

# WHEN $R(X)$ AND $R\langle X \rangle$ ARE CLEAN: A CONSTRUCTIVE TREATMENT

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ABSTRACT. For  $R$  a commutative ring, we give constructive proofs that  $R(X)$  is clean exactly when  $R$  is clean, and that  $R\langle X \rangle$  is clean exactly when  $R$  is zero dimensional. We also give a constructive proof of the known result that  $R(X) = R\langle X \rangle$  exactly when  $R$  is zero dimensional. By a *constructive proof* we mean one that is carried out within the context of intuitionistic logic. In practice this means that the arguments are arithmetic rather than ideal theoretic.

## 1. WHEN $R(X)$ IS CLEAN

Throughout this article, all rings are assumed to be commutative with identity.

In [9], Nicholson defined a *clean ring* to be a ring in which every element is the sum of a unit and an idempotent. These rings have garnered much attention in the last decade. Many authors have determined when specific kinds of rings are clean, others have carried out the same program for various generalizations of the notion of a clean ring. Local rings and von Neumann regular rings are clean. For more information and a history see [8].

Our first theorem provides a useful characterization of a clean ring. Recall that two ideals are said to be *comaximal* if their sum is the whole ring.

**Theorem 1** (Theorem 1.7 [8]). *A ring  $R$  is clean if and only if for any two comaximal ideals  $I$  and  $J$  of  $R$ , there is an idempotent  $e \in I$  with  $1 - e \in J$ .*

*Proof.* Suppose  $R$  is clean. If  $I + J = R$ , then there exists  $a \in I$  such that  $1 - a \in J$ . Write  $1 - a = u + e$  where  $u$  is a unit and  $e$  is an idempotent. Then  $u^{-1}(1 - a) = 1 + u^{-1}e$  whence  $u^{-1}(1 - a)(1 - e) = 1 - e$ . Thus  $1 - e \in J$ . Also  $a = 1 - u - e$ , so  $ae = -ue$  whence  $-u^{-1}ae = e$ , so  $e \in I$ .

Conversely, suppose the second condition holds. If  $a \in R$ , then  $Ra + R(1 - a) = R$ , so there is an idempotent  $e \in Ra$  with  $1 - e \in R(1 - a)$ . Thus  $e = ra$  and  $1 - e = s(1 - a)$ , whence  $re - s(1 - e)$  and  $ea + (1 - e)(a - 1)$  are inverses, so the latter is a unit which when added to the idempotent  $1 - e$  is equal to  $a$ .  $\square$

The *content ideal* of a polynomial  $f(x) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$  is the ideal of  $R$  generated by  $a_0, \dots, a_n$ . We denote the content ideal of  $f$  by  $c(f)$  (The content ideal is sometimes denoted by  $A_f$  in the literature.) Let  $U = \{f \in R[X] : c(f) = R\}$ .

The *Dedekind-Mertens lemma* says that if  $\deg g \leq m$ , then  $c(f)^{m+1}c(g) = c(f)^m c(fg)$ . A constructive proof can be found in [3] or in [7]. In particular, if  $c(f) = R$ , then  $c(g) = c(fg)$ . This result is needed to conclude that polynomials with content  $R$  are regular elements of  $R[X]$  and that  $U$  is closed under multiplication.

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The Nagata ring over  $R$ , denoted  $R(X)$ , is the localization of  $R[X]$  with respect to  $U$ , that is,  $R(X) = U^{-1}R[X]$ . Thus every element of  $R(X)$  has the form  $f/g$  where  $f, g \in R[X]$  and  $c(g) = R$ . The Nagata ring is a subring of the classical ring of quotients of  $R[X]$ . For more information on the Nagata ring see [1].

We will need the following result about idempotents in  $R(X)$ .

**Theorem 2** (Theorem 2.4 [1]). *Every idempotent of  $R(X)$  lies in  $R$ .*

*Proof.* The proof in [1] appeals to the fact that if  $c(f) = R$ , then  $f$  is regular and  $c(g) = c(fg)$ , which follows from the Dedekind-Mertens lemma. It also uses the fact that a finitely generated idempotent ideal is generated by an idempotent. The proof of this latter result in [4] is purely computational.  $\square$

We can now prove our main theorem regarding the Nagata ring.

