Topology and its Applications 97 (1999) 165–174

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

The separated cellularity of a topological space and finite separation spaces

R.T. Finn, J. Martinez *, W.W. McGovern

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

Received 13 January 1997; received in revised form 13 March 1997

Abstract

A separated cell $\mathfrak{A}$ of a topological space X is a family of pairwise disjoint open sets with the property that, for any partition $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$ of $\mathfrak{A}$, $\bigcup \mathfrak{A}_1$ and $\bigcup \mathfrak{A}_2$ are completely separated. The separated cellularity $sc(X)$ of X is the supremum of all cardinals of separated cells. A space X has finite separation if it has no infinite separated cells. With X compact, every separated cell of X has size $< \kappa$ if and only if no regular closed subset has a continuous surjection onto $\beta\kappa$. It is shown that for normal, first-countable spaces finite separation is equivalent to sequential compactness. Compact finite separation spaces are examined, and compared to other classes of compact spaces which occur in the literature. It is shown that all dyadic spaces have finite separation. Likewise, every compact scattered space, every compact countably tight space, and every compact hereditarily paracompact space has finite separation. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Separated cellularity; Finite separation; $\pi\chi$-spaces; Independence character

AMS classification: Primary 54A25; 54D30, Secondary 54D99; 06E15; 54G05

1. Introduction

Spaces with finite separation arose in connection with the study of global dimension of rings of functions. The concept is associated with a cardinal invariant, called separated cellularity, which we shall introduce presently. In [3] the global dimension of a boolean ring is discussed, and there it emerges that when a boolean ring has global dimension 2 then its independence character cannot exceed $\aleph_1$, and from this realization it is not far to the concept of finite separation. In this paper we first characterize what it means for all separated cells of a compact space to be of size $< \kappa$; applying this to ω, a number of

* Corresponding author. E-mail: martinez@math.ufl.edu.

PII: S0166-8641(98)00075-3

straightforward observations reveal that many interesting classes of compact spaces have finite separation.

We shall assume, unless the contrary is stated, that all topological spaces are *Tychonoff*; that is, X is Hausdorff and there is a base of cozerosets for the open sets of X; the reader should consult [5] for any unexplained, basic terminology. Our main reference for general topology is [2].

Definition and comments 1.1. Let X be a space. A family of nonempty pairwise disjoint open sets will be called an *open cell* of X. If $\mathfrak{A}$ is an open cell such that, for any subset $\mathfrak{A}_0$ of $\mathfrak{A}$, $U = \bigcup \mathfrak{A}_0$ and $V = \bigcup \mathfrak{A} \setminus \mathfrak{A}'$ are completely separated—meaning that there is a continuous real-valued function f on X which maps U to 1 and V to 0—the family $\mathfrak{A}$ will be called a *separated cell*. The supremum of cardinals of open cells is the *cellularity* of X, denoted $c(X)$. The supremum over all cardinals of separated cells is the *separated cellularity* of X, and it is denoted $sc(X)$. Clearly, $sc(X) \leqslant c(X)$.

If $sc(X) = \aleph_0$ this may happen because X has no uncountable separated cell while it does have a countable one. This occurs, for example in $\beta\omega$, the Stone–Čech compactification of the discrete natural numbers. Or $sc(X) = \aleph_0$ because there are no infinite separated cells at all. If X has this feature we say that X *has finite separation* or is a *finite separation space*. Most of the time we will abbreviate, and say that X is an *fs*-space. Explicitly then, X is an *fs*-space if for each infinite cell $U_1, U_2, \ldots$ there is a partition of the naturals into A and B so that the sets $U = \bigcup_{n\in A} U_n$ and $V = \bigcup_{n\in B} U_n$ are not completely separated. It is not hard to see directly that every compact metric space is an *fs*-space, and that the one-point compactification of any discrete space D, αD, is also an *fs*-space.

Before going any further let us record an important example of an *fs*-space.

