theorem 2.1: the freeness of meets is used on a nucleus defined on $L$. ■

**Remark 2.6.** Wilson's proof that, with free meets, a frame $L$ has a boolean assembly uses a result of Beazer and Macnab ([BM79]), which states that $\mathbb{N}L$ is boolean if and only if each upset of $L$ has a smallest dense element.

## 3 Atomic Assemblies.

The objective in this section is to characterize the frames with free meets, for which the assembly is an atomic boolean algebra. The goal is Theorem 3.6; the crucial step is to describe the atoms of the assembly.

We begin with a closer look at the join-irreducible elements of a frame. As in previous sections, $L$ denotes a frame.

**Remark 3.1.** For each join-irreducible element $c$, let $m(c)$ denote the supremum of all elements *strictly* less than $c$. Then $m(c) < c$ and clearly the largest element under $c$. Now consider

$$v(c) = c \to m(c).$$

It is a routine matter to check that

- $v(c)$ is meet-irreducible and maximal with respect to $c \not\leq v(c)$;
- $v(c)$ is the unique meet-irreducible element which is maximal with respect to $c \not\leq y$.

- For any two special elements $a, b \in L$,

$$a < b \iff m(a) < m(b) \iff v(a) < v(b).$$

Now assume that $L$ satisfies the dual frame law. Reversing the roles of meets and joins, we have the inverse of the map $c \mapsto v(c)$; explicitly, starting with the meet-irreducible $m$, let $m^*$ denote the least element $> m$. Finally, put

$$c(m) = \wedge \{ x \in L : m^* \leq m \vee x \}.$$

Then $m \mapsto c(m)$ inverts $c \mapsto v(c)$.

Let $\mathcal{J}(L)$ stand for the set of join-irreducible elements of $L$.

Next, we proceed to identify the atoms in the assembly of $L$. It should be clear that if $j \in \mathbb{N}L$ is an atom then it is of the form $c_a \wedge o_b$. Moreover, $c_a \wedge o_b > 0$ precisely when $a \not\leq b$.

The proof of the lemma that follows is adapted from an argument provided by the referee of this paper. The insights he provided have made this a better paper.

**Lemma 3.2.** *Suppose that $a \in L$ is join-irreducible and $v = v(a)$. Then $c_a \wedge o_v$ is an atom.*

**Proof.** It is enough to show that if $0 < c_x \wedge o_y \leq c_a \wedge o_v$, then the two nontrivial nuclei agree. To this end, observe that $c_x \wedge o_y \leq c_a$, which implies that $c_x \leq c_a \vee c_y$ – because $c_y$ and $o_y$ are complements of one another in $\mathbb{N}L$. Thus, $x \leq a \vee y$, and since $x \not\leq y$, we also have $a \not\leq y$, which means that $y \leq v$, and so $o_v \leq o_y$. Starting with $c_x \wedge o_y \leq o_v$, a similar argument leads to $x \wedge v \leq y$, whence $x \not\leq v$ and, therefore, $a \leq x$ and $c_a \leq c_x$. It follows that $c_a \wedge o_v \leq c_x \wedge o_y$, and hence the desired equality. ■

**Remark 3.3.** These comments are recorded by way of motivation for what lies immediately ahead.

It is not enough, in general, that every nontrivial nucleus exceed an atom of the assembly, for the latter to be boolean. One must have the feature that each nucleus be a join of atoms. The following example is telling.

Let $L$ denote the inversely well-ordered natural numbers, together with a least element 0. In [Wi94], Wilson argues that $\mathbb{N}L$ is not boolean; note that the character of 0, namely, the nucleus $j$ defined by $j(n) = 1$, for every natural number $n$, and $j(0) = 0$, does not preserve the meet 0 of the natural numbers. On the other hand, for each $n$ (which is special) $v(n) = n + 1$, and

$$j_{n,n+1}(m) = m, \quad \text{for each} \quad m \neq n + 1,$$

while $j_{n,n+1}(n + 1) = n$. And the reader will readily verify that

$$\bigvee_n j_{n,n+1}(m + 1) = m,$$

for each natural number $m$, so that $l = \bigvee_n j_{n,n+1} < j$. Thus, $j$ is not a join of atoms.

