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Singular f-rings which are α-G.C.D. rings

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Abstract

This article introduces α-G.C.D. rings. Let α be a cardinal. A, a commutative ring with identity, is an α-G.C.D. ring if for each $S \subseteq A$ of cardinality less than α there exists a g.c.d. for S, i.e., an element $a \in A$ such that $(s) \leqslant (a)$ for all $s \in S$ and if $b \in A$ also satisfies $(s) \leqslant (b)$ for all $s \in S$ then $(a) \leqslant (b)$. In Section 2, it is shown that for a zero-dimensional space X, $C(X,\mathbb{Z})$ is an ω^+-G.C.D. ring if and only if X is a P-space. In Section 3, it is shown that the validity of the statement: for every zero-dimensional space X, $C(X,\mathbb{Z})$ is an α-G.C.D. ring if and only if X is a P_α-space, is equivalent to the nonexistence of a measurable cardinal. An example is given of a zero-dimensional space X, and a cardinal α, for which $C(X,\mathbb{Z})$ is an α-G.C.D. ring yet X is not a P_α-space.

Keywords: P_α-space; Ring of continuous functions; Measurable cardinals; Singular f-rings

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1. Introduction

We begin with some standard notation which may be found in [2,4,13]. Throughout, A will denote a commutative ring with identity, and Min(A) will stand for its set of minimal prime ideals. We may, and often will view Min(A) as a topological space under the hull-kernel topology. What this means is that the collection of subsets of the form $U(a) = \{P \in \text{Min}(A) \mid a \notin P\}$, for $a \in A$, forms a base for the open sets. For an element $a \in A$, (a) signifies the ideal generated by a. It has been shown that Min(A) is a zero-dimensional Hausdorff space [11]. In this paper, *semiprime* means that A has no nonzero nilpotent elements.

Our basic premise is that A is an f-ring. As such we denote its supremum and infimum by $\vee$ and $\wedge$, respectively. We assume the reader is familiar with the terms *ℓ-subgroup,*

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(order) convex ℓ-subgroup, and *value*, as well as the definition and some basic facts about f-rings. Since we are assuming our rings are commutative it is safe to say that an ℓ-ideal is a ring ideal which also happens to be a convex ℓ-subgroup. The collection of maximal ℓ-ideals is denoted by $\mathrm{Max}_l(A)$, which is a compact Hausdorff space when endowed with the hull-kernel topology. We let $\mathrm{Yos}(a)$ denote the set of values of the element a. We view $\mathrm{Yos}(a)$ as a topological space under the hull-kernel topology, for which it is a compact Hausdorff space. In particular, we shall use $Y(A)$ to denote the set of values of 1 and call this the *Yosida space* of A.

There are two classes of f-rings which shall be of importance: the Archimedean f-rings and the singular f-rings. An f-ring A is said to be *Archimedean* if whenever $0 \leqslant a$, $b \in A$ and $na \leqslant b$ for all natural numbers n, then $a = 0$. The f-ring A is said to be *singular* if whenever $0 \leqslant a \leqslant 1$, then $a \wedge (1 - a) = 0$. Any singular f-ring is necessarily semiprime. In particular, we may represent $A \leqslant \prod_{P \in \mathrm{Min}(A)} A/P$ as a subdirect product of totally ordered integral domains each with least positive element $1 + P$, and thus, we speak of the elements of A as functions. Observe that singular f-rings are *saturated*, i.e., if $B \subseteq \mathrm{Min}(A)$ is a clopen subset, then the characteristic function on B (from the above product) is in A. In [6], it is shown that if A is singular, then $\mathrm{Min}(A) = Y(A) = \mathrm{Max}_l(A)$ and so is a zero-dimensional compact Hausdorff space.