Example 1.2. Observe that if κ is any infinite cardinal, then 2^κ is an *fs*-space. In this space the clopen sets are precisely the finite unions of basic clopen sets of the Tychonoff topology on the direct product. Now, if 2^κ has an infinite separated cell, it must be countable since it is well known that Cantor spaces have countable cellularity (see 3R in [9]). On the other hand, separation only involves at most countably many indices in κ. That, together with the fact that projection of 2^κ onto any 2^λ, where $\lambda \leqslant \kappa$, sends basic open sets to basic open sets, and therefore clopens to clopens, imply that a separated cell in 2^κ must map by projection to a separated cell in some 2^ω, which is the ordinary Cantor set, which is metric and hence an *fs*-space. Thus, 2^κ too is an *fs*-space. This example shows that *fs*-spaces can have arbitrarily large independence character. (For additional discussion on independence, see Section 3.)

Recall that a space is *dyadic* if it is a continuous image of a Cantor space. We show presently (Proposition 2.2.2) that each dyadic space is an *fs*-space.

Before proceeding with the study of *fs*-spaces per sé, let us prove a basic relationship between separated cells and continuous images of the form $\beta\kappa$. We begin with two lemmas.

Lemma 1.3. *Suppose that $f : X \to Y$ is a continuous surjection, and κ is an infinite cardinal. If every separated cell of X has size $< \kappa$ then the same is true of Y.*

Proof. Simply note that if $\mathfrak{A}$ is a separated cell in Y then $\mathfrak{A}'$, the family of inverse images of sets in $\mathfrak{A}$, is a separated cell in X of the same cardinality. □

Lemma 1.4. *Suppose that X is a normal space in which every separated cell has size $< \kappa$.* (κ *is an infinite cardinal.*) *Then every regular closed set K has the same property.*

Proof. Suppose that $\mathfrak{A}$ is a separated cell of K-open sets. Since K is regular, $U \cap \operatorname{int}_X K \neq \emptyset$ for each $U \in \mathfrak{A}$. Evidently, the family $\{U \cap \operatorname{int}_X K\colon U \in \mathfrak{A}\}$ is a separated cell in K (of size κ). Since K is closed and X is a normal space, this is also a separated cell in X, a contradiction. □

Here is the characterization alluded to earlier.

Theorem 1.5. *Suppose that X is a compact space and κ is an infinite cardinal. The following are equivalent*:

(1) *Every separated cell of X has size $< \kappa$.*
(2) *No regular closed subset of X maps continuously onto $\beta\kappa$.*
(3) *No regular closed subset of X retracts onto $\beta\kappa$.*

Proof. That (1) implies (2) follows from Lemmas 1.3 and 1.4, and the fact that $\beta\kappa$ evidently has a separated cell of size κ, namely the collection of singletons of isolated points. (2) $\Rightarrow$ (3) is obvious. As to the remaining implication, suppose that $\mathfrak{A}$ is a separated cell. The closure $K = \operatorname{cl}_X(\bigcup \mathfrak{A})$ is clearly regular. Well order the sets in $\mathfrak{A} = (U_\alpha)_{\alpha<\kappa}$. Consider now the map h from $\bigcup \mathfrak{A}$ onto κ which sends all elements of U_α to α. The fact that the U_α form a separated cell implies that the Stone extension

$$h^\beta : \beta\Big(\bigcup \mathfrak{A}\Big) \to \beta\kappa$$

of h, which maps onto $\beta\kappa$, factors through K. The second factor, h', out of K, then maps continuously onto $\beta\kappa$. Now pick a point p_α out of each U_α, and let $Y = (p_\alpha)_{\alpha<\kappa}$. It should be clear that $Y' = \operatorname{cl}_X Y$ is a copy of $\beta\kappa$, and that the restriction of h' to Y' is a homeomorphism. This suffices to show that (3) implies (1). □

We now turn to *fs*-spaces. We are especially interested in compact *fs*-spaces.

2. Compact *fs*-spaces

Let us recall some familiar forms of compactness.

