A generalization of the Jaffard-Ohm-Kaplansky Theorem

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To Jorge Martinez on his 60^{th} *birthday*

Abstract

The well-known Jaffard-Ohm-Kaplansky Theorem states that every abelian ℓ-group can be realized as the group of divisibility of a commutative Bézout domain. To date there is no realization (except in certain circumstances) of an arbitrary, not necessarily abelian, ℓ-group as the group of divisibility of an integral domain. We show that using filters on lattices we can construct a nice quantal frame whose "group of divisibility" is the given ℓ-group. We then show that our construction when applied to an abelian ℓ-group gives rise to the lattice of ideals of any Prüfer domain assured by the Jaffard-Ohm-Kaplansky Theorem. Thus, we are assured of the appropriate generalization of the Jaffard-Ohm-Kaplansky Theorem.

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1 Introduction

The Jaffard-Ohm-Kaplansky Theorem states that the group of divisibility of a Bézout domain is an abelian lattice-ordered group and conversely, given any abelian lattice-ordered group there is a Bézout domain whose group of divisibility is the given group. In this article we tackle the problem of generalizing this result to arbitrary, and not necessarily abelian, lattice-ordered groups. In this section we recall the appropriate definitions needed for our construction. These definitions come from the theory of rings, the theory of lattice-ordered groups, as well as the theory of quantales (and frames). For those readers familiar with groups of divisibility, lattice-ordered groups, and frames skipping the introduction would be appropriate action.

If R is a commutative ring with identity we let $\mathcal{L}(R)$ denote its lattice of ideals (ordered by inclusion) and $U(R)$ its set of units which is a group under multiplication.

For any $r, r_1, \cdots, r_n \in R$, we employ the notation rR to denote the principal ideal generated by r, and $r_1R + \cdots + r_nR$ to denote the finitely generated ideal of R generated by the $r_1, \cdots, r_n$.

Definition 1.1 Let $(G, \cdot, e)$ be a group (not necessarily abelian). A partial order on G, say $\leq$, makes G into a *partially-ordered group* if for all $g, h, x, y \in G$ with $g \leq h$

$$xgy \leq xhy.$$

A partially-ordered group G is called a *lattice-ordered group* (or *ℓ-group* for short) if the partial-order is a lattice order. The set

$$G^+ := \{g \in G : g \geq e\}$$

is called the *positive cone of* G, and G is said to be a *directed* partially-ordered group when G^+ generates G as a group. Lattice-ordered groups are directed. We cite [3] as our main reference for the theory of ℓ-groups.

Definition 1.2 Let R be a commutative integral domain and let $q(R)$ denote its classical field of fractions. The set $q(R)^* := q(R) \smallsetminus \{0\}$ is a group under multiplication. An obvious subgroup of $q(R)^*$ is $U(R)$, the set of units of R. The *group of divisibility* of R is the factor group $G(R) := q(R)^*/U(R)$.

$G(R)$ can be endowed with a partial order as follows. For any $xU(R), yU(R) \in G(R)$, set $xU(R) \leq yU(R)$ if and only if $yx^{-1} \in R$. This partial order makes $G(R)$ into a directed partially-ordered group. Furthermore,

$$G(R)^+ = \{dU(R) : d \in R\}.$$

$G(R)$ is an ℓ-group precisely when R is a GCD-domain, that is, when every pair of elements possess a great common divisor. A nice subclass of the class of GCD-domains is the class of Bézout domains. A Bézout domain is a domain for which every finitely generated ideal is principally generated.

Another way of constructing the group of divisibility of R is to consider the collection of principal ideals of R, say $Princ(R)$, and to partial order this set by reverse inclusion. This set is a submonoid of the collection of invertible ideals of R and thus is a cancellative abelian monoid. It follows that there is a group of quotients of $Princ(R)$, say $G'(R)$, for which the partial order on $Princ(R)$ can be extended to $G'(R)$, making $G'(R)$ into a directed partially-ordered group. One can then check that the identification $rR \mapsto rU(R)$ is an isomorphism between the groups $G'(R)$ and $G(R)$. (Notice that $Princ(R)$ is a lattice precisely when R is a GCD-domain, and $Princ(R)$ is a sublattice of $\mathcal{L}(R)$ exactly when R is a Bézout domain.) A nice source for information on GCD-domains (and Bézout domains, in particular) as well as the group of divisibility is [4].

In [1] the authors take a Prüfer domain R and observe that the group of invertible ideals of R, denoted $(Inv(R), \cdot, R)$, becomes a lattice-ordered group when ordered by reverse inclusion. The positive cone of $Inv(R)$ is the set of integral invertible ideals, i.e., $Inv(R)^+ = Inv(R) \cap \mathcal{L}(R)$. When R is a Bézout domain, then $Inv(R)^+ = Princ(R)$ and so $Inv(R)$, $G(R)$, and $G'(R)$ are all isomorphic as ℓ-groups. It is the construction of Brewer and Klingler which interests us and leads us into the theory quantal frames.