**Theorem 3.** *A ring  $R$  is clean if and only if  $R(X)$  is clean.*

*Proof.* Let  $R$  be a clean ring. Suppose  $a = f/g \in R(X)$  with  $c(g) = R$ . Write

$$f = f_0 + f_1X + \cdots + f_nX^n \text{ and } g = g_0 + g_1X + \cdots + g_nX^n.$$

(Note: we are not assuming that  $f_n \neq 0$  or that  $g_n \neq 0$ .)

Let  $I = (f_0, \dots, f_n)$  and  $J = (f_0 - g_0, \dots, f_n - g_n)$  be ideals of  $R$ . Then  $I + J = R$ . Since  $R$  is a clean ring, by Theorem 1 there exists an idempotent  $e \in J$  with  $1 - e \in I$ . Let  $K = (f_0 - eg_0, \dots, f_n - eg_n)$ . We will show that  $K = R$ .

Consider the ideals  $K + Re$  and  $K + R(1 - e)$ . Clearly  $K + Re \supseteq I + Re = R$ . Also,  $K + R(1 - e) \supseteq J + R(1 - e) = R$  because  $f_i - g_i = f_i - eg_i - (1 - e)g_i$ . So

$$R = (K + Re)(K + R(1 - e)) \subseteq K,$$

whence  $K = R$ .

Notice that

$$a - e = \frac{f}{g} - e = \frac{f}{g} - \frac{eg}{g} = \frac{f_0 - eg_0 + (f_1 - eg_1)X + \cdots + (f_n - eg_n)X^n}{g}.$$

Now  $c(f - eg) = K = R$  and so  $a - e$  is a unit of  $R(X)$ . Therefore,  $R(X)$  is a clean ring.

Conversely, suppose  $R(X)$  is a clean ring. For any  $a \in R$ , we can write  $a = (f/g) + e$  where  $f/g$  is a unit of  $R(X)$  and  $e^2 = e \in R(X)$ . Thus  $c(f) = c(g) = R$ . By Theorem 2,  $e \in R$ , so  $f/g \in R$ . This means that there is a  $u \in R$  such that  $f = ug$ , whence, by the Dedekind-Mertens lemma,  $R = c(f) = c(ug) = c(u)$  so  $u$  is a unit of  $R$  and  $a = u + e$ .  $\square$

## 2. WHEN $R(X)$ IS CLEAN

We say that an element  $r$  in a ring  $R$  is *zero-dimensional* if either of the following equivalent conditions holds

- We can write  $R = A \times B$  where  $r$  is nilpotent in  $A$  and a unit in  $B$ .
- There is  $n$  such that  $r^n \in Rr^{n+1}$ .

A constructive proof that these two conditions are equivalent may be found in [2, Lemma 1]. We say that a ring is *zero-dimensional* exactly when all of its elements are zero dimensional (see [6, Lemma 3.1] for the classical equivalence of this with prime ideals being maximal). Note that if  $R$  is a finite product  $\prod R_i$ , then  $r \in R$  is zero dimensional if and only if the coordinates of  $r$  in each  $R_i$  are zero dimensional.

**Theorem 4.** *If  $I$  is nil and  $R/I$  is zero dimensional, then  $R$  is zero dimensional.*

*Proof.* If  $r \in R$ , then there exists  $n$  and  $s \in R$  such that  $r^n(1 - sr) \in I$ . So  $r^{mn}(1 - sr)^m = 0$ . But  $(1 - sr)^m = 1 - tr$  for some  $t \in R$ , so  $r^{mn}(1 - tr) = 0$ .  $\square$

The converse of this theorem is immediate from the definition in the second bullet above.

**Theorem 5.** *Zero-dimensional elements are clean.*

*Proof.* If  $r \in R$  is zero-dimensional, then we can write  $R = A \times B$  where  $r$  is nilpotent in  $A$  and a unit in  $B$ . As nilpotents and units are both clean, and an element in a product is clean when it is clean in each factor, we get that  $r$  is clean.  $\square$

The ring  $R\langle X \rangle$  is the localization of the polynomial ring  $R[X]$  with respect to the set of monic polynomials. We will need to know when a polynomial is a factor of a monic polynomial. The characterization involves invertible polynomials. A polynomial  $c \in R[X]$  has an inverse in  $R[X]$  exactly when the constant term of  $c$  is invertible in  $R$  and all other coefficients are nilpotent. For a constructive proof of this, rather than the usual proof using prime ideals, see [11, Theorem 3] or [7, Lemme II.2.6.4].

