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Feebly projectable ℓ-groups

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In memory of Paul F. Conrad

Abstract. In the article [17], we introduced and investigated feebly and flatly projectable frames. In this article, we apply these two properties to lattice-ordered groups. An example is constructed to illustrate that the two properties are distinct, which solves a question from [17]. We also investigate these properties with respect to archimedean ℓ-groups with weak order unit, as well as commutative semiprime f-rings.

1. Introduction

Our setting is the theory of lattice-ordered groups. We assume some familiarity with the area, but use the first section of this article to lay the foundation on which the article is based. References best suited to the present purposes are [5], [2], and [1]. In the fourth section, we will consider commutative semiprime f-rings, e.g., $C(X)$, the f-ring of real-valued continuous functions on the topological space X. An appropriate reference for this topic is [13].

For the record, a *lattice-ordered group* (or *ℓ-group* for short) is a group $(G, +, 0)$ that is also equipped with a lattice-order, say $\leq$, which is compatible with the group operations. That is, for all $a, b, x, y \in G$ with $a \leq b$, we have $y + a + x \leq y + b + x$. (Even though we use additive notation, we do not assume that our ℓ-groups are abelian.) We denote the infimum and supremum of G by $\wedge$ and $\vee$, respectively. The set of *positive* elements of G is $G^+ = \{g \in G \mid g \geq 0\}$. For $g \in G$, we define $g^+ = g \vee 0$, $g^- = (-g) \vee 0$, and $|g| = g^+ + g^-$, and call these the *positive part*, the *negative part*, and the *absolute value* of g, respectively. An element $u \in G^+$ is called a *weak order unit* if whenever $u \wedge g = 0$, then $g = 0$.

A subgroup H of G is called an ℓ-subgroup if it is also a sublattice of G. The ℓ-subgroup $H \leq G$ is said to be *convex* if whenever $0 \leq g \leq h$ and $h \in H$, then $g \in H$. The collection of all convex ℓ-subgroups of G is denoted $\mathcal{C}(G)$. When partially-ordered by inclusion, $\mathcal{C}(G)$ becomes a complete distributive lattice. In particular, an arbitrary intersection of convex ℓ-subgroups is again a convex ℓ-subgroup. This allows us to discuss the convex ℓ-subgroup generated by a set.

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For a subset $S \subseteq G$, we use $G(S)$ to denote the convex ℓ-subgroup generated by S. When $S = \{s\}$, we instead write $G(s)$ and call this the *principal convex ℓ-group generated by s*. It is known that

$$G(s) = \{g \in G \mid |g| \leq n|s| \text{ for some } n \in \mathbb{N}\}.$$

Using this, we gather that $G(s) = G(|s|)$ and for any finite subset $\{s_1, \ldots, s_n\}$ of G^+,

$$G(s_1) \cap \cdots \cap G(s_n) = G(s_1 \wedge \cdots \wedge s_n)$$

and

$$G(s_1) \vee \cdots \vee G(s_n) = G(s_1 \vee \cdots \vee s_n) = G(s_1 + \cdots + s_n).$$

For every ℓ-group G, $\mathcal{C}(G)$ is an algebraic frame. Therefore, the techniques used in frame theory are useful in studying $\mathcal{C}(G)$. Recall that a *frame* is a complete distributive lattice $(L, \leq, \wedge, \vee)$ that satisfies the strengthened distributive law (also known as the *frame law*)

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I} (a \wedge b_i)$$

for any arbitrary index set I. Frames are also known as both complete Heyting algebras or complete Brouwerian lattices. (The latter is what Birkhoff called them in his book [3] on lattices.) It is known that $\mathcal{C}(G)$ satisfies the strengthened distributive law, and thus, $\mathcal{C}(G)$ is a frame. We now recall some pertinent definitions and facts from the theory of frames.