Definition 1.3 A *quantale* is an algebra of type $(Q, \wedge, \vee, \cdot)$ where $(Q, \wedge, \vee)$ is a complete lattice, $(Q, \cdot)$ is a semigroup, and such that for any subset $S \subseteq Q$ we have

$$a \cdot \bigvee S = \bigvee \{a \cdot s : s \in S\} \qquad \text{and} \qquad \bigvee S \cdot a = \bigvee \{s \cdot a : s \in S\}.$$

The above equalities will be called the *quantale laws.* Unless there is ambiguity we shall assume that $\cdot$ is always the operation of a given quantale. When $\cdot$ is a commutative operation then we say Q is a *commutative quantale.* The top and bottom element of Q will be denoted by 1 and 0 respectively.

Our basic reference for the theory of quantales is [5]. Our aim is to include all the definitions and results needed for completeness sake.

Definition 1.4 A *frame* is a complete lattice $(L, \wedge, \vee)$ for which finite meets distribute through arbitrary supremum, that is, for any subset $S \subseteq L$ the following equality holds for any $a \in L$

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}.$$

This equality is known as the *frame law.* If we consider $\cdot = \wedge$ as an extra operation in $(L, \wedge, \vee, \wedge)$, then it follows that a frame is a quantale.

More generally, if $(Q, \wedge, \vee, \cdot)$ is a quantale for which $(Q, \wedge, \vee)$ is a frame then we call Q a *quantal frame.* Notice that here we are not assuming that the binary product and infimum are equal.

Lemma 1.5 *Suppose Q is a quantale. Then the operation $\cdot$ is order preserving (that is, for all $a, b, c \in Q$ with $a \leq b$, $ac \leq bc$ and $ca \leq cb$).*

Proof. Suppose $a \leq b$ and $c \in Q$. We show that $c \cdot a \leq c \cdot b$ and note that the proof that $a \cdot c \leq b \cdot c$ is similar. Notice that

$$c \cdot b = c \cdot (b \vee a) = (c \cdot b) \vee c \cdot a$$

where the last equality follows from the quantale law. It follows that $c \cdot a \leq c \cdot b$.■

Definition 1.6 Let $(Q, \wedge, \vee, \cdot)$ be a quantale and consider the following properties.

- Q is called *right-sided* if for every $a \in Q$, $a \cdot 1 \leq a$.
- Q is called *strictly right-sided* if for every $a \in Q$, $a \cdot 1 = a$.
- Q is called *left-sided* if $1 \cdot a \leq a$ for all $a \in Q$.
- Q is called *strictly left-sided* if $1 \cdot a = a$ for all $a \in Q$.
- A quantale that is both (strictly) left-sided and (strictly) right-sided is called *(strictly) two-sided.*

Lemma 1.7 *Suppose L is a quantale. L is right-sided if and only if for all $a, b \in L$, $a \cdot b \leq a$. Similarly, L is left-sided if and only for all $a, b \in L$ $a \cdot b \leq b$.*

Proof. The sufficiency of the first statement is patent. As for the reverse suppose L is right-sided and that $a, b \in L$. The result follows from the following string of inequalities and equalities.

$$\begin{array}{rcll} a & \geq & a \cdot 1 & \\ & = & a \cdot (1 \vee b) & \\ & = & (a \cdot 1) \vee (a \cdot b) & \quad \text{(quantale law)} \\ & \geq & a \cdot b & \end{array}$$

The other statements are proved similarly.■

Definition 1.8 Suppose $(Q, \wedge, \vee, \cdot)$ is a quantale and that $c \in Q$. We say that c is a *compact element of* Q if whenever $c \leq \bigvee S$ for some subset $S \subseteq Q$, then there are finitely many $s_1, \cdots, s_n \in S$ for which $c \leq s_1 \vee \cdots \vee s_n$. The collection of compact elements of Q is denoted by $\mathfrak{K}(Q)$. Q is said to be a *compact quantale* if 1 is a compact element of Q. We call Q an *algebraic quantale* if every element of Q is the supremum of compact elements. The notions of a compact element, compact frame, and algebraic frame are defined similarly.

It is straightforward to check that $\mathfrak{K}(Q)$ is closed under finite supremum. A compact algebraic quantale Q for which $\mathfrak{K}(Q)$ is closed under finite products is called a *coherent quantale.* By a *coherent quantal frame* we shall mean a quantal frame whose quantale structure is coherent and the finite infimum of compact elements is compact. (For those familiar with the terminology this means that the frame structure on Q is coherent as well.)

Example 1.9 Let R be a not necessarily commutative ring and let $\mathcal{L}_\ell(R)$ denote the lattice of left ideals of R. The binary product of left ideals makes $(\mathcal{L}_\ell(R), \wedge, \vee, \cdot)$ into a strictly two-sided algebraic quantale.