The following observation further underscores the point raised just now. Lemma 3.4 is surely known; yet, as it is so useful, leading up to Theorem 3.6, we give the short proof anyway.

**Lemma 3.4.** *Suppose that $L$ is a frame in which 1 is a join of atoms. Then each nontrivial element of $L$ is a join of atoms, and $L$ is boolean.*

**Proof.** Let $A(L)$ denote the set of atoms of $L$; the hypothesis is that $1 = \bigvee A(L)$. So if $x \in L$ is not zero, then $x = \vee \{ a \in A(L) : a \le x \}$, by the frame law.

Next, for each $y \in L$, $y \vee y^{\perp}$ is a supremum of atoms, and hence of all the atoms; that is, $y \vee y^{\perp} = 1$. ∎

Here is the final preliminary to Theorem 3.6.

**Lemma 3.5.** *Suppose $\mathcal{M}(L)$ freely generates $L$ and satisfies the DCC. Suppose $a \in L$, $p \in \mathcal{M}(L)$ such that $p > a$ and minimal in $\mathcal{M}(L)$ with this property. Then*

$$p \to a \nleq a.$$

**Proof.** On account of the DCC, $a$ is the infimum of meet-irreducible elements which are minimal in $\mathcal{M}(L)$ with respect to being above $a$. If $\mathcal{M}(a)$ is the set of such meet-irreducible elements, then owing to the dual frame law (by Theorem 2.4),

$$a < \bigwedge \mathcal{M}(a) \setminus \{q\}, \quad \forall q \in \mathcal{M}(a).$$

Thus, if $p \in \mathcal{M}(a)$ and $p \to a \le a$,

$$a < \bigwedge \mathcal{M}(a) \setminus \{p\} \le p \to a \le a,$$

which is a contradiction. ∎

The groundwork is now prepared for the central theorem.

**Theorem 3.6.** *For a frame $L$, the following are equivalent.*

(a) *$L$ has free meets and $\mathbb{N}L$ is atomic.*

(b) *$L$ has free meets for surjective frame maps, and is spatial.*

(c) *$L$ is freely generated by $\mathcal{M}(L)$, which satisfies the DCC.*

(d) *$L$ satisfies the dual frame law, and $\mathbb{N}L$ is an atomic boolean frame.*

**Proof.** (a) $\Longrightarrow$ (b): It is enough to note that, since $\mathbb{N}L$ is an atomic boolean frame, it is spatial, which implies that $L$, as a subframe, is also spatial.

(b) $\Longrightarrow$ (c) is a consequence of Proposition 2.5(c), together with Theorem 2.4, and it is clear that (d) $\Longrightarrow$ (a).

Finally, (c) $\Longrightarrow$ (d): Theorem 2.4 gives us that the dual frame law holds. Further, by Lemma 3.4, it suffices to show that $1 \in \mathbb{N}L$ is a join of the atoms $c_a \wedge o_v$, with $a \in \mathcal{J}(L)$ and $v = v(a)$. Let $f$ be the pointwise supremum of all such atoms; by Lemma 1.1, we must show that $f(x) > x$ for each $x < 1$ in $L$.

Suppose, to the contrary, that $f(b) = b < 1$; let $p \in \mathcal{M}(b)$, using the notation of Lemma 3.5. Then, for a join-irreducible $a$ such that $p = v(a)$ we have:

$$p \to b \leq c_a(b) \wedge o_p(b) \leq f(b) = b,$$

which contradicts Lemma 3.5. Thus,

$$1 = \bigvee_{a \in \mathcal{J}(L)} c_a \wedge o_{v(a)},$$

which concludes the proof. ∎

## 4 Algebraic Frames.

In this section we consider algebraic frames, in the light of the foregoing material. Recall that a complete lattice $L$ is *algebraic* if each member of $L$ is a join of compact ones. We use $\mathfrak{k}(L)$ to denote the join-subsemilattice of all compact elements of the frame $L$. We say that $L$ *has the finite intersection property (FIP)* if the meet of two compact elements of $L$ is compact.

The first observation to make is the following. It suffices to sketch the proof.