Definition and remarks 2.1. Let X be a Hausdorff space. X is *countably compact* if every countable open cover of X has a finite subcover. X is *sequentially compact* if every sequence in X has a convergent subsequence. It is well known that every sequentially compact space is countably compact [2, 3.10.30]. The converse is false: $\beta\omega$ is compact but not sequentially compact. It is well known that for metric spaces these notions are equivalent to compactness [2, 4.1.17]. In fact, if X is first countable then X is sequentially compact if and only if it is countably compact [2, 3.10.31].

From Lemma 1.3 we immediately derive two results. Recall that a space X is *pseudocompact* if every continuous real-valued function out of X is bounded.

Proposition 2.2.1. *Any continuous image of an fs-space is an fs-space. In particular, every fs-space is pseudocompact.*

Proof. If X is not pseudocompact, there is a continuous map of X onto an unbounded subset of the reals, which is clearly not an *fs*-space. □

Proposition 2.2.2. *Every dyadic space is an fs-space.*

The first connection between *fs*-spaces and classical notions of compactness is this:

Proposition 2.3. *Every sequentially compact space is an fs-space.*

Proof. Suppose X is sequentially compact, yet $U_1, U_2, \ldots$ is an infinite separated cell in X. Pick a point $p_n \in U_n$. By assumption, $\{p_n\colon n < \omega\}$ has a subsequence, indexed by the infinite subset L of ω, which converges to, say, p. Now partition $L = L_1 \cup L_2$ into two infinite subsets, and let $\omega = N_1 \cup N_2$ be a partition so that $L_i \subseteq N_i$, with $i = 1, 2$. Let $V_i = \bigcup\{U_n\colon n \in N_i\}$, and note that $p \in \operatorname{cl} V_1 \cap \operatorname{cl} V_2$, which contradicts the separation of the cell $U_1, U_2, \ldots.$ □

Recall that a space X is *sequential* if $p \in \operatorname{cl} S$ $(S \subseteq X)$ precisely when there is a sequence of points in S converging to p. Now collect the Remarks 2.1, Propositions 2.2.1 and 2.3, and recall [2, 3.10.21 and 3.10.31] that every normal sequential, pseudocompact space is countably compact. We then have the following result.

Proposition 2.4. *For a normal, sequential space X the following are equivalent.*
(a) *X is sequentially compact*;
(b) *X is countably compact*;
(c) *X is an fs-space*;
(d) *X is pseudocompact.*

At this stage a few remarks are in order.

Remarks 2.5. (a) It should be observed that in Proposition 2.4, (b) implies (d) for any Tychonoff space [2, 3.10.20].

(b) Recall the example Ψ [5, 5I]: consider a maximal *almost pairwise disjoint* family $\mathcal{E}$ of subsets of ω; that is, for each $X, Y \in \mathcal{E}$, $X \cap Y$ is finite, and $\mathcal{E}$ is maximal with respect to this condition. One constructs a space $\Psi = \omega \cup \{p_X\colon\ X \in \mathcal{E}\}$, by adjoining to ω a point p_X for each $X \in \mathcal{E}$. The points of ω are isolated in Ψ, while a neighborhood of p_X is of the form $\{p_X\} \cup K$, with $K \subseteq \omega$, and $X \setminus K$ is finite. Then Ψ is easily shown to be an *fs*-space. But, while Ψ is first countable, it is not countably compact and not normal, thus showing that, in the absence of normality, (c) does not imply (b) in Proposition 2.4.

(c) The well ordered space ω_1 (under the interval topology) is normal, countably compact and first countable. Thus, Proposition 2.4 applies to it, and ω_1 is an *fs*-space. However, ω_1 is not metrizable since it is not paracompact [2, 5.1.3].

(d) By Theorem 1.5, no compact *fs*-space can map continuously onto $\beta\omega$. On the other hand, since a Cantor space is an *fs*-space, and $2^{\mathfrak{c}}$ contains a copy of $\beta\omega$ [2, 3.6.20], we see that a compact *fs*-space can contain a copy of $\beta\omega$. This stands in contrast to the situation with compact sequentially compact spaces, which cannot contain such copies. Observe as well that an arbitrary closed subspace of a compact *fs*-space need not be an *fs*-space.