When R is commutative we can say more. Since the compact elements of $\mathcal{L}(R)$ are precisely the finitely generated ideals, and since the product of finitely generated ideals is again finitely generated it follows that $\mathcal{L}(R)$ is a coherent quantale. $\mathcal{L}(R)$ is a quantal frame precisely when $\mathcal{L}(R)$ is a distributive lattice. Such a ring is called an arithmetical ring. Recall that the ring R is arithmetical precisely when every localization R_M at a maximal ideal is a valuation domain. The classes of arithmetical domains and Prüfer domains coincide. Therefore, for a domain R we gather that $\mathcal{L}(R)$ is a coherent quantal frame if and only if R is a Prüfer domain.

2 The Frame of Filters of a Lattice

In this section we shall consider the collection of filters of a distributive lattice. Throughout this section $(L, \wedge, \vee)$ will denote a distributive lattice. By a *filter* on L we mean a subset F of L which has the following properties:

i) F is closed under binary meets, (that is, for all $x, y \in F$, $x \wedge y \in F$);

ii) if $x \in F$ and $x \leq y$, then $y \in F$.

First of all we do consider L and $\emptyset$ to be (trivial) filters on L. We denote the collection of all filters on L by $\mathfrak{F}_L$. We endow $\mathfrak{F}_L$ with the partial order given by inclusion. Since the intersection of filters is again a filter it follows that $(\mathfrak{F}_L, \leq)$ is a complete lattice where meet is given by intersection and join is given by the intersection of all filters containing each member of the family in question. Thus, $\mathfrak{F}_L$ is a complete bounded lattice where $1 = L$ and $0 = \emptyset$. Our next result gives a more useful description of the join in $\mathfrak{F}_L$.

Lemma 2.1 *Let $S \subseteq \mathfrak{F}_L$. Then*

$$\bigvee S = \{x \in L : \exists F_1, \cdots, F_n \in S, \exists s_i \in F_i, i = 1, \cdots, n \textit{ for which } x \geq s_1 \wedge \cdots \wedge s_n\}.$$

In particular, for any filters $F_1, F_2 \in \mathfrak{F}_L$,

$$F_1 \vee F_2 = \{x \in L : x \geq f_1 \wedge f_2 \textit{ for some } f_1 \in F_1, f_2 \in F_2\}.$$

Proof. We begin by demonstrating that the set T defined in the statement of the lemma is in fact a filter. Clearly, it satisfies property (ii) of the definition of a filter. Next, let $x, y \in T$. Then there exists a collection of filters in S, say $F_{j_1}, \cdots, F_{j_n}, F_{i_1}, \cdots, F_{i_m}$ and $s_j \in F_j, t_i \in F_i$ ($j = j_1, \cdots, j_n$ and $i = i_1, \cdots, i_m$) so that $x \geq s_{j_1} \wedge \cdots s_{j_n}$ and $y \geq t_{i_1} \wedge \cdots \wedge t_{i_m}$. Since $x \wedge y \geq s_{j_1} \wedge \cdots s_{j_n} \wedge t_{i_1} \wedge \cdots \wedge t_{i_m}$ it follows that T is a filter, and so $T \in \mathfrak{F}_L$. It is apparent that $F \leq T$ for all $F \in S$. Finally, if H is a filter for which $F \leq H$ for all $F \in S$, then for any finite subset $F_1, \cdots, F_n \subseteq S$ and $s_i \in F_i$ we have that $s_i \in H$, whence $s_1 \wedge \cdots \wedge s_n \in T$. Therefore, $T \leq H$.■

The nicest of all kinds of filters are the principal filters. For an element $x \in L$, we let $F_x = \{y \in L : y \geq x\}$ and notice that F_x is the smallest filter of L containing x, and therefore we call F_x the *principal filter generated by* x. Notice that $F_x = F_y$ if and only if $x = y$. We leave the proof of the next lemma for the interested reader.

Lemma 2.2 *For any $x, y \in L$,*

$$F_x \vee F_y = F_{x \wedge y} \qquad \textit{and} \qquad F_x \wedge F_y = F_{x \vee y}.$$

Lemma 2.3 *The compact elements of $\mathfrak{F}_L$ are precisely the principal filters together with the empty filter; namely,*

$$\mathfrak{K}(\mathfrak{F}_L) = \{F_x : x \in L\} \cup \{0\}.$$

Moreover, $\mathfrak{F}_L$ is an algebraic lattice.

Proof. First off, notice that 0 is always compact in any lattice. We now move on to show that each F_x is a compact element of $\mathfrak{F}_L$. Suppose that $S \subseteq \mathfrak{F}_L$ and that $F_x \leq \bigvee S$. Then by Lemma 2.1, $x \geq s_1 \wedge s_2 \wedge \cdots \wedge s_n$ for some $s_i \in F_i \in S$. Consequently, $x \in F_1 \vee \cdots \vee F_n$ and so $F_x \leq F_1 \vee \cdots \vee F_n$. We conclude that each F_x is a compact element of $\mathfrak{F}_L$.