As for the topological preliminaries, if X is a topological space, then $C(X,\mathbb{Z})$ is the Archimedean, singular f-ring of integer-valued continuous functions on X. If $f \in C(X,\mathbb{Z})$, then $\mathrm{coz}(f) = \{x \in X\colon f(x) \neq 0\}$ and is called the cozeroset of f. Similarly, $Z(f) = \{x \in X\colon f(x) = 0\}$ is called the zeroset of f. We wish to point out that as we shall focus our attention on the ℓ-group and ring structure of $C(X,\mathbb{Z})$ we assume that X is zero-dimensional. In this case, $\mathfrak{B}(X)$ shall stand for its Boolean algebra of clopen sets. Recall that $\mathrm{Min}(C(X,\mathbb{Z})) = \beta_0 X$, the zero-dimensional compactification of X. For additional information of f-rings and singular f-rings we recommend the reader consult [7,6,12].

One of our main goals is to characterize those spaces X for which $C(X,\mathbb{Z})$ is a particular type of ring. The following definition shall be useful.

Definition 1.1. For an arbitrary cardinal α, X is said to be a P_α-space if every G_α-set is open. (By G_α-set we mean an intersection of less than α many open sets.) If $\alpha = \omega^+$ then we say X is a P-space.

We recall that in [10, 14,18], P-spaces were characterized as those spaces whose rings of real-valued continuous functions, $C(X)$, are von Neumann regular rings. The following description of P_α-spaces shall be useful in this paper. Its proof is simple and so we leave it out.

Proposition 1.2. *Let α be a cardinal. A zero-dimensional space X is a P_α-space if and only if every union of less than α many clopen sets is again clopen.*

Another useful definition which we shall use throughout is the following:

Definition 1.3. We say a space is *α-disconnected* if each α-cozeroset of X has clopen closure. (We say a subset of X is an α-cozeroset if it is the union of less than α many cozerosets.) Equivalently, (see [10, 1H]) X is α-disconnected if disjoint open sets, one of which is an α-cozeroset, have disjoint closures. Notice that if $\alpha = \omega^+$, then the α-disconnected spaces are precisely the *basically disconnected* spaces. Additionally, if every open set has clopen closure then X is called *extremally disconnected.* Note that some authors use the term *∞-disconnected*. For a delightful exposition of α-disconnected spaces see [3].

Finally, we wish to point out the following notation. As the collection of ideals of a ring forms a lattice under inclusion, we shall use $\leqslant$ for this partial order. When two sets are being compared we shall employ $\subseteq$.

2. ω^+-G.C.D. rings

There is much in the literature recognizing the interplay of ideal theoretic features of $C(X)$ versus topological features of the space X. In studying $C(X,\mathbb{Z})$, one first realizes the difficulty in realizing this interplay between algebra and topology when one considers the algebraic notion of a Bézout ring. Recall that a ring is called Bézout if every finitely generated ideal is principal. For instance, it is well known, dating back to 1956 (see [9]), that $C(X)$ is Bézout if and only if X is an F-space, i.e., every cozeroset is C^*-embedded. As for $C(X,\mathbb{Z})$ the result is much simpler; $C(X,\mathbb{Z})$ is always Bézout [14]. Therefore, with respect to the algebra, we need more restrictive conditions to distinguish between the spaces. It is natural to consider the condition on $C(X,\mathbb{Z})$ that all countably generated ideals are principal. But as we shall soon seen this is far too strong. Therefore, we consider a less restrictive definition, but one which shall carry us through this paper.

Definition 2.1. If α is a cardinal, a ring A is said to be an α-G.C.D. ring if for each subset $\{a_\sigma\}_{\sigma<\beta} \subseteq A$, where $\beta < \alpha$, there is a g.c.d. for the $\{a_\sigma\}_{\sigma<\beta}$, i.e., an $a \in A$ such that $(a_\sigma) \leqslant (a)$ for all $\sigma < \beta$ and if b also satisfies this property, then $(a) \leqslant (b)$. Note that the ω-G.C.D. rings are precisely the G.C.D. rings. (An ample source on G.C.D. rings is Gilmer's book [8].)