If X is totally ordered space (that is to say, a space which is a totally ordered set, endowed with the interval topology) then it is an *fs*-space precisely when it is sequentially compact. Let us indicate why.

Theorem 2.6. *Let X be a totally ordered space. Then the following are equivalent.*

(a) *X is sequentially compact.*

(b) *X is an fs-space.*

(c) *For any strictly increasing or decreasing sequence $\{x_i\}_{i<\omega}$, the limit $\lim x_i$ exists.*

Proof. We already know that (a) implies (b). Now assume (b). Let $(x_i)_{i<\omega}$ be a strictly increasing sequence in X. Note that any ordered space is normal. Since X is assumed to be an *fs*-space, it is pseudocompact, and therefore countably compact. This means that $(x_i)_{i<\omega}$ has an accumulation point, which is necessarily the limit of the sequence. Thus, (b) implies (c).

Finally, if any strictly increasing or strictly decreasing sequence has a limit, pick any sequence $(x_i)_{i<\omega}$. If it contains just a finite number of terms there is nothing to prove. Otherwise, let $x_{i_1}, \ldots, x_{i_n}, \ldots$ be a subsequence of distinct terms. This subsequence must contain either a strictly increasing or a strictly decreasing subsequence, and in either event it converges, by assumption. Thus X is sequentially compact. □

Immediate from Theorem 2.6 is the following:

Corollary 2.6.1. *Any compact totally ordered space is an fs-space.*

For the rest of this article all spaces will be compact and Hausdorff, unless the contrary is specified. Before proceeding here, let us enunciate a simple principle, used to obtain that a certain class $\mathfrak{C}$ of compact spaces consists of *fs*-spaces. The proof is obvious in view of Theorem 1.5. We also note that every class is assumed to be closed under formation of homeomorphic copies.

Proposition 2.7. *Suppose that $\mathfrak{C}$ is a class of compact spaces, closed under taking regular closed subspaces, and that $\beta\omega$ is not a retract of a member of $\mathfrak{C}$. Then every space in $\mathfrak{C}$ is an fs-space.*

Remark 2.7.1. Proposition 2.7 furnishes an alternative proof of Corollary 2.2.2. Indeed, dyadic spaces are preserved by regular closed sets, and $\beta\omega$ is not dyadic.

We proceed with some routine applications of Proposition 2.7. First, recall that a space X is said to be *scattered* if every closed subspace Y contains an isolated point of Y.

Proposition 2.8. *Every scattered compact space is an fs-space.*

Proof. By definition, this class is closed under taking all closed subspaces. It is also clear that $\beta\omega$ is not scattered, and so, by Proposition 2.7, we are done. □

In the next proposition, the proof of which is obvious by an application of Proposition 2.7, we leave it to the reader to review the definition of paracompactness, say, in [2]. A space is *hereditarily* paracompact if every subspace is paracompact.

Proposition 2.9. *Every hereditarily paracompact space is an fs-space.*

Here is another application of Proposition 2.7; a definition to begin with.

Remark 2.10. Recall, that a (not necessarily compact) space X is *countably tight* if for each $x \in X$ and whenever $x \in \operatorname{cl} K$, there is a countable subset L of K, so that $x \in \operatorname{cl} L$. The class of compact countably tight spaces is obviously closed under taking closed subspaces. The notion of tightness is due to Arkhangel'skiĭ [1]. To show that $\beta\omega$ is not countably tight, we apply a result of Kunen [8] showing that in $\beta\omega \setminus \omega$ there are "so-called" weak P-points which are not P-points.

Recall that $p \in X$ is a *P-point* if for each F_σ-set D so that $p \notin D$ it follows that $p \notin \operatorname{cl} D$. p is a *weak P-point* if for each countable subset L excluding p, $p \notin \operatorname{cl} L$. Now since every countable subset is an F_σ-set in a Hausdorff space, it is clear that every P-point is a weak P-point. The result of Kunen referred to above establishes that $\beta\omega$ is not countably tight.