Next, notice that for any nonempty $F \in \mathfrak{F}_L$, $F = \bigvee_{x \in F} F_x$ so $\mathfrak{F}_L$ is an algebraic lattice. Moreover, if F is a compact element of $\mathfrak{F}_L$, then $F = F_{x_1} \vee \cdots \vee F_{x_n}$ for some finite subset of F. By Lemma 2.2, we arrive at the conclusion that $F = F_{x_1 \wedge \cdots \wedge x_n}$ is a principal filter.■

Lemma 2.4 *$\mathfrak{F}_L$ is a compact lattice if and only if L has a bottom element.*

Proof. For $\mathfrak{F}_L$ to be compact it means that $1 = F_x$ for some $x \in L$. Therefore $x \leq y$ for all $y \in L$, and so x is the bottom element of L. Conversely, $L = F_0$.■

Theorem 2.5 *The set $\mathfrak{F}_L$ of all filters on L is an algebraic frame if and only if L is a distributive lattice. Moreover, $\mathfrak{F}_L$ is a coherent frame if and only if L is a distributive lattice with bottom element.*

Proof. The second statement follows from the first since L has a bottom element if and only if $\mathfrak{F}_L$ is compact.

Next, if L is distributive, then it follows that the set of compact elements of $\mathfrak{F}_L$ distribute, and therefore, since $\mathfrak{F}_L$ is an algebraic lattice, $\mathfrak{F}_L$ is distributive.

Suppose that $\mathfrak{F}_L$ is an algebraic frame. This means that $\wedge$ and $\vee$ distribute over each other. Then in particular, for any $x, y, z \in L$ we apply Lemma 2.2 repeatedly

$$\begin{aligned} F_{x \wedge (y \vee z)} &= F_x \vee (F_{y \vee z}) \\ &= F_x \vee (F_y \wedge F_z) \\ &= (F_x \vee F_y) \wedge F_x \vee F_z) \\ &= F_{x \wedge y} \wedge F_{x \wedge z} \\ &= F_{(x \wedge y) \vee (x \wedge z)} \end{aligned}$$

We conclude that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, whence L is distributive.

Conversely, if L is distributive then it follows that the set of compact elements of $\mathfrak{F}_L$ distribute. A complete distributive algebraic lattice is necessarily a frame.■

Definition 2.6 Throughout the rest of this section we assume that $(L, \wedge, \vee, \cdot)$ is a distributive lattice together with an extra associative binary operation which is order preserving. We use this operation to define an operation on $\mathfrak{F}_L$ as follows. For $F, H \in \mathfrak{F}_L$ we define

$$F * H = \{x \in L : x \geq [(f_1 \cdot g_1) \wedge \cdots \wedge (f_n \cdot g_n)] \text{ for some } f_1, \cdots, f_n \in F, g_1 \cdots, g_n \in G\}.$$

Notice that even though our definition looks a little strange it is needed to ensure that our product is again a filter. The proof of this fact is straightforward and left to the interested reader. Furthermore, observe that $F * 0 = 0 * F = 0$ for all $F \in \mathfrak{F}_L$.

We now show that our operation is actually given by something simpler.

Proposition 2.7 *Suppose $(L, \wedge, \vee, \cdot)$ is a distributive lattice with an extra binary operation which is order preserving. Then for any $F, H \in \mathfrak{F}_L$*

$$F * H = \{x \in L : x \geq f \cdot h \text{ for some } f \in F, h \in H\}.$$

Furthermore, if $\cdot$ is associative, then so is $*$. In particular, $(\mathfrak{F}_L, 0, *)$ is a monoid.

Proof. Clearly, the right side of the equality is contained in the left side. As to the reverse let $x \in F * H$ so $x \geq f_1 h_1 \wedge \cdots \wedge f_n h_n$ for appropriate $f_1, \cdots, f_n \in F$ and $h_1, \cdots, h_n \in H$. Since the operation is order preserving it follows that for each $1 \leq i, j \leq n$

$$f_i h_j \geq (f_1 \wedge \cdots \wedge f_n) h_j \geq (f_1 \wedge \cdots \wedge f_n)(h_1 \wedge \cdots \wedge h_n).$$

Setting $f = f_1 \wedge \cdots \wedge f_n$ and $h = h_1 \wedge \cdots \wedge h_n$ and observing that $f \in F$ and $h \in H$ we conclude that $x \geq fh$, whence $F * H = \{x \in L : x \geq fh$ for some $f \in F$ and $h \in H\}$.

Thus, we are left to show that $*$ is an associative operation on $\mathfrak{F}_L$. Towards that goal let $F, G, H \in \mathfrak{F}_L$, and $x \in (F * G) * H$. What this means (by what we just proved above) is that $x \geq x'h$ for some $x' \in F * G$ and $h \in H$. Then $x' \geq fg$ for some $f \in F$ and $g \in G$, and once again using that the operation is order preserving we gather that $x \geq (fg)h$. Now, we use that the extra operation on L is associative to conclude that $x \geq f(gh) \in F * (G * H)$. The reverse containment is similar.■

Definition 2.8 Consider $(L, \wedge, \vee, \cdot)$ where $(L, \wedge, \vee)$ is a lattice and $(L, \cdot)$ is a semigroup. If L has the property that for any $x, y, z \in L$, $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$ and $(y \vee z) \cdot x = (y \cdot x) \vee (z \cdot x)$ (and dually for $\wedge$), then we say $(L, \wedge, \vee, \cdot)$ is a *lattice-ordered semigroup* (or *ℓ-semigroup* for short). Notice that this implies that the product is order preserving and so we can apply the above results. By a *distributive ℓ-semigroup* we mean an ℓ-semigroup whose lattice is distributive.