With this concept in hand, we shall briefly consider, the notion of an ω^+-Bézout ring. A ring A is *ω^+-Bézout* if every countably generated ideal is principal. As we shall see, for our purposes, there is no need to define an α-Bézout ring for an arbitrary cardinal α greater than ω^+. Observe that the ring A is ω^+-Bézout if and only if A is an ω^+-G.C.D. ring and each g.c.d. may be written as a (finite) linear combination of elements from the countable family in question. Equivalently, A is ω^+-Bézout if and only if A is Bézout and satisfies the ascending chain condition for principal ideals. These translate, at least for singular f-rings, to easily describe the ω^+-Bézout rings. Now, in an arbitrary ring whenever $(a_1, a_2, \ldots) = (a)$ it follows that $\bigcup U(a_i) = U(a)$. This together with the fact that singular f-rings are saturated leads us to the following proposition.

Proposition 2.2. *Let A be a singular f-ring. If A is ω^+-Bézout then every countable union of clopen sets in* Min(A) *is again clopen, in which case* Min(A) *is finite.*

Proof. Observe that the second statement follows from Proposition 1.2 and the fact that compact P-spaces are finite. Now, let $B_1, \ldots \in \mathfrak{B}(\text{Min}(A))$. Denote their union by B. We wish to show that B is clopen. For each natural number i let χ_i denote the characteristic function on B_i. Since A is saturated, $\chi_i \in A$. By our assumption, there exists a $\chi \in A$ such that $(\chi_1, \chi_2, \ldots) = (\chi)$. As observed above, we have that B is clopen. □

If we assume that A is Archimedean then the converse holds since "Min(A) finite" forces A to be a finite product of copies of $\mathbb{Z}$. Notice that Example 2.4 demonstrates the necessity of Archimedeaneity in the hypothesis.

Formally, we now state what happens in the Archimedean case.

Corollary 2.3. *Let A be an Archimedean singular f-ring. The following are equivalent.*
(i) *A is ω^+-Bézout;*
(ii) Min(A) *is finite;*
(iii) *$A = \mathbb{Z}^F$, where F is finite.*

Example 2.4. Let $A = \mathbb{Z}[X]$, the totally ordered f-domain of integer-valued polynomials, where $1 \ll X \ll X^2 \ll \cdots$. Since A is an integral domain we have Min$(A) = \{(0)\}$, a singleton. Next, a standard result in ring theory is that A is not Bézout and so cannot be ω^+-Bézout. Thus, A is an example of a non-Archimedean, non-ω^+-Bézout singular f-ring whose minimal prime ideal space is a singleton.

We now turn our attention to singular f-rings which happen to be α-G.C.D. rings, and study their space of minimal prime ideals. In particular, our quintessential example: $C(X, \mathbb{Z})$ will play a pivotal role. To this end and in the same vein as Proposition 2.2, we have:

Proposition 2.5. *Let A be a singular f-ring which is an α-G.C.D. ring. Then* Min(A) *is α-disconnected.*

Proof. To prove Proposition 2.5 we recall a standard result concerning Stone duality; a "Stone–Nakano-like Theorem": that a zero-dimensional compact Hausdorff space is α-disconnected if and only if its Boolean algebra of clopen sets is α-complete.

Notice that if $a \in A$ is a g.c.d. for a collection $\{a_\sigma\}_{\sigma<\beta} \subseteq A$ where $\beta < \alpha$, then in $\mathfrak{B}(\text{Min}(A))$, $U(a) = \bigvee_{\sigma<\beta} U(a_\sigma)$. Again, we use the fact that A is saturated. Now, if we are given arbitrary clopen sets $\{U_\sigma\}_{\sigma<\beta}$ of $\mathfrak{B}(\text{Min}(A))$, then letting $a \in A$ be a g.c.d. for the appropriate characteristic functions $\{\chi_{U_\sigma}\}$ we have that in $\mathfrak{B}(\text{Min}(A))$, $U(a) = \bigvee U(a_\sigma)$. Thus, $\mathfrak{B}(\text{Min}(A))$ is α-complete and so Min(A) is α-disconnected. □

We are now in position to prove the main result of this section. We turn our attention to $C(X, \mathbb{Z})$ and find we may apply Proposition 2.5.