The upshot of this discussion is, via Proposition 2.7:

Proposition 2.11. *Every compact countably tight space is an fs-space.*

Definition and remarks 2.12. (a) We wish to recall another important cardinal invariant of a topological space. We have already mentioned the cellularity. We assume that the reader is familiar with the concept of weight.

(1) The *π-character of a point $p \in X$*: the least cardinal of a π-base of open sets at p. Recall that a *π-base* at p is a collection $\mathfrak{A}$ of open sets with the feature that if U

is a neighborhood of p, then U contains a member of $\mathfrak{A}$. The π-character at p is denoted $\pi\chi(p, X)$.

(2) The *π-character* $\pi\chi(X)$ of X: it is the supremum over all $\pi\chi(p, X)$, with $p \in X$.

For any space, $c(X)$, $\pi\chi(X) \leqslant w(X)$, but in general the cellularity and the π-character of a space are not comparable.

(b) Call a space a *$\pi\chi$-space* if for each regular uncountable cardinal κ,

$$w\big(\{x \in X\colon\ \pi\chi(x, X) < \kappa\}\big) < \kappa.$$

If every continuous image of the space X is a $\pi\chi$-space we say that X is a *strong* $\pi\chi$-space. It is shown in [10] that if X is any $\pi\chi$-space then also

$$w\big(\operatorname{cl}\big(\{x \in X\colon\ \pi\chi(x, X) < \kappa\}\big)\big) < \kappa,$$

for each regular uncountable cardinal κ. Note that for any $\pi\chi$-space X, $\pi\chi(X) = w(X)$.

(c) The class of $\pi\chi$-classes encompasses a great deal. Dyadic spaces are, in fact, strong $\pi\chi$-spaces; this is due to Ščepin [11]. It is also well known (see p. 93, Chapter 2 of [7]) that the underlying space of a compact group is dyadic. This fact is originally due to Kuz'minov and Ivanovskiĭ (1959).

The proof of the next result is due to Uspenskiĭ. We observe that I will denote the ordinary closed unit interval throughout.

Theorem 2.13. *Every compact strong $\pi\chi$-space is an fs-space.*

Proof. Let X be a compact space which is not an *fs*-space, and suppose that the open cell $\{U_n\colon\ n \in \omega\}$ witnesses this. We must show that X admits a map onto a space which is not a $\pi\chi$-space. For every $n \in \omega$ pick a nonempty zeroset $B_n \subseteq U_n$ and a function $f_n : X \to I$ such that $f_n(x) = 0$ if and only if $x \in B_n$ and $f_n(x) = 1$ for every $x \in \bigcup\{U_k\colon\ k \in \omega,\ k \neq n\}$. For every $A \subseteq \omega$ pick a continuous function $G_A : X \to I$ which separates $\bigcup\{U_n\colon\ n \in A\}$ from $\bigcup\{U_n\colon\ n \in \omega \setminus A\}$; that is, $G_A = 0$ on the first set and $G_A = 1$ on the second. Let

$$\mathcal{F} = \{f_n\colon\ n \in \omega\} \cup \{G_A\colon\ A \subseteq \omega\},$$

and let $H : X \to I^{\mathcal{F}}$ be the diagonal product of the the family $\mathcal{F}$. In other words, $H(x) = \{f(x)\colon\ f \in \mathcal{F}\}$. Since each function $h \in \mathcal{F}$ is constant on each set B_n, the set $H(B_n)$ is a singleton, say $\{p_n\}$. As $B_n = f_n^{-1}(0)$, we also have $B_n = H^{-1}(p_n)$.

Now the reader should have no trouble verifying that (i) the image of a G_δ-set which is the fibre of a point under a continuous surjection of compact spaces is itself a G_δ-point, and (ii) that any G_δ-point in a compact space has a countable local π-base (and, indeed, a countable local *base*, although we do not need this).