**Proposition 4.1.** *Any frame with free meets and atomic assembly is algebraic. The compact elements are the finite joins of join-irreducible elements.*

**Proof.** Suppose $x > 0$ in $L$; we show that $x$ is a join of join-irreducible elements. Let $q \in \mathcal{M}(L)$ such that $x \not\leq q$; then $c(q) \leq x$, and it is easy to verify that

$$x = \bigvee \Big\{ c(q) \,:\, q \in \mathcal{M}(L),\ x \not\leq q \Big\}.$$

That the compact elements are as specified is obvious. ∎

However, a frame having the properties of Theorem 3.6 need not have the FIP; consider the following example.

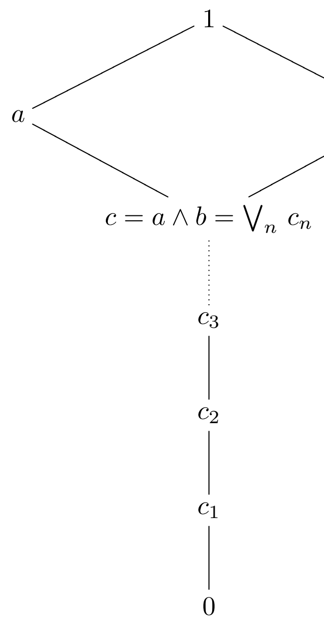
**Example 4.2.** $L$ is the frame depicted below. The salient features of this frame are:

1. Each element, except $a \wedge b$, is compact. Each compact element but the top is join irreducible.

2. $L$ is algebraic and distributive, and, therefore, a frame. The FIP fails.

3. Every meet is a finite meet, and so it is obvious that in $L$ the dual frame law holds. It is also clear that $L$ satisfies the DCC. Thus, $L$ has free meets and an atomic boolean assembly.

The knowledgeable reader will notice that the dual frame – by turning the picture upside down – is also algebraic, and the frame of all convex $\ell$-subgroups of an $\ell$-group.




The last comment on the foregoing example prompts the next remark.

**Remark 4.3.** Any frame $L$ having free meets and an atomic assembly is generated by its meet-irreducible elements, so that $L$ with the dual ordering is also algebraic.

Next, we record an observation which should generate further study; namely, Theorem 3.6 and Proposition 4.4 seem to suggest that for a spatial frame, free meets for all frame maps will follow from the same property for a suitable class of surjective frame maps. One ought to wonder how restrictive such a class of surjections may be.

Recall that a frame homomorphism $h$ is *coherent* if it maps compact elements to compact elements.

**Proposition 4.4.** *An algebraic frame $L$ satisfies each of the conditions of Theorem 3.6 if and only if $L$ is spatial and has free meets for all coherent frame surjections of $L$.*

**Proof.** For each $x \in L$, $c_x$ is coherent, and the map in the proof of Proposition 2.5(c)(i) obviously is coherent as well. This means that if $L$ $L$ is spatial and has free meets for all coherent frame surjections of $L$, then every prime is meet-irreducible, whence (c) of Theorem 3.6 follows. ■

Finally, in this section, we discuss when the embedding $x \mapsto c_x$ is itself coherent.

**Proposition 4.5.** *For a frame $L$ with free meets and atomic assembly, the following are equivalent.*

(a) *The map $c : L \longrightarrow \mathbb{N}L$ is coherent.*

(b) *For each compact $a \in L$, $(\downarrow a) \cap \mathcal{J}(L)$ is finite.*

(c) *For each compact $a \in L$, $\downarrow a$ is finite.*

(d) *For each $q \in \mathcal{M}(L)$, $\downarrow q$ is finite.*

*If the above are satisfied, then $L$ has the FIP.*

**Proof.** Let $a \in \mathfrak{k}(L)$ and $b \in \mathcal{J}(L)$, with $v = v(b)$. In a calculation similar to the proof of Lemma 3.2, we have that

$$c_b \wedge o_v \leq c_a \quad \Longleftrightarrow \quad b \leq a.$$

Since in $\mathbb{N}L$ the compact elements are the finite joins of atoms, the equivalence of (a) and (b) is clear.

The rest is easy, and is left to the reader. ■

## 5 Concluding Remarks.

To begin, we have a question.