Proposition 2.9 *Suppose* $(L, \wedge, \vee, \cdot)$ *is a distributive* ℓ*-semigroup. Then* $(\mathfrak{F}_L, \wedge, \vee, *)$ *is a distributive quantale. Furthermore, the following properties are satisfied.*

1. *For any* $y, z \in L$, $F_y * F_z = F_{yz}$.
2. $(\mathfrak{F}_L, *)$ *is a commutative semigroup if and only if* $(L, \cdot)$ *is a commutative semigroup.*

Proof. First notice that since L is distributive so is $(\mathfrak{F}_L, \wedge, \vee)$. Next, to show that $(\mathfrak{F}_L, \wedge, \vee)$ is a quantale we need to demonstrate that for any $F \in \mathfrak{F}_L$ and $\{H_i\}_{i \in I} \subseteq \mathfrak{F}_L$

$$F * \bigvee_{i \in I} H_i = \bigvee_{i \in I} F * H_i.$$

Let $y \in F * \bigvee_{i \in I} H_i$. This means that $y \geq f \cdot x$ for some $f \in F$ and $x \in \bigvee_{i \in I} H_i$. Recall that $x \in \bigvee_{i \in I} H_i$ means there is a finite collection of I, say $i_1, \cdots, i_n \in I$ and $h_j \in H_{i_j}$ $(j = 1, \cdots, n)$ for which

$$x \geq h_{i_1} \wedge \cdots \wedge h_{i_n}$$

Therefore,

$$\begin{aligned} y &\geq f \cdot x \\ &\geq f \cdot (h_{i_1} \wedge \cdots \wedge h_{i_n}) \\ &= f \cdot h_{i_1} \wedge \cdots \wedge f \cdot h_{i_n}. \end{aligned}$$

Now for each $j = 1, \cdots, n$ we know that $f \cdot h_{i_j} \in F * H_{i_j}$, and therefore, $y \in \bigvee_{i \in I} F * H_i$. Thus, $F * \bigvee_{i \in I} H_i \leq \bigvee_{i \in I} F * H_i$.

Conversely, if $y \in \bigvee_{i \in I} F * H_i$, then $y \geq x_{i_1} \wedge \cdots \wedge x_{i_n}$ for some $x_{i_j} \in F * H_{i_j}$. Then $x_{i_j} \geq f_j \cdot h_{i_j}$ for appropriate $f_j \in F$ and $h_{i_j} \in H_{i_j}$. We conclude that $(\mathfrak{F}_L, \wedge \vee,)$ is a quantale.

1. Let $y, z \in L$. Recall that $x \in F_{y \cdot z}$ precisely when $x \geq y \cdot z$, and therefore $x \in F_y * F_z$. Conversely, let $x \in F_y * F_z$. By Proposition 2.7 there are $y_1 \in F_y$ and $z_1 \in F_z$ such that $x \geq y_1 \cdot z_1$. Note that $y_1 \geq y$ and $z_1 \geq z$. Since L is an ℓ-semigroup it follows that $x \geq y \cdot z$, whence $x \in F_{yz}$.

2. The necessity easily follows from 1. since if * is commutative then for any $x, y \in L$ we have

$$F_{x \cdot y} = F_x * F_y = F_y * F_x = F_{y \cdot x}$$

and therefore, we have that $xy \in F_{y \cdot x}$, so $xy \geq yx$. Similarly, we derive that $yx \geq xy$ and so we conclude that $yx = xy$, whence $\cdot$ is a commutative operation.

As for the reverse suppose that $\cdot$ is a commutative operation and let $F, H \in \mathfrak{F}_L$. Let $x \in F * H$ which means that $x \geq fh$ for appropriate $f \in F$ and $h \in H$. Now, since $\cdot$ is commutative we get that $x \geq hf$ and so $x \in H * F$. Hence $F * H \subseteq H * F$. A similar argument yields that $H * F \subseteq F * H$ and so $H * F = F * H$.■

Example 2.10 The most natural example of ℓ-semigroup, and the one we are most interested in, is the positive cone of a lattice-ordered group. It is known that an ℓ-group is a distributive ℓ-semigroup, and hence its positive cone is a distributive ℓ-monoid with a least element. We arrive at our two main theorems of this section. Combined, the two theorems give us our generalization of the Jaffard-Ohm-Kaplansky Theorem to arbitrary lattice-ordered groups. For our purposes here we shall consider an isomorphism of quantal frames to be a bijection which preserves finite products, finite meets, and arbitrary joins.

