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Bazzoni's conjecture

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Abstract

A conjecture posed 11 years ago by S. Bazzoni is solved by showing that a Prüfer domain with the property that every locally finitely generated ideal is finitely generated is, in fact, a Prüfer domain of finite character.

Keywords: Prüfer domain; Finite character; Abelian ℓ-group; Finite Basis Theorem

By a *Prüfer domain* we mean a commutative integral domain R for which every finitely generated ideal of R is invertible. In her paper [1] S. Bazzoni investigated the class semigroup of a Prüfer domain and aimed at classifying when the class semigroup of R, denoted $\mathscr{S}(R)$, is a Clifford semigroup (that is, for every element $a \in \mathscr{S}(R)$ there is an element $x \in \mathscr{S}(R)$ such that $a = a^2x$). In her work she isolated the following property.

(∗) An ideal I of R is finitely generated if and only if every localization IR_M is principally generated for every maximal ideal M of R.

It is said that the ideal I of R is *locally principal* if IR_M is a principal ideal of R_M for every maximal ideal M of R. Bazzoni proved that if a Prüfer domain has finite character (that

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is, every nonzero element belongs to a finite number of maximal ideals) then $\mathscr{S}(R)$ is a Clifford semigroup, and in turn, if $\mathscr{S}(R)$ is a Clifford semigroup, then R satisfies $(*)$. In a later article, [2], she was able to show that for a Prüfer domain R, if $\mathscr{S}(R)$ is a Clifford semigroup, then R has finite character. In [1] and then again in [2] she proposed the following.

Conjecture. *A Prüfer domain satisfies property $(*)$ if and only if R has finite character.*

We prove that the conjecture is true by showing that each of the two properties can be translated, via the group of invertible ideals, into a statement about lattice-ordered groups, and then show that for lattice-ordered groups these statements are equivalent. We denote the group of invertible ideals of a Prüfer domain R by $\mathrm{Inv}(R)$. In [3] the authors showed that once the set of integral invertible ideals is partially-ordered by reverse inclusion, then this partial-order extends to a partial-order on all of $\mathrm{Inv}(R)$. Moreover, this order is a lattice-order and makes $\mathrm{Inv}(R)$ into a lattice-ordered group (or ℓ-group, for short) whose positive cone is precisely the set of integral invertible ideals. The positive cone of $\mathrm{Inv}(R)$ will be denoted by $\mathrm{Inv}(R)^+$. Terms not defined explicitly below may be found in [1,5,6], and [3].

For an ideal I of R we let $F_I = \{J \in \mathrm{Inv}(R)\colon\ J \subseteq I\}$. It is straightforward to check that F_I is a nonempty filter on $\mathrm{Inv}(R)^+$. (Recall that a nonempty *filter* on a lattice $(L, \wedge, \vee)$ is a nonempty proper subset $F \subset L$ such that if $f, g \in F$, then so is $f \wedge g$, and if $f \in F$ and $h \in L$ with $f \leqslant h$, then $h \in F$.) Conversely, if F is a filter on $\mathrm{Inv}(R)^+$, then the ideal $I = \sum_{J \in F} J$ satisfies $F_I = F$. This leads to a one-to-one correspondence between the ideals of R and nonempty filters on $\mathrm{Inv}(R)^+$. A *principal filter* is a filter of the form $\{x \in L\colon\ x \geqslant a\}$ for some $a \in L$.

Lemma 1. *Suppose R is a Prüfer domain. The ideal I is finitely generated if and only if F_I is a principal filter.*

Now, suppose $(G, +, 0)$ is an abelian lattice-ordered group and let H be a subgroup. H is called a *convex ℓ-subgroup* if it is a sublattice and also has the property that if $h_1 \leqslant g \leqslant h_2$ with $h_1, h_2 \in H$, then $g \in H$. The intersection of a collection of convex ℓ-subgroups is again a convex ℓ-subgroup and so one may speak of the convex ℓ-subgroup generated by a given set. In particular, for $g \in G^+$, the principal convex ℓ-subgroup generated by g is denoted by $G(g)$ and is the set $G(g) = \{h \in G\colon\ |h| \leqslant ng \text{ for some } n \in \mathbb{N}\}$.

If P is a convex ℓ-subgroup of G then we say P is a *prime subgroup* if whenever $a \wedge b = 0$, then either $a \in P$ or $b \in P$. It is known that the convex ℓ-subgroup P is prime if and only if the factor ℓ-group G/P is a totally-ordered ℓ-group. Zorn's Lemma may be applied to show that every prime subgroup contains a minimal prime subgroup. We let the set of minimal prime subgroups be denoted by $\mathrm{Min}(G)$. For an element $g \in G$, we let $U(g) = \{P \in \mathrm{Min}(G)\colon\ g \notin P\}$.