The top element and bottom element of a frame L are denoted by 1 and 0, respectively. The frame law guarantees that for each $a \in L$, there is a largest element which is disjoint from a. We denote this element by $a^\perp$ and call this the *polar of a*. (The polar of a is also known as the *pseudo-complement of a*.) An element $a \in L$ is called *dense* if $a^\perp = 0$. Throughout the rest of this section, we assume that $(L \leq, \wedge, \vee)$ is a frame.

The element $a \in L$ is called *compact* if whenever $a \leq \bigvee_{i\in I} b_i$, there is a finite subset $i_1, \ldots, i_n \in I$ such that $a \leq b_{i_1} \vee \cdots \vee b_{i_n}$. A frame for which every element is the supremum of compact elements is known as an *algebraic* frame. The collection of compact elements of L is denoted by $\mathfrak{K}(L)$. This set is closed under finite joins, and when it is also closed under finite meets, we say that L satisfies the *finite intersection property on compact elements*, abbreviated *the FIP*. A compact dense element of L is called a *unit of L*.

An element $p \in L$ is said to be *prime* if whenever $a \wedge b = p$, then either $a = p$ or $b = p$. We let $\mathrm{Spec}(L)$ denote the collection of prime elements of L. The usual Zorn's Lemma argument assures us of the following two results. One, for any compact element $z \in \mathfrak{K}(L)$, there is an element m which is maximal with respect to $z \nleq m$. Such a maximal element is called a *value of z* and is known to be a prime element. The set of all values of z is denoted by $\mathrm{Yos}_L(z)$ and is called the *Yosida space of z*. Two, minimal prime elements exist and every prime element contains a minimal prime. The set of minimal prime elements of L is denoted by $\mathrm{Min}(L)$. When $\mathrm{Spec}(L)$ satisfies the property that the collection of primes containing a given prime form a chain, then $\mathrm{Spec}(L)$ is called a *root system*. It is known that under Zorn's Lemma, if L is an algebraic frame with FIP, then $\mathrm{Spec}(L)$ is a root system if and only if L satisfies *disjointification*, that is, for each $a, b \in \mathfrak{K}(L)$ there are disjoint $c, d \in \mathfrak{K}(L)$ such that $c \leq a$, $d \leq b$, and $a \vee d = a \vee b = b \vee c$.

Both $\mathrm{Yos}_L(z)$ and $\mathrm{Min}(L)$ are topological spaces when equipped with the Zariski topology. In particular, for $a \in \mathfrak{K}(L)$, we denote

$$U(a) = \{v \in \mathrm{Yos}(z) \mid a \not\leq v\} \text{ and } U_m(a) = \{p \in \mathrm{Min}(L) \mid a \not\leq p\}.$$

The collection $\{U(a) \mid a \in \mathfrak{K}(L)\}$ (respectively, $\{U_m(a) \mid a \in \mathfrak{K}(L)\}$) forms a base for the Zariski topology on $\mathrm{Yos}_L(z)$ (respectively, $\mathrm{Min}(L)$). With respect to the Zariski topology, $\mathrm{Yos}_L(z)$ is a compact Hausdorff space, while $\mathrm{Min}(L)$ is a zero-dimensional Hausdorff space. (Here *zero-dimensional* means the space has a base of clopen sets).

We now return to ℓ-groups and the frame $\mathcal{C}(G)$. The compact elements of $\mathcal{C}(G)$ are precisely the principal convex ℓ-subgroups, and therefore, $\mathcal{C}(G)$ is an algebraic frame satisfying the FIP. Moreover, $\mathcal{C}(G)$ satisfies disjointification and so $\mathrm{Spec}(G)$ forms a root system. The polars of $\mathcal{C}(G)$ are sets of the form

$$S^{\perp} = \{g \in G \mid |g| \wedge |s| = 0 \text{ for all } s \in S\}$$

for some $S \subseteq G$. These are also called the polar subgroups of G. When $S = \{s\}$ we write $s^{\perp}$. An element $u \in G^+$ for which $G(g)$ is a unit of $\mathcal{C}(G)$ is called a *weak order unit*. Equivalently, $u \in G^+$ is a weak order unit if $u \wedge g = 0$ implies $g = 0$.