Theorem 2.11 *Let $(G, \cdot, e)$ be an arbitrary ℓ-group. Then $\mathfrak{F}_{G^+}$ is a strictly two-sided coherent quantal frame for which $F * (H \wedge J) = (F * H) \wedge (F * J)$ and $(H \wedge J) * F = (H * F) \wedge (H * F)$ for all $F, H, J \in \mathfrak{F}_{G^+}$. Moreover, the nonzero compact elements of $\mathfrak{F}_{G^+}$ are cancellative.*

Proof. As we mentioned previously the lattice structure on G^+ is distributive and has a least element so by Theorem 2.5 and Proposition 2.9, $\mathfrak{F}_{G^+}$ is a coherent quantal frame. That $\mathfrak{F}_{G^+}$ is strictly two-sided stems from the equality $1 * H = F_e * H = H = H * F_e = H * 1$.

In general, we always have $F*(H\wedge J) \leq (F*H)\wedge(F*J)$, so let $x \in (F*H)\wedge(F*J)$. This means that for some $f_1, f_2 \in F$, $h \in H$, and $j \in J$, $x \geq f_1 h$ and $x \geq f_2 j$. It follows that $x \geq f'h$ and $x \geq f'j$ where $f' = f_1 \wedge f_2 \in F$. Thus, $x \geq f'h \vee f'j = f'(h \vee j)$. Notice that $h \vee j \in H \wedge J$ and so $x \in F * (H \wedge J)$. The other equalities also hold.

To see that the compact elements are cancellative let $F_x \in \mathfrak{K}(\mathfrak{F}_{G^+})$ and $H, J \in \mathfrak{F}_{G^+}$ satisfy $F_x * H = F_x * J$. We aim to prove that $H = J$. To that end let $h \in H$. Then $xh \in F_x * H = F_x * J$ and so $xh \geq xj$ for some $j \in J$. Since this inequality takes place in an ℓ-group we conclude that $h \geq j$ and so $h \in J$, whence $H \subseteq J$. By symmetry we conclude that $H = J$. Similarly, if $H * F_x = J * F_x$ then $H = J$. Consequently, every compact element is a cancellative element of $\mathfrak{F}_{G^+}$.■

Theorem 2.12 *Suppose $(Q, \wedge, \vee, \cdot)$ is a strictly two-sided coherent quantal frame for which the nonzero compact elements of Q are cancellative and $a \cdot (b \wedge c) = (a \cdot b) \wedge (a \cdot c)$ and $(b \wedge c) \cdot a = (b \wedge a) \cdot (c \wedge a)$ for all $a, b, c \in \mathfrak{K}(Q)$. Then there exists an ℓ-monoid L for which $Q \cong \mathfrak{F}_L$.*

Proof. First note that the hypothesis on Q forces the set of nonzero compact elements of Q to be both a lattice and a semigroup. Denote the partial order on Q by $\leq$ and define the reverse partial-order on $L = \mathfrak{K}(Q) \setminus \{0\}$ (the set of nonzero compact elements of Q) by $x \preceq y$ precisely when $x \geq y$. It follows that $(L, \preceq, \sqcap, \sqcup)$ is a lattice where the supremum (infimum)of x and y, denoted by $x \sqcup y$ ($x \sqcap y$), is $x \wedge y$ ($x \vee y$). By hypothesis $1 \in \mathfrak{K}(Q)$ and so 1 is the least element of L. Moreover, since Q is strictly two-sided it follows that bottom element of L is an identity of L. Therefore, L is a lattice with bottom element as well as a monoid. The hypothesis that $\cdot$ distributes over meets guarantees that L is an ℓ-monoid.

Next, we show that $Q \cong \mathfrak{F}_L$ as quantal frames. But this is straightforward once we notice that the nonzero compact elements of $\mathfrak{F}_L$ are in 1-1 order reversing correspondence with $L = \mathfrak{K}(Q) \setminus \{0\}$. Consequently, the compact elements of $\mathfrak{F}_L$ are in 1-1 order preserving correspondence with $\mathfrak{K}(Q)$. Since both $\mathfrak{F}_L$ and Q are coherent, this correspondence extends to a frame isomorphism between $\mathfrak{F}_L$ and Q (and hence it preserves finite meets and arbitrary joins). Therefore, it suffices to show that this correspondence preserves the binary operation $*$. To that end let $F, H \in \mathfrak{F}_L$ and denote the correspondence just described by $\Phi : \mathfrak{F}_L \to Q$. Then

$$\begin{aligned}
\Phi(F * H) &= \Phi(\bigvee_{x\in F} x \bigvee_{h\in H} h) \\
&= \Phi(\bigvee_{x\in F, h\in H} x \cdot h) \\
&= \bigvee_{x\in F, h\in H} x \cdot h \\
&= \bigvee_{x\in F} x \bigvee_{h\in H} h \\
&= \Phi(F) * \Phi(H)
\end{aligned}$$

Therefore, $Q \cong \mathfrak{F}_L$.■

Corollary 2.13 *Let R be a commutative Prüfer domain and let G be its group of invertible ideals. Then $\mathcal{L}(R) \cong \mathfrak{F}_{G^+}$ as quantal frames.*

3 Complemented ℓ-Groups

In this final section we give a nice application of our results from the previous section.