Any ring map  $\theta : R \rightarrow S$  induces a map  $\theta_* : R[X] \rightarrow S[X]$  that takes monic polynomials to monic polynomials. If  $R$  is a finite product  $\prod R_i$ , let  $\pi_i$  denote the projection map from  $R$  onto  $R_i$ .

**Theorem 6.** *Let  $R$  be a ring and  $f \in R[X]$ . The following conditions are equivalent:*

- (1)  *$f$  is a factor of a monic polynomial in  $R[X]$ ,*
- (2)  *$R$  is a finite product  $\prod R_i$  with  $\pi_{i*}f = c_i d_i$  where  $c_i \in R_i[X]$  invertible and  $d_i \in R_i[X]$  monic for each  $i$ ,*
- (3)  *$R$  is a finite product  $\prod R_i$  with  $\pi_{i*}f$  a factor of a monic polynomial in  $R_i[X]$  for each  $i$ .*

*Proof.* Gilmer and Heinzer [5] proved that (1) and (2) are equivalent for reduced rings  $R$ , and Yengui [12] proved that constructively. A constructive proof of the equivalence of (1) and (2) for arbitrary rings can be found in [10].

As (1) and (2) are equivalent, and (2) obviously implies (3), we need only show that (3) implies (2). Suppose that  $\pi_{i*}f$  is a factor of a monic polynomial in  $R_i[X]$  for each  $i$ . We use (1) implies (2) to get a decomposition  $R = \prod R_{ij}$  such that  $\pi_{ij*}f = c_{ij} d_{ij}$  with  $c_{ij}$  invertible and  $d_{ij}$  monic. But that is (2) for this finer decomposition.  $\square$

**Corollary 7.** *If  $rX + s \in R[X]$  is a factor of a monic polynomial, then  $r$  is zero dimensional.*

*Proof.* We can write  $R$  as a finite product  $\prod R_i$  so that the image of  $rX + s$  in  $R_i[X]$  can be written as  $r_i X + s_i = c_i d_i$  where  $c_i$  is an invertible polynomial and  $d_i$  is monic. If  $d_i = 1$ , then  $r_i$  is nilpotent. If  $\deg d_i = 1$ , then  $r_i = c_i$  is invertible in  $R_i$ . So we can write  $R = A \times B$  where the coordinate of  $r$  in  $A$  is nilpotent and the coordinate of  $r$  in  $B$  is invertible. Thus  $r$  is zero dimensional.  $\square$

We want to identify  $R\langle X \rangle$  with  $\prod R_i\langle X \rangle$ . The maps  $\pi_{i*} : R[X] \rightarrow R_i[X]$  define a map  $\varphi : R[X] \rightarrow \prod R_i[X]$  which is clearly an isomorphism. As  $\pi_{i*}$  takes monic polynomials to monic polynomials, we get maps  $\pi_{i**} : R\langle X \rangle \rightarrow R_i\langle X \rangle$  which define a map  $\varphi_* : R\langle X \rangle \rightarrow \prod R_i\langle X \rangle$ . This will be our isomorphism.

Suppose we are given  $f_i/g_i$  in  $R_i\langle X \rangle$ , with  $g_i$  monic for each  $i$ . We may assume that all the  $g_i$  have the same degree by multiplying them, and the  $f_i$ , by appropriate powers of  $X$ . Define  $f$  and  $g$  in  $R[X]$  by  $f = \varphi^{-1}(f_1, \dots, f_n)$  and  $g = \varphi^{-1}(g_1, \dots, g_n)$  in  $R[X]$ , noting that  $g$  is monic because the  $g_i$  all have the same degree. Thus  $\varphi_*$  is a natural isomorphism from  $R\langle X \rangle$  to  $\prod R_i\langle X \rangle$ .

We can now prove the main result of this section.

**Theorem 8.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is zero dimensional.
- (2)  $R\langle X \rangle$  is a zero-dimensional ring.
- (3)  $R\langle X \rangle$  is a clean ring.
- (4)  $R(X) = R\langle X \rangle$ .

*Proof.* Suppose  $R$  is zero-dimensional. Let  $N$  be the ideal of nilpotent elements of  $R$ , the nilradical of  $R$ , and let  $S = R/N$ . The map  $\pi : R \rightarrow S$  induces a map  $\pi_{**} : R\langle X \rangle \rightarrow S\langle X \rangle$  which is clearly onto. We want to show that  $\ker \pi_{**}$  is the nilradical of  $R\langle X \rangle$ , so by Theorem 4 it will suffice to show that  $S\langle X \rangle$  is zero dimensional. As  $S$  is reduced, it follows that  $S\langle X \rangle$  is reduced, so  $\ker \pi_{**}$  contains the nilradical of  $R\langle X \rangle$ . Conversely, suppose  $\pi_{**}(f/g) = 0$ . Then  $\pi_*(f) = 0$  so each coefficient of  $f$  is nilpotent whence  $f/g$  is nilpotent.