For a Prüfer domain R there is a one-to-one order reversing correspondence between the nonzero prime ideals of R and the prime subgroups of $\mathrm{Inv}(R)$. This correspondence takes a prime ideal P and produces the prime subgroup $\bar{P}$ obtained by taking the convex ℓ-subgroup of $\mathrm{Inv}(R)$ generated by the set $\{J \in \mathrm{Inv}(R)^+\colon\ J \nsubseteq P\}$. In particular, this correspondence restricts to a bijection between the set of maximal ideals of R and the minimal prime subgroups of $\mathrm{Inv}(R)$. Furthermore, for an invertible ideal $I \in \mathrm{Inv}(R)^+$, the set of maximal ideals of R containing I corresponds to the minimal primes of $\mathrm{Inv}(R)^+$ not containing I, i.e., $U(I)$. Therefore, we obtain the result:

Theorem 2. *Suppose R is a Prüfer domain. R has finite character if and only if for every $I \in \mathrm{Inv}(R)^+$, $U(I)$ is a finite set.*

We continue to assume that R is a Prüfer domain. Recall that for any localization R_P at a nonzero prime ideal P, $\mathrm{Inv}(R_P)$ is ℓ-isomorphic to the totally-ordered group $\mathrm{Inv}(R)/\bar{P}$. This map is given by first defining the map $\psi : \mathrm{Inv}(R)^+ \to \mathrm{Inv}(R_P)^+$ by $\psi(I) = IR_P$ and then extending this in the natural way to an ℓ-group homomorphism $\hat{\psi} : \mathrm{Inv}(R) \to \mathrm{Inv}(R_P)$. Since every finitely generated ideal of R_P is an extension of a finitely generated ideal of R it follows that $\hat{\psi}$ is a surjective homomorphism. Consequently, $\mathrm{Inv}(R)/\ker\hat{\psi} \cong \mathrm{Inv}(R_P)$. It is straightforward to check that $\ker\hat{\psi} = \bar{P}$. To understand the $(*)$ condition in terms of ℓ-groups we come to our first definition.

Definition 3. Let $(G, +, 0)$ be an abelian ℓ-group and suppose F is a filter on G^+. We say F is a *cold filter* if for every $P \in \mathrm{Min}(G)$ the filter $P_F = \{g + P\colon\ g \in F\}$ on $(G/P)^+$ has a minimum. (By a *minimum* we mean there is some $f \in F$ such that for every $g \in F$, $f + P \leqslant g + P$.) Notice that a principal filter is cold.

Lemma 4. *Suppose I is an ideal of the Prüfer domain R. I is locally principal if and only if F_I is a cold filter.*

Proof. Suppose that I is a locally principal ideal of R. As defined above F_I is a filter on $\mathrm{Inv}(R)^+$. We aim to show that F_I is a cold filter. To that end let $P \in \mathrm{Min}(G)$. As we previously mentioned there is some $M \in \mathrm{Max}(R)$ such that $P = \bar{M}$. For the sake of notation we let $[J]$ denote the congruence class of $\mathrm{Inv}(R)/P$ represented by the element $J \in \mathrm{Inv}(R)$. In this notation $P_{F_I} = \{[J]\colon\ J \in F_I\}$. By the correspondence $\mathrm{Inv}(R)/P \cong \mathrm{Inv}(R_M)$ defined above we get that P_{F_I} is in one-to-one order preserving correspondence with the set $\{JR_M\colon\ J \in F_I\}$. Because we are assuming that I is locally principal it follows by the usual localization arguments that there is some $K \in F_I$ such that $IR_M = KR_M$. Working back through the correspondence yields that $[K]$ is the minimum of P_{F_I}. Since P was arbitrarily chosen we conclude that F_I is a cold filter.

Conversely, suppose that F_I is a cold filter and let M be any maximal ideal. We again use the notation $[J]$ for the congruence class of $\mathrm{Inv}(R)/\bar{M}$ represented by $J \in \mathrm{Inv}(R)$; we also use the correspondence between $\bar{M}_{F_I}$ and $\{JR_M\colon\ J \in F_I\}$. Since F_I is a cold filter it follows that there is some $K \in F_I$ such $[K]$ is a minimum of $\bar{M}_{F_I}$. We aim to demonstrate that $IR_M = KR_M$. Clearly, $KR_M \subseteq IR_M$. On the other hand, for any $i \in I$ the inequality $[K] \leqslant [iR]$ is valid. Therefore, $(iR)R_M \subseteq KR_M$, whence $IR_M = KR_M$. Since KR_M is a finitely generated ideal of the valuation domain R_M we conclude that I is locally principal. □

We now are in position to translate the $(*)$ condition to ℓ-groups.

Proposition 5. *Suppose R is a Prüfer domain. R satisfies $(*)$ if and only if the ℓ-group $\mathrm{Inv}(R)$ satisfies the property that every cold filter is principal.*

Remark 6. Observe that in order to prove that Bazzoni's conjecture is true it suffices to prove the following analog statement:

- *Suppose G is an abelian ℓ-group for which every cold filter of G is principal. Then for every $g \in G^+$, $U(g)$ is finite.*

The work of Paul Conrad on lattice-ordered groups with a finite basis is pivotal. Conrad [4] investigated ℓ-groups with property (F):

(F) For each $g \in G^+$, g is greater than at most a finite number of disjoint elements.