The prime elements of $\mathcal{C}(G)$ are known as the prime subgroups of G. They are characterized as follows. The lemma can be found as Theorem 9.1 in [5].

Lemma 1.1. *For $P \in \mathcal{C}(G)$, the following are equivalent.*

(a) *P is a prime subgroup.*
(b) *If $a \wedge b = 0$, then $a \in P$ or $b \in P$.*
(c) *If $A, B \in \mathcal{C}(G)$, $P \subset A$, and $P \subset B$, then $P \subset A \cap B$;*
(d) *If $a, b \in G^+ \smallsetminus P$, then $a \wedge b \in G^+ \smallsetminus P$.*

For any $g \in G^+$, the convex ℓ-subgroups of G that are maximal with respect to not containing g are called the *values of g*. The collection of all values of g is denoted by $\mathrm{Yos}(g)$. (Observe that $\mathrm{Yos}(g) = \mathrm{Yos}_{\mathcal{C}(G)}(G(g))$.)

The collection of all minimal prime subgroups of G is denoted by $\mathrm{Min}(G)$ and is endowed with the hull-kernel topology. Recall that the collection of sets of the form $U_m(g) = \{M \in \mathrm{Min}(G) \mid g \notin M\}$ forms an open base for the Zariski topology. We use $V_m(g)$ to denote the complement of $U_m(g)$. We will have more to say about $\mathrm{Yos}(g)$ and $\mathrm{Min}(G)$ in later sections. Minimal primes are useful in characterizing weak order units. It is known that $u \in G^+$ is a weak order unit if and only if $u \notin P$ for all $P \in \mathrm{Min}(G)$. Finally, when $G = G(g)$, then g is called a *strong order unit*. Notice that a strong order unit is, in fact, a weak order unit.

Example 1.2. Let X be a Tychonoff space, that is, completely regular and Hausdorff. For a subgroup $\mathbb{G} \leq \mathbb{R}$ of real numbers, we let $C(X, \mathbb{G})$ denote the collection of continuous $\mathbb{G}$-valued functions on X. Under the pointwise operations, $C(X, \mathbb{G})$ is a lattice-ordered group and the constant function $\mathbf{1}$ is a weak order unit. We denote the principal convex ℓ-group generated by $\mathbf{1}$ by $C^*(X, \mathbb{G})$ and observe that $\mathbf{1}$ is a strong order unit of $C^*(X, \mathbb{G})$. We write $C(X)$ and $C^*(X)$ to denote the special case $C(X, \mathbb{R})$ and $C^*(X, \mathbb{R})$. The reference [7] is still the best source for information on $C(X)$. We also suggest [25] as a reference for topological spaces. In particular, we take our definition of strongly zero-dimensional space from [25]. We say X is *strongly zero-dimensional* if disjoint zerosets of X can be separated by a clopen set. (In general, we say two subsets U and V of a space X can be *separated by a clopen set* if there is a clopen subset K of X such that $U \subseteq K$ and $V \subseteq X \smallsetminus K$.)

We work within the foundational principles of ZFC. In particular, we do assume the Axiom of Choice and its more useful equivalent form, Zorn's Lemma.

2. Feebly projectable ℓ-groups

We begin this section by recalling the definition of a projectable ℓ-group.

Definition 2.1. An ℓ-group G is called *projectable* if for every $g \in G^+$, $G = g^\perp \oplus g^{\perp\perp}$.

The class of all projectable ℓ-groups is a well-understood class. For example, from [4] we know that every projectable ℓ-group G is representable, that is, G can be viewed as a subdirect product of totally ordered groups. Also, every projectable ℓ-group has stranded primes, where an ℓ-group G is said to have *stranded primes* if every prime subgroup contains a unique minimal prime subgroup. Since $\mathrm{Spec}(G)$ is a root system, it follows that the stranded primes property means that $\mathrm{Spec}(G)$ is a disjoint union of maximal chains of prime subgroups. The property of having stranded primes is also known by many different names: semi-projectable, weakly projectable, and normal (see page 94 of [5] for more information on this terminology). Here is a useful characterization (see Proposition 7.5.1, [2]).