Let  $f/g \in S\langle X \rangle$ . We can write  $S$  as a finite product  $\prod S_i$  where each coefficient of  $f_i$  is either zero or invertible. Consequently, each  $f_i$  is either zero or has an invertible leading coefficient. Thus  $f_i/g_i$  is either zero or invertible, hence zero dimensional, for all  $i$ . Therefore,  $f/g$  is zero dimensional. Consequently,  $R\langle X \rangle$  is a zero-dimensional ring.

If  $R\langle X \rangle$  is a zero-dimensional ring, then  $R\langle X \rangle$  is a clean ring by Theorem 5.

Next, suppose that  $R\langle X \rangle$  is a clean ring. Let  $r \in R$  and write  $rX = f/g + e$  where  $f$  is a factor of a monic polynomial in  $R[X]$  and  $e$  is an idempotent. Then  $e \in R$  and  $(rX - e)g$  is a factor of a monic polynomial in  $R[X]$ . Hence  $rX - e$  is a factor of a monic polynomial in  $R[X]$ , so  $r$  is a zero-dimensional element of  $R$  by Corollary 7. Therefore,  $R$  is a zero-dimensional ring.

We have shown that the first three conditions are equivalent. For the equivalence of the last condition, we rely in part on an argument from [6, Lemma 17.8].

Suppose that  $R$  is a zero-dimensional ring. We will show that any  $f \in R[X]$  of content  $R$  is a factor of a monic polynomial. Because  $R$  is zero dimensional, we can write  $R = \prod R_i$  so that the coefficients of each  $f_i$  are either nilpotent or invertible (and, of course, at least one is invertible). Let  $u$  be a unit of  $R_i$  such that  $uf_i = g + h$  where  $g$  is monic and  $h$  is nilpotent. Then  $(uf_i - g)^k = 0$  for some  $k$ , whence  $f_i$  divides the monic polynomial  $g^k$ . This is true for each  $i$ , so  $f$  divides some monic polynomial in  $R[X]$  by Theorem 6. It follows that  $R(X) = R\langle X \rangle$ .

Conversely, suppose that polynomials with content  $R$  are factors of monic polynomials. Then for any  $r \in R$ , the polynomial  $rX + 1$  is a factor of a monic polynomial. Consequently,  $r$  is zero dimensional by Corollary 7.

□

## REFERENCES

- [1] D. D. Anderson, D. F. Anderson, and R. Markanda, *The rings  $R\langle X \rangle$  and  $R\langle X \rangle$* , J. Algebra **95** (1985), no. 1, 96-115.
- [2] J. Brewer and F. Richman, *Subrings of zero-dimensional rings*, Multiplicative ideal theory in commutative algebra, Springer 2006, pp. 73–88.
- [3] T. Coquand, *A direct proof of the Dedekind-Mertens Lemma*, <http://www.cse.chalmers.se/~coquand/mertens.pdf>
- [4] R. Gilmer, *Multiplicative Ideal Theory*, Decker, New York, 1972.
- [5] R. Gilmer and W. Heinzer, *On the divisors of monic polynomials over commutative rings*, Pacific J. Math. **78** (1978), 121–123.
- [6] J. A. Huckaba, *Commutative Rings With Zero Divisors*, Marcel Decker 1988.
- [7] H. Lombardi and C. Quitté, *Algèbre Commutative, Méthodes Constructive*, Calvage & Mounet, Paris, 2011.
- [8] W. Wm. McGovern, *Neat rings*, J. Pure Appl. Algebra **205** (2006), no. 2, 243-265.
- [9] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269-278.
- [10] F. Richman, *A division algorithm*, J. Alg. and its Appl. **4** (2005), 441–449.
- [11] \_\_\_\_\_, *Nontrivial uses of trivial rings*, Proc. Amer. Math. Soc., **103** (1988), 1012–1014.
- [12] I. Yengui, *An algorithm for the divisors of monic polynomials over a commutative ring*, Math. Nacht. **260** (2003), 1–7.

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