Theorem 2.6. *For a zero-dimensional space X, $C(X,\mathbb{Z})$ is an ω^+-G.C.D. ring if and only if X is a P-space.*

Proof. We begin with the sufficiency. Suppose X is a P-space and let $f_1, f_2, \ldots \in C(X,\mathbb{Z})$. Without loss of generality we assume that each $f_i > 0$. By the well-ordering of the naturals we may define the following function $d \in C(X,\mathbb{Z})$. First, let

$$B = \bigcap_{i<\omega} Z(f_i)$$

a clopen subset of X. Next, define

$$d(x) = \begin{cases} \text{g.c.d.}\,\{f_j(x)\colon\ x \in \operatorname{coz}(f_j)\}, & \text{if } x \notin B; \\ 0, & \text{otherwise.} \end{cases}$$

To see that $d \in C(X,\mathbb{Z})$ we fix $x \in X$ and set $U_i = f_i^{-1}(f_i(x))$ for each natural i. Then as X is a P-space $U = \bigcap U_i$ is the clopen neighbourhood of x for which $f_i(U) = f_i(x)$ for all natural i. Therefore, $d(U) = d(x)$ and $d \in C(X,\mathbb{Z})$. By construction d is a common divisor of the f_i.

Now, if g divides all the f_i, then let $h(x) = d(x)g(x)^{-1}$ where $x \in \operatorname{coz}(g)$ and 0 otherwise. It follows that $h \in C(X,\mathbb{Z})$. Observe, that if $g(x) = 0$ then so does $f_i(x)$, for all i, i.e., $x \in B$. Therefore $gh = d$ and so d is a g.c.d. for the f_i.

As for the necessity: let Π denote the collection of prime numbers. Now, let $\{B_i\colon\ i \in \Pi\}$ be a collection of clopen sets of X and let $B = \bigcup B_i$. We wish to show that B is clopen. By disjointifying, we may assume that the B_i are pairwise disjoint. Consider the following bounded continuous functions:

$$d_i(B_j) = \begin{cases} j, & \text{if } j \leqslant i; \\ 0, & \text{otherwise} \end{cases}$$

for every $i, j \in \Pi$. Since $C(X,\mathbb{Z})$ is an ω^+-G.C.D. ring there is a $d = \text{g.c.d.}\{d_i | i \in \Pi\}$. By Proposition 2.5, $\operatorname{coz}(d) = \operatorname{cl}\bigcup \operatorname{coz}(d_i) = \operatorname{cl} B$. Since we want to show that $\operatorname{cl} B = B$, we suppose there is some $x \in \operatorname{cl} B - B$. Let $n = d(x)$. As $d^{-1}(n) - B_n$ is an open neighbourhood of x choose $y \in (d^{-1}(n) - B_n) \cap B$. Let p be a prime different than n with $y \in B_p$.

Using the fact that $d | d_i$ obtain an $s_i \in C(X,\mathbb{Z})$ for which $d_i = s_i d$, for each $i \in \Pi$. Then the equation $d_p(y) = p = s_p(y)d(y) = s_p(y)n$ together with $p \neq n$ forces $n = 1$. Express the nonempty clopen set $d^{-1}(n) \cap B_p = C$ and let $e = p\chi_C \vee d \in C(X,\mathbb{Z})$. Finally, for each $i \in \Pi$, let $s_i' = s_i$ if $i < p$ and $s_i' = s_i + (1-p)\chi_C$ otherwise. A quick check shows that $d_i = s_i' e$ yet $e \nmid d$, contradicting the fact that $d = \text{g.c.d.}\{d_i | i \in \Pi\}$. $\square$

Remark 2.7. We now turn our attention to generalizing Theorem 2.6 to an arbitrary cardinal. (The existence of) Ulam-measurable cardinals shall be of utmost importance. Do note though that the proof of the sufficiency does reveal that if X is a P_α-space then $C(X,\mathbb{Z})$ is an α-G.C.D. ring.