In [2] the authors investigated complemented ℓ-groups. They defined an ℓ-group $(G, \cdot, e)$ to be *complemented* if for every $g \in G^+$ there is an $h \in G^+$ for which $g \wedge h = e$ and $g \vee h$ is a weak-order unit. (Recall that a *weak-order unit* is a positive element $u \in G$ for which $u \wedge g = e$ implies $g = e$ for any $g \in G^+$.) They investigated the topological structure of the Zariski topology on the collection of minimal prime subgroups of G, denoted $Min(G)$. They classified complemented ℓ-groups as those ℓ-groups for which $Min(G)$ is compact. Their proof involved a transfinite induction and by the authors' own admission seemed overly complicated. In [6] the proof for abelian ℓ-groups was shortened considerably and did not use transfinite induction. In this section we show that the argument used in [6] can now be applied to arbitrary ℓ-groups using our generalization to the Jaffard-Ohm-Kaplansky Theorem.

Definition 3.1 Let G be an ℓ-group and H a subgroup. We say H is a *convex ℓ-subgroup of* G if H is a sublattice that also has the property that whenever $e \leq g \leq h$ for some $h \in H$, then $g \in H$. A prime subgroup of G is a convex ℓ-subgroup P for which $a \wedge b \in P$ implies $a \in P$ or $b \in P$. The convex ℓ-subgroup P is prime if and only if the set of cosets G/P is a totally-ordered set. Zorn's Lemma ensures that prime subgroups exist in ℓ-groups and, moreover, every prime subgroup of G contains a minimal prime subgroup. The collection of all minimal prime subgroups is denoted $Min(G)$. Weak order units can be characterized using minimal prime subgroups. The element u is a weak order unit if and only if it does not belong to any minimal prime subgroup.

$Min(G)$ is endowed with two topologies. The first is the *Zariski topology* (a.k.a. the hull-kernel topology) which is constructed using the base of open sets $\{U_m(g)\}_{g \in G}$ where

$$U_m(g) = \{P \in Min(G) : g \notin P\}.$$

The set-theoretic complement of $U_m(g)$ is denoted $V_m(g)$. These sets have the property that $U_m(g) = U_m(|g|)$ and so we might as well assume $g \in G^+$. Furthermore, for any $g, h \in G^+$, $U_m(g) \cap U_m(h) = U_m(g \wedge h)$ and $U_m(g) \cup U_m(h) = U_m(g \vee h) = U_m(gh)$. We cite [2] as a reference for the Zariski topology on $Min(G)$. It is known that each $U_m(g)$ is a clopen set in the Zariski topology and that $Min(G)$ is a zero-dimensional Hausdorff space. (Zero-dimensional space means that it has a base of clopen sets.)

The *inverse topology on $Min(G)$* is the one obtained by taking sets of the form $V_m(g)$ ($g \in G^+$) as a base for a topology. This topology is always T_1 and compact, yet not T_2 in general. So that there is no confusion we denote the inverse topology on $Min(G)$ by $Min(G)^{-1}$. Since each basic open set of $Min(G)^{-1}$, $V_m(g)$ is also open in the Zariski topology on $Min(G)$, it follows that the Zariski topology is finer than the inverse topology. For more information on $Min(G)^{-1}$ when G is abelian we recommend the reader check [6].

For a Bézout domain R it is known that there is one-to-one order reversing correspondence between the prime ideals of R and the prime subgroups of its group of divisibility $G(R)$ (recall Definition 1.2). This correspondence also holds for Prüfer domains but in the more general context; namely, between the minimal prime ideals of R and the prime subgroups of $Inv(G)$. We put this in its appropriate context.

Definition 3.2 Let $(Q, \wedge, \vee, \cdot)$ be a coherent strictly two-sided quantal frame. The element $p \in Q$ is called a *prime element* if whenever $a \cdot b \leq p$, then $a \leq p$ or $b \leq p$. Since Q is compact it follows that every element $x < 1$ lies beneath a maximal element (use Zorn's Lemma). By a *maximal element* we mean an element $m < 1$ for which $m \leq n \leq 1$ implies $m = n$ or $1 = n$. Moreover, since Q is strictly two-sided it follows that these maximal element are in fact prime elements of Q. We denote the set of maximal elements of Q by $Max(Q)$.

We endow $Max(Q)$ with two topologies: the Zariski topology and the inverse topology. For any compact element $x \in \mathfrak{K}(Q)$ we define $U_M(x) = \{m \in Max(Q) : x \not\leq m\}$ (and its set-theoretic complement by $V_M(x)$). These sets have the following properties. For any $x, y \in \mathfrak{K}(Q)$, $U_M(x) \cap U_M(y) = U_M(x \cdot y)$ and $U_M(x) \cup U_M(y) = U_M(x \vee y)$. It follows that both sets $\{U_M(x) : x \in \mathfrak{K}(Q)\}$ and $\{V_M(x) : x \in \mathfrak{K}(Q)\}$ form bases for topologies on $Max(Q)$. The one endowed with the former is called the *Zariski topology*, while the latter is called the *inverse topology* and is denoted $Max(Q)^{-1}$.

