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Free topological groups of weak P-spaces

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Abstract

It is known that the free topological group over the Tychonoff space X, denoted $F(X)$, is a P-space if and only if X is a P-space. This article is concerned with the question of whether one can characterize when $F(X)$ is a weak P-space, that is, a space where all countable subsets are closed. Our main result is that $F(X)$ is a weak P-space if and only if X is a weak P-space and every countable subset is C-embedded. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We assume that all of our topological spaces are Tychonoff, that is, completely regular and Hausdorff. In fact, we are interested in Tychonoff weak P-spaces, and the aim of this paper is to determine when the free topological over a space X lies in this class.

Definition 1.1. A point $p \in X$ is called a *P-point* if its filter of neighbourhoods is closed under countable intersections. In a Tychonoff space this is equivalent to saying every zeroset containing the point is a neighbourhood of the point.

A space X is called a *P-space* if every point in X is a P-point. X is a P-space if and only if every zeroset is open. See [5] for more information on P-spaces.

Definition 1.2. A point $p \in X$ is called a *weak P-point* if it is separated from every countable set not containing it. A space X is called a *weak P-space* if every point of X is a weak P-point. Observe that X is a weak P-space if and only if every countable subset of X is closed. Every P-space is a weak P-space.

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It is known that the free topological group over a space X is a P-space if and only if X is a P-space. We question, address, and resolve this question whether the same can be said for weak P-spaces. Presently, we conclude this section by supplying some definitions and examples which should aid the casual reader.

Definition 1.3. For a space X, $C(X)$ denotes the set of continuous real-valued functions on X. Let $Y \subseteq X$. We say Y is *C-embedded in X* if every continuous function on Y has a continuous extension to X.

We now end this section with some examples. We shall give a brief description of the free topological group over the space X at the beginning of the next section.

Example 1.4.

(i) The following information may be found in [9]. Let $r \in \mathbb{R}$ and M a Lesbegue measurable set. r is called a *point of density of M* if
$$\lim_{h\to 0^+} \frac{\lambda(M \cap (r-h, r+h))}{2h} = 1.$$
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *approximately continuous* at a point r if
$$\lim_{x\in M\to r} f(x) = f(r)$$
for some measurable set M, where r is a point of density of M.
The *density topology* on $\mathbb{R}$ is the weak topology generated by all approximately continuous functions on $\mathbb{R}$. It is known that $\mathbb{R}$ with the density topology is a connected weak P-space. Since every point in $\mathbb{R}$ with the density topology is a G_δ-point and hence a zero-set it follows that $\mathbb{R}$ with the density topology is not a P-space.

(ii) Let $\boldsymbol{D}$ be a discrete space of uncountable, yet non-measurable cardinality. Choose any non-principal uniform ultrafilter p on $\boldsymbol{D}$, i.e., $p \in \beta\boldsymbol{D} - \boldsymbol{D}$ and every set in the ultrafilter has the same cardinality as $\boldsymbol{D}$. Next, let $X = \boldsymbol{D} \cup \{p\}$ whose topology is the subspace topology inherited from $\beta\boldsymbol{D}$. Then as p is a uniform ultrafilter, every countable set of X is closed. As $\boldsymbol{D}$ has non-measurable cardinality it follows that there exists a countable collection of sets in the ultrafilter whose intersection is empty. Translating these properties to X, this means that X is a weak P-space and p is a G_δ-point, hence X is not a P-space.

(iii) The space X, consisting of all the weak P-points of $\beta\mathbb{N} - \mathbb{N}$, forms a weak P-space which is not a P-space and such that no point of X is a G_δ-point.

2. Free topological groups

We shall concern ourselves with the Markov free topological group over the space X. As the theory of free topological groups is well-developed we leave out any preliminary information on them and suggest [10,6]. Let us just say that given a Tychonoff space X there exists a (Tychonoff) topological group $F(X)$ and a continuous embedding

$i_X : X \to F(X)$ such that if $f : X \to G$ is any continuous function from X into an arbitrary topological group then there exists a continuous extension $\hat{f} : F(X) \to G$ for which $\hat{f} \circ i_X = f$. Observe that the application of F is in fact a functor; that is, given a continuous map $\alpha : X \to Y$, there exists a natural group homomorphism $F(\alpha) : F(X) \to F(Y)$ which makes the following diagram commute.

$$\begin{array}{ccc} X & \hookrightarrow & F(X) \\ \alpha \downarrow & & \downarrow F(\alpha) \\ Y & \hookrightarrow & F(Y) \end{array}$$

The group structure of $F(X)$ is precisely the free group over X. We let e denote the identity element of $F(X)$ (the empty word). The topology on $F(X)$ is the coarsest topology making i_X and all extensions $\hat{f}$ continuous. Though easily defined, this topology is very complicated and often involves using norms. For example, only when X is discrete is $F(X)$ metrizable (and in fact, discrete). Though introduced independently in the forties by Markov and Graev (see [10,6]), only recently have researchers discovered when, for a metrizable space X, $F(X)$ is countably tight (see [3]); precisely when X is separable or discrete.

As we mentioned in the first section it is known that X is a P-space if and only if $F(X)$ is a P-space. This is due to the fact that given a topological group G, the P-ification of G is again a topological group [4]. Since there is no such "weak P-ification" of an arbitrary Tychonoff space, it is natural to question whether X is a weak P-space if and only if $F(X)$ is a weak P-space. We shall show that the answer to this question is negative and characterize those weak P-spaces X for which $F(X)$ is also a weak P-space. The theorem follows from what we know about weak P-spaces together with a corollary to a theorem of Uspenskii [11]. To properly apply said theorem we shall need a few definitions.