3. Ulam-measurable cardinals

As stated in Remark 2.7, we are after a generalization of Theorem 2.6. The goal of this section is to decide which additional set-theoretic axioms (along with ZFC) will enable us to do so. As the title of this section foreshadows, Ulam-measurable cardinals will play a crucial role. We suggest [5] for a good source on Ulam-measurable cardinals. We assume basic knowledge of them. Throughout this section μ will denote the least Ulam-measurable cardinal. To start us off we state the following two lemmas without proof. The first lemma may be found in [5] and the second is the natural generalization of exercise 12H, [10], to an arbitrary cardinal.

Lemma 3.1. *Let α be a cardinal. Then α is Ulam-measurable if and only if $\mu \leqslant \alpha$.*

Lemma 3.2. *Suppose X is a zero-dimensional space such that there exists an uncountable cardinal α for which X is α-disconnected yet X is not a P_α-space. We assume further that X is a P_β-space for all cardinals $\beta < \alpha$. Then α is measurable.*

We immediately obtain the following.

Proposition 3.3. *Let X be a zero-dimensional space and α a cardinal such that $C(X,\mathbb{Z})$ is an α-G.C.D. ring but X is not a P_α-space. Then both α and $|X|$ are measurable cardinals.*

Proof. First, let α' be the least uncountable cardinal such that $C(X,\mathbb{Z})$ is an α'-G.C.D. ring, X is a P_σ-space for all $\sigma < \alpha'$, yet X is not a $P_{\alpha'}$-space. Observe that $\alpha' \leqslant \alpha$. By Lemma 3.2 and Proposition 2.5, α' and α are both measurable. Thus, the only thing left to be shown is that $|X|$ is measurable. For simplicity let β denote the cardinality of X. If β were not Ulam-measurable, then we would have $\beta < \mu \leqslant \alpha' \leqslant \alpha$. Since $\beta < \alpha$, $C(X,\mathbb{Z})$ is a β-G.C.D. ring. Since $\beta < \alpha'$, X is a P_β-space. But the only space of cardinality β that is a P_β-space is the discrete space, contradicting the fact that X is not a P_α-space. □

We now have our sought after generalization.

Theorem 3.4. *Let X be a zero-dimensional space, whose cardinality is not Ulam-measurable. If α is an arbitrary cardinal, then $C(X,\mathbb{Z})$ is an α-G.C.D. ring if and only if X is a P_α-space.*

As indicated in the beginning of the section, we now turn our attention to an example which shows that if one assumes the existence of measurable cardinals then there is a space X which satisfies Proposition 3.3. Of course, our space will have measurable cardinality.

Example 3.5. Recall that μ is the least measurable cardinal. Choose an ω^+-complete ultrafilter on μ, say p, and let $X = \mu \cup \{p\}$ with the subspace topology inherited from $\beta\mu$.

Since p is in fact μ^+-complete it follows that X is a P_β-space for all β, nonmeasurable, yet X is not a P_{μ^+}-space. We show that $C(X,\mathbb{Z})$ is a μ^+-G.C.D. ring.

Let $\{f_\sigma\}_{\sigma<\mu}$ be a collection of functions from $C(X,\mathbb{Z})$. Let $g \in C(\mu,\mathbb{Z})$ be the g.c.d. of the restrictions of the f_σ to μ. Since $p \in v\mu$ we know there exists an extension to p, say $g(p)$. It is evident g is a common divisor for the f_σ. Thus, all that is left, is to show that if $h \in C(X,\mathbb{Z})$ and h divides each f_σ then h divides g.

Clearly, by restricting to μ we have that at each $x \in \mu$, $h(x)|g(x)$. We argue as before. If $h(p) \nmid g(p)$ then letting $j = \text{g.c.d.}(h,g)$ we have that $j(x) = h(x)$, for all $x \in \mu$, but then $j(p) = h(p)$; otherwise the restriction of h to μ has two continuous extensions to X, which we know cannot happen. Thus, g is the g.c.d. of the f_σ and so $C(X,\mathbb{Z})$ is a μ^+-G.C.D. ring, whereas X is not a P_μ-space.

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