Applying this to the G_δ-set B_n, we obtain that $\{p_n\}$ is a G_δ-point in $Y = H(X)$, and each $\{p_n\}$ has a countable local π-base. On the other hand, it follows from our construction that, for every $A \subseteq \omega$, the sets $\{p_n\colon\ n \in A\}$ and $\{p_n\colon\ n \in \omega \setminus A\}$ are completely separated. Hence, the closure of the set $P = \{p_n\colon\ n \in \omega\}$ in Y is homeomorphic to $\beta\omega$ and, therefore, has uncountable weight. This shows that Y is not a $\pi\chi$-space, and the proof is complete. □

Remark 2.13.1. (a) A $\pi\chi$-space need not be an *fs*-space, at least not with the assumption of the Continuum Hypothesis (CH), as we shall see in Example 2.15.2 below.

(b) In view of Proposition 2.8, for example, we have many *fs*-spaces which are not $\pi\chi$-spaces.

We now consider when a class of spaces intersects the class of *fs*-spaces "trivially".

Definition 2.14. Recall that a space X is an *F-space* (respectively, *quasi F-space*) if every cozeroset (respectively, every dense cozeroset) of X is C^*-embedded. For the basic information on these topological spaces we refer the reader to [5] and [12].

Now we have:

Proposition 2.15. *Every* (*Tychonoff*) *F-space which is an fs-space must be finite.*

Proof. Suppose that X is an infinite compact F-space which is also an *fs*-space. Let $U_1, U_2, \ldots$ be any disjoint countable family of open sets. Without loss of generality we may suppose that each U_n is a cozeroset. Suppose that A and B are subsets defining a partition of ω. Then $U_A = \bigcup_{n\in A} U_n$ and $U_B = \bigcup_{n\in B} U_n$ are disjoint cozerosets (in an F-space) and therefore completely separated [5, Theorem 14.25]. This contradicts the assumption that X is an *fs*-space. The claim of the theorem then follows. □

Remark 2.15.1. Note that there are infinite compact *fs*-spaces which are quasi F-spaces: αD, the one-point compactification of a discrete uncountable space D.

Example 2.15.2. *With* CH, *a compact $\pi\chi$-space which is not an fs-space.*

It is well known that $w(\beta\omega\setminus\omega) = \mathfrak{c}$ (see [12, 3.17]), and folklore that every point of $\beta\omega\setminus\omega$ has uncountable π-character. Applying CH, it then follows that $\beta\omega\setminus\omega$ is a $\pi\chi$-space. Observe, however, that, since a strong $\pi\chi$-space must have countable cellularity (see [10] or [7, Chapter 18, p. 599]), $\beta\omega\setminus\omega$ is not strongly $\pi\chi$.

In light of Proposition 2.15, we offer the following observation, leaving the proof to the reader.

Proposition 2.16. *Suppose that X is a compact F-space, which is also a $\pi\chi$-space. The set of points having countable π-character is finite.*

We conclude this section with a comment.

Remark 2.17. Let $I(\aleph_0)$ stand for the class of all spaces which do not admit a continuous surjection onto I^α, for any uncountable cardinal α. Then $I(\aleph_0)$ is a proper subclass of the class of *fs*-spaces. To see that $I(\aleph_0)$ is contained in the class of *fs*-spaces, apply the ideas of Proposition 2.7: the class $I(\aleph_0)$ is hereditary and it does not contain $\beta\omega$.

On the other hand, per Example 1.2 a Cantor space of any weight is an *fs*-space; if the weight is uncountable, the Cantor space is not in $I(\aleph_0)$.

3. Concluding remarks

We conclude the paper with a comment about the separated cellularity and its relationship to the independence character of a compact, Hausdorff and zero-dimensional space, which we shall refer to as a *Stone space*.