For an ℓ-group G the prime elements of $\mathfrak{F}_{G^+}$ are precisely the prime filters on G^+. The maximal elements of $\mathfrak{F}_{G^+}$ are precisely the ultrafilters on G^+. In general there is a one-to-one order reversing correspondence between the set of prime elements of $\mathfrak{F}_{G^+}$ and the set of prime subgroups of G. An excellent reference for this is Proposition 14.3 of [3]. It is useful to note that for a prime element $p \in \mathfrak{F}_{G^+}$ the corresponding prime subgroup P_p is the convex ℓ-subgroup generated by the set $\{g \in G^+ : g \notin p\}$. The above correspondence theorem is sometimes referred to as the *Lemma on Ultrafilters.*

Lemma 3.3 *The correspondence $\Psi : Max(\mathfrak{F}_{G^+}) \to Min(G)$ defined by $\Psi(p) = P_p$ is a bijection for which $\Psi(U_M(F_x)) = V_m(x)$ and $\Psi(V_M(F_x)) = U_m(x)$.*

Proof. Let $M \in U_m(F_x)$. This means that $F_x \not\leq M$ and so $x \notin M$. Therefore, $x \in P_M$. It follows that $\Psi(M) = P_M \in V_m(x)$. Conversely, let $P \in V_m(x)$ and let $M \in Max(\mathfrak{F}_{G^+})$ satisfy $P_M = P$. Then $x \notin M$ and so $F_x \not\leq M$, whence $M \in U_M(\mathfrak{F}_{G^+})$.■

Corollary 3.4 *For any ℓ-group G:*

1) $\Psi : Max(\mathfrak{F}_{G^+}) \to Min(G)$ is a homeomorphism between $Max(\mathfrak{F}_{G^+})$ and $Min(G)^{-1}$. Therefore, $Max(\mathfrak{F}_{G^+})$ is a compact T_1-space.

2) $\Psi : Max(\mathfrak{F}_{G^+}) \to Min(G)$ is a homeomorphism between $Max(\mathfrak{F}_{G^+})^{-1}$ and $Min(G)$. Therefore, $Max(\mathfrak{F}_{G^+})^{-1}$ is a zero-dimensional Hausdorff space.

We now share an easier proof of Theorem 2.2 of [2]. Notice that we avoid using transfinite induction.

Theorem 3.5 *For an ℓ-group G, the following statements are equivalent.*

1. *G is a complemented ℓ-group.*
2. *$Min(G) = Min(G)^{-1}$, i.e. the Zariski topology on $Min(G)$ equals the inverse topology on $Min(G)$.*
3. *$Min(G)$ is compact.*

Proof. 1. $\Rightarrow$ 2. Suppose G is a complemented ℓ-group. In general, the Zariski topology is finer than the inverse topology on $Min(G)$. Therefore to show equality it suffices to show that every set of the form $U_m(g)$ is inverse open. To that end let $g \in G^+$. By hypothesis, there is some $h \in G^+$ such that $g \wedge h = e$ and $g \vee h$ is a weak order unit. Since $U_m(g) \cap U_m(h) = U_m(g \wedge h) = U_m(e) = \emptyset$ and $U_m(g) \cup U_m(h) = U_m(g \vee h) = Min(G)$ where the latter equality follows from the fact that a weak order unit is contained in no minimal prime subgroup. We have thus demonstrated that $U_m(g) = V_m(h)$ and therefore every basic Zariski open set is an inverse open set, whence $Min(G) = Min(G)^{-1}$.

2. $\Rightarrow$ 3. This follows from the fact that the inverse topology is always compact.

3. $\Rightarrow$ 1. Suppose $Min(G)$ is compact and let $g \in G^+$. Since $V_m(g)$ is an open subset of $Min(G)$ it follows that we can cover $V_m(g)$ by basic open sets, say

$$V_m(g) = \bigcup_{b \in I} U_m(b)$$

for some subset $I \subseteq G^+$. Because $Min(G)$ is compact so is the closed set $V_m(G)$ and therefore the above cover can be reduced to a finite subcover, say $V_m(g) = U_m(b_1) \cup \cdots \cup U_m(b_n) = U(b_1 \vee \cdots \vee b_n)$. What we have show is that for every $g \in G^+$, there is some $b \in G^+$ such that $U_m(g) = V_m(b)$. We claim that b is a complement of g. First note that $U_m(g \wedge b) = U_m(g) \cap U_m(b) = \emptyset$. Since the intersection of all minimal prime subgroups of G is e it follows that $g \cap b = e$. Next, $Min(G) = U_m(g) \cup U_m(b) = U_m(g \vee b)$ and therefore $g \vee b$ does not belong to any minimal prime subgroup of G, whence $g \vee b$ is a weak-order unit of G. We conclude that G is a complemented ℓ-group.$\blacksquare$

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