Definition 2.1. If $Y \subseteq X$, then there is a natural continuous injection of $F(Y)$ into $F(X)$. In general, this injection may not be an embedding. If it is, we say that Y is *F-embedded* in X and that $F(Y)$ is a *topological subgroup of* $F(X)$.

Definition 2.2. Let $S \subseteq F(X)$. The *carrier of S* is defined to be the subset of X consisting of all elements of X taking part in an irreducible expression of an element of S. If $\langle S \rangle$ denotes the subgroup of $F(X)$ generated by S, then $\operatorname{car} S = \operatorname{car}\langle S \rangle$.

Corollary 2.3 [11, Theorem 2]. *Suppose $Y \subseteq X$ is countable. Then Y is F-embedded in X if and only if Y is C-embedded in X.*

We are now ready to prove the main theorem of our paper.

Theorem 2.4. *The following are equivalent for a topological space X.*

(i) *X is a weak P-space and every countable subset of X is C-embedded.*
(ii) *X is a weak P-space and every countable subset of X is F-embedded.*
(iii) *$F(X)$ is a weak P-space.*

Proof. The corollary to Uspenskii's Theorem implies that (i) and (ii) are equivalent.

(i) $\Rightarrow$ (iii) We would like to show that e is a weak P-point of $F(X)$ (and so every point of the (homogeneous) topological group $F(X)$ is a weak P-point). Let $S = \{a_1, a_2, \ldots\}$ be a subset of $F(X)$ such that $e \notin S$. Observe that $\operatorname{car} S$ is countable. By our assumptions and prior corollary $\operatorname{car} S$ is C- and F-embedded in X. Thus, $F(\operatorname{car} S)$ is a topological subgroup of $F(X)$.

As X is a weak P-space it follows that $\operatorname{car} S$ is a discrete space and so $F(\operatorname{car} S)$ is equipped with the discrete topology, hence $F(\operatorname{car} S)$ is a discrete subspace of $F(X)$. Next, since S is a subset of the discrete subspace $F(\operatorname{car} S)$ it follows that e is not in the closure of S in $F(X)$, whence e is a weak P-point of $F(X)$.

(iii) $\Rightarrow$ (i) Conversely, if $F(X)$ is a weak P-space, then first, X is a weak P-space. Next, let $S \subseteq X$ be a countable subset, then $\langle S \rangle$ is a countable subset of $F(X)$ and so is a discrete subspace. Thus, $F(S)$ is a topological subgroup of $F(X)$, i.e., S is F-embedded in X. By the corollary it follows that S is C-embedded. □

Remark 2.5. The known fact that every countable subset of a P-space is C-embedded may be easily derived from Theorem 2.4 and the fact that P-spaces are weak P-spaces.

As we may apply our functor F denumerably many times it is natural to wonder whether $F^2(X)$ is a weak P-space given that X is a space satisfying condition (i) of Theorem 2.4, or equivalently given that $F(X)$ is a weak P-space. We obtain the following result.

Theorem 2.6. *Suppose X is a weak P-space and every countable subset is C-embedded. Then every countable subset of $F(X)$ is C-embedded in $F(X)$.*

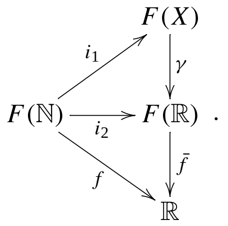
Proof. Without loss of generality, we assume that our set in question is of the form $\langle S \rangle$, for some countable subset $S \subseteq X$. Enumerate $S = \{x_n\}$. We shall need the following lemma.

Lemma 2.7. *Suppose $S \subseteq X$ is a countable, discrete and closed subset of X. If S is C-embedded in X, then there exists a continuous map $\gamma : F(X) \to F(\mathbb{R})$ whose restriction to $F(S)$ is a homeomorphism onto $F(\mathbb{N})$.*

Proof. Let $f : S \to \mathbb{N}$ be defined by $f(x_n) = n$. Since $S \subseteq X$ is a C-embedding there is a continuous extension $\bar{f} : X \to \mathbb{R}$. The free functor gives rise to the continuous homomorphism $F(\bar{f}) : F(X) \to F(\mathbb{R})$. From here it is an easy exercise to show that the restriction of $F(\bar{f})$ to $F(S)$ is in fact a homeomorphism onto $F(\mathbb{N})$, which concludes the proof of the lemma. □

Returning to the proof of Theorem 2.6, we would like to show that any function $f : F(S) \to \mathbb{R}$ may be extended to $F(X)$ (here $F(S) = \langle S \rangle$). First, we identify $F(S)$ with $F(\mathbb{N})$. Next, we let $\bar{f} : F(\mathbb{R}) \to \mathbb{R}$ be any continuous function extending f. Such a function exists as $F(\mathbb{N}) \subseteq F(\mathbb{R})$ is a C-embedding.

Since the following diagram commutes, where γ is the continuous map supplied by Lemma 2.7 and i_1, i_2 are the natural embeddings of $F(\mathbb{N})$ into $F(X)$ and $F(\mathbb{R})$, respectively, the result follows.



□

Corollary 2.8. *Suppose X is a weak P-space for which every countable subspace is C-embedded. Then $F^n(X)$ is a weak P-space for every natural number n.*

We end this section with an example of a weak P-space with no countable C-embedded subspace.

Example 2.9. The set of weak P-points of $\beta\mathbb{N} - \mathbb{N}$ forms a pseudocompact weak P-space (see [2, Proposition 6.3]). As pseudocompact spaces are characterized by having no C-embedded discrete, countable subsets (see [12]), it follows that this space does not satisfy the conditions of Theorem 2.4 and so there exist weak P-spaces whose free topological groups are not weak P-spaces.

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