Definition and remarks 3.1. The independence character of a boolean ring was studied in [3], in particular, because of its relationship to the separated cellularity of its Stone dual. We restrict to Stone spaces. A family $\mathfrak{A}$ of clopen sets in X is *independent* if for any two disjoint finite subsets $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of $\mathfrak{A}$, $(\bigcap \mathfrak{A}_1) \cap (\bigcap \mathfrak{A}_2') \neq \emptyset$, where $\mathfrak{A}_2'$ denotes the set of all complements of the sets in $\mathfrak{A}_2$. The *independence character* of X, $\mathrm{ind}(X)$ is the supremum of all cardinals of independent families of clopen sets. By Stone duality, to say that there is an independent family of cardinality κ is to say that X maps continuously onto the Cantor space 2^κ.

Now it is shown in [4], that, for any Stone space $2^{<sc(X)} \leqslant \mathrm{ind}(X)$. (Note: for cardinals κ and λ, $\kappa^{<\lambda}$ stands for the supremum of all the cardinals κ^α, where $\alpha < \lambda$.) Thus, if X is a Stone space having countable independence character then it must be an *fs*-space. We now use Theorem 1.5 to give a different proof of this inequality.

Proposition 3.2. *For any Stone space X we have*

$$2^{<sc(X)} \leqslant \mathrm{ind}(X).$$

Proof. Suppose, to the contrary, that there is a cardinal $\lambda < sc(X)$ so that $\mathrm{ind}(X) < 2^\lambda$. A separated cell $\mathfrak{A}$ of size λ then exists; the closure K of the union of all members of this family is regular, and there is a continuous surjection $f: K \to \beta\lambda$. Now by a theorem of Hausdorff [6], $\beta\lambda$ has an independent family of clopen sets of size 2^λ, and so f may be composed with a continuous surjection g of $\beta\lambda$ onto the Cantor space 2^{2^λ}. Now extend the composite gf to X (by extending each of the projections to the two-element space 2, which is possible since K is compact). This gives us a continuous surjection of X onto 2^{2^λ}, which contradicts the supposition that $\mathrm{ind}(X) < 2^\lambda$. □

Acknowledgements

We owe a great deal to V.V. Uspenskiĭ, for his help in pointing us to literature which was essential in this work, and for his many comments and suggestions. As we noted, the proof of Theorem 2.13 is his. We also thank the referee for the suggestions which prompted us to reorganize the paper considerably.

References

[1] A.V. Arkhangel'skiĭ, Structure and classification of topological spaces and cardinal invariants, Russian Math. Surveys 33 (6) (1978) 33–96.
[2] R. Engelking, General Topology, Monograf. Mat. 60 (Polish Scientific Publishers, Warsaw, 1977).
[3] R.T. Finn and J. Martinez, Cardinal invariants and their relationship to the global dimension of a boolean ring, submitted.
[4] R.T. Finn and J. Martinez, Global dimension of rings of continuous functions and related f-rings, submitted.
[5] L. Gillman and M. Jerison, Rings of Continuous Functions, GTM 43 (Springer, New York, 1976).
[6] F. Hausdorff, Über zwei Sätze von G. Fichtenholz und L. Kantorovich, Studia Math. 6 (1936) 18–19.
[7] M. Hušek and J. van Mill (eds.), Recent Progress in General Topology (North-Holland, Amsterdam, 1992).
[8] K. Kunen, Weak P-points in $\mathbb{N}^*$, in: Topology, Colloq. Math. Soc. János Bolyai 23 (Budapest, 1978) 741–749.
[9] J.R. Porter and R.G. Woods, Extensions and Absolutes of Hausdorff Spaces (Springer, Berlin, 1989).
[10] B.È. Šapirovskiĭ, On a class of spaces containing all dyadic and all weakly Noetherian bicompacta, in: Proc. Leningrad Intern. Topology Conf. (Leningrad, Aug. 23–27, 1982) pp. 119–134 (in Russian).
[11] E.V. Ščepin, On κ-metrizable spaces, Math. USSR-Izv. 14 (2) (1980) 407–440.
[12] R.C. Walker, The Stone–Čech Compactification, Ergebn. Math. 83 (Springer, Berlin, 1974).