Theorem 2.2. *The following statements are equivalent for an ℓ-group G.*

(a) *G has stranded primes.*
(b) *For every pair of incomparable minimal prime subgroups M_1 and M_2, $G = M_1 \vee M_2$.*
(c) *Whenever $a, b \in G^+$ and $a \wedge b = 0$, then $G = a^\perp \vee b^\perp$.*

The stranded primes property, as well as projectability, can be characterized using frame-theoretic properties of $\mathcal{C}(G)$. To describe these characterizations, we need to recall a few more definitions from the theory of frames.

Definition 2.3. Suppose L is an algebraic frame. L is said to be *projectable* if for every $a \in \mathfrak{K}(L)$, $L = a^{\perp\perp} \vee a^{\perp}$. In [22], the authors studied regularity conditions and showed that being projectable is equivalent to what they called Reg(2) (and equivalent to Reg(3)). They also investigated the condition

Reg(4) for all $a, b \in \mathfrak{K}(L)$, if $a \wedge b = 0$, then $a^{\perp} \vee b^{\perp} = 1$.

The proofs of the following lemmas are straightforward and can be found in [22].

Lemma 2.4. *Suppose G is an ℓ-group. G is projectable if and only if $\mathcal{C}(G)$ is a projectable frame.*

Lemma 2.5. *Suppose G is an ℓ-group. G has stranded primes if and only if $\mathcal{C}(G)$ satisfies Reg(4).*

In [17] we investigated two frame-theoretic properties that lie between the properties of projectability and Reg(4).

Definition 2.6. Suppose L is an algebraic frame. We say L is *feebly projectable* if whenever $a, b \in \mathfrak{K}(L)$ and $a \wedge b = 0$, there is an $x \in \mathfrak{K}(L)$ such that $a \leq x^{\perp\perp}$, $b \leq x^{\perp}$ and $x^{\perp} \vee x^{\perp\perp} = 1$.

A weaker concept is that of flat projectability. L is called *flatly projectable* if whenever $a, b \in \mathfrak{K}(L)$ and $a \wedge b = 0$, there is a $t \in L$ such that $a \leq t$, $b \leq t^{\perp}$ and $t \vee t^{\perp} = 1$. Every feebly projectable frame is flatly projectable, and every flatly projectable frame satisfies Reg(4). In [17], we were unable to construct a counterexample of a flatly projectable frame which is not feebly projectable. We were able to prove that if the frame possesses a unit, then the two notions are equivalent. We address this situation later in the article.

We apply the definitions just stated and the results of [17] to ℓ-groups. The ℓ-group G is *feebly projectable* if whenever $a, b \in G^{+}$ are disjoint, then there is some $x \in G^{+}$ such that $a \in x^{\perp\perp}$, $b \in x^{\perp}$, and $G = x^{\perp\perp} \oplus x^{\perp}$. We say G is *flatly projectable* if for any pair of disjoint elements a and b, there is a polar subgroup S of G such that $a \in S$, $b \in S^{\perp}$, and $G = S \oplus S^{\perp}$.

Lemma 2.7. *Suppose G is an ℓ-group. Then G is feebly projectable if and only if $\mathcal{C}(G)$ is a feebly projectable frame.*

Proof. First assume $\mathcal{C}(G)$ is a feebly projectable frame. Let $a, b \in G^{+}$ be disjoint elements of G; then $G(a)$ and $G(b)$ are disjoint, compact elements of the frame $\mathcal{C}(G)$. So, there exists $x \in G^{+}$ with $G(a) \leq G(x)^{\perp\perp}$, $G(b) \leq G(x)^{\perp}$, and $G(x)^{\perp} \vee G(x)