**Question 5.1.** *If $L$ is a frame subject to the properties of Theorem 3.6, it is a complete sublattice of an atomic boolean frame, and it is algebraic (albeit not necessarily coherently embedded). Is the converse true?*

If $L$ is a complete sublattice of such a boolean frame, then it too must be completely distributive ([MB89, Theorem 14.5]), and hence be a dual frame. And since atomic boolean frames are spatial, $L$ is, perforce, spatial as well.

If one invokes Zorn's Lemma, then every algebraic latice is generated by the meet-irreducible elements ([M73, Lemma 1.3]). And thus we have the following conclusion: *If $L$ is an algebraic frame which is completely embedded in an atomic boolean frame, then – in ZFC – then $\mathcal{M}(L)$ freely generates $L$ (Theorem 2.4 again).*

That is half of Theorem 3.6(c); so what about the DCC on $\mathcal{M}(L)$? The best answer we can provide is the following.

**Proposition 5.2.** (ZFC) *Suppose $L$ is an algebraic frame which is coherently and completely embedded in an atomic boolean frame. Then every prime of $L$ is meet-irreducible.*

**Proof.** Suppose $B$ is an atomic boolean frame containing $L$ as a complete subframe, and suppose the inclusion $L \subseteq B$ is coherent. Let $g : B \longrightarrow L$ denote the right adjoint of the inclusion; thus, for each $x \in B$, $g(x)$ is the supremum of all $z \in L$, with $z \leq x$. It is well known that $g$ preserves all infima, and, therefore, maps primes to primes; in [HM07, Lemma 3.2] it is shown (using Zorn's Lemma) that $g$ always induces a surjective map from one spectrum onto the other – that is to say, for any coherent embedding of algebraic frames.

From the comments in 5.1, $\mathcal{M}(L)$ freely generates $L$. The following calculation then demonstrates that $g$ maps meet-irreducibles to meet-irreducibles:

$$\bigwedge S \leq g(x) \iff \bigwedge S \leq x,$$

for all subsets $S$ of $L$, each $x \in B$, and by the assumption that $L$ is completely embedded in $B$.

Now suppose that $p$ is prime in $L$; let $q$ be a prime of $B$ such that $g(q) = p$. Observe that since $B$ is boolean, the primes are precisely the complements of atoms, and, therefore, clearly meet-irreducible. Hence, $p \in \mathcal{M}(L)$, completing the proof. ∎

And, thus, we have the expected corollary of Propositions 5.2 and 4.5.

**Corollary 5.3.** (ZFC) *Suppose $L$ is a frame. The following are equivalent.*

- *$L$ has free meets and an atomic boolean assembly, in which it is coherently embedded.*
- *$L$ is algebraic, and it can be coherently and completely embedded in an atomic boolean frame.*

To conclude, and place this effort in a kind of perspective, in part historical, but also strongly tied to our development, it seems appropriate to highlight Theorem 2.4 once more. When we first considered frames with free meets, and saw that the dual frame law entered the picture, we were reminded, immediately, of Theorem 2.4. Our conditioning then brought to mind that this was but a stepping stone for Conrad to his famous theorem on finite-valued $\ell$-groups – [C65] and, more expansively, [D95, §46]. Martínez generalized Conrad's Theorem in [M72] to a frame-theoretic setting, though not in the language of frame theory. Our mindset then became attached to algebraic frames, the ambient structures of [M72].

The referee of this paper disabused us of the notion that the ambient structures *here* should be algebraic, and for that we are grateful. Nonetheless, since this work was inspired by Conrad, we conclude by stating one of the generalizations of his theorem from [M72], for algebraic frames.

Somehow, this inspiration seems to be contagious, as some version or other of Theorem 5.4 keeps resurfacing, such as in [ST93].

**Theorem 5.4.** ([M72, Corollary 3.1.1]) *Let $L$ be an algebraic frame. Assume that $\mathcal{M}(L)$ generates $L$. Then the following are equivalent.*

(a) *$\mathcal{M}(L)$ freely generates $L$.*

(b) *$L$ is completely distributive.*

(c) *The dual frame law holds for members of $\mathcal{M}(L)$.*

(d) *Each meet-irreducible element $q$ of $L$ is maximal with respect to not exceeding some join-irreducible element* (namely, $c(q)$).

(e) *Each $x \in \mathfrak{k}(L)$ can be expressed, uniquely, as a finite join of pairwise incomparable join-irreducible elements.*

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