The main property to be gathered from such ℓ-groups is [4, Theorem 5.2] which states that if G satisfies (F), then G has a basis. (An element of $b \in G^+$ is said to be *basic* if the set $\{g \in G\colon\ 0 \leqslant g \leqslant b\}$ is totally-ordered. A *basis* is a maximal pairwise disjoint set of elements each of which is basic.) Moreover, when $G = G(g)$ for some $g \in G^+$ then G does not have an infinite set of pairwise disjoint elements, and so G has a finite basis. Therefore, by the Finite Basis Theorem [5, Theorem 46.12] this means that G has a finite number of minimal prime subgroups. We explain shortly why this is useful.

Proposition 7. *Let G be an abelian ℓ-group. If every cold filter on G^+ is principal, then G satisfies* (F).

Proof. Suppose every cold filter of G^+ is principal. Let $g \in G$ have the property that it lies above every element of the pairwise disjoint family, say $U = \{u_\alpha\}_{\alpha \in A}$. Let $\hat{U} = \{u_{\alpha_1} \vee \cdots \vee u_{\alpha_n}\colon\ \alpha_1, \ldots, \alpha_n \in A\}$ and

$$F = \{h \in G^+\colon\ h \geqslant g - u \text{ for some } u \in \hat{U}\}.$$

Then F is a filter on G^+ since if $(g - u_1) \wedge (g - u_2) = g - (u_1 \vee u_2)$. Let $P \in \mathrm{Min}(G)$ and consider P_F. Suppose there is some $u_\alpha \in U$ with $u_\alpha \notin P$. Then since the set U is pairwise disjoint it follows that for every $\beta \in A$, $u_\beta \in P$. Thus, for any minimal prime subgroup P at most one of the u_α does not belong to P, whence for any $u \in \hat{U}$, $(g - u) + P = g + P$ or $(g - u) + P = g - u_\alpha + P$ depending on whether every element of U belongs to P or u_α is the unique element not belonging to P. This forces the set $P_F = \{f + P\colon\ f \in F\}$ to have either $g + P$ or $g - u_\alpha + P$ as a minimum, whence F is a cold filter.

By hypothesis F must be a principal filter. Let $h \in G^+$ be a generator for F. Since the set $\{g - u\colon\ u \in \hat{U}\}$ is a filter base for F it follows that $h = g - u$ for some $u \in U$. Let $u = u_{\alpha_1} \vee \cdots \vee u_{\alpha_n}$. Then for any $\beta \neq \alpha_1, \ldots, \alpha_n$ we have

$$g - u = h \leqslant g - (u \vee u_\beta) \leqslant g - u = h.$$

We conclude that $u_\beta = 0$ and so U is actually a finite set. □

Proposition 8. *Suppose G is an ℓ-group for which every cold filter on G^+ is principal and let $g \in G^+$. Then $G(g)$ also has the property that every cold filter on G^+ is principal.*

Proof. Let F be a cold filter on $G(g)^+$ and extend F to a filter on G^+ by taking $\hat{F} = \{h \in G\colon\ h \geqslant f \text{ for some } f \in F\}$. We claim that $\hat{F}$ is a cold filter on G^+. Let $P \in \mathrm{Min}(G)$. Then $P' = P \cap G(g)$ is a minimal prime subgroup of $G(g)$ and so the set P'_F has a minimum, say $k + P' \in F$. For any $h \in \hat{F}$, $h \geqslant f$ for some $f \in F$ and so by the Isomorphism Theorem

$$h + P \geqslant f + P \geqslant k + P.$$

It follows that $k + P$ is a minimum for $P_{\hat{F}}$, whence $\hat{F}$ is a cold filter. By hypothesis, $\hat{F}$ is principally generated, say by $k \in G^+$. Now, for any $f \in F$ we have that $k \leqslant f$ and so by convexity of $G(g)$ it follows that $k \in G(g)^+$ and that F is the principal filter generated by k. □

Theorem 9. *Suppose G is an ℓ-group for which every cold filter on G^+ is principal. For every $g \in G^+$, $U(g)$ is a finite set.*

Proof. We use the fact that for any nonzero $g \in G^+$ the collection of minimal prime subgroups of $G(g)$ is in one-to-one correspondence with $U(g)$ (the map sends $P \in U(g)$ to $P \cap G(g)$). By Proposition 8 if G has said property then so does $G(g)$. By Proposition 7, $G(g)$ satisfies (F). Now, since any infinite pairwise disjoint set of $G(g)$ would yield an infinite pairwise disjoint family of elements beneath g it follows that $G(g)$ does not have an infinite family of pairwise disjoint elements. Therefore, $G(g)$ has a finite basis and so by the Finite Basis Theorem, $G(g)$ has a finite number of minimal prime subgroups. We conclude that $U(g)$ is finite. □

Theorem 10. *Suppose R is a Prüfer domain. R satisfies $(*)$ if and only if R has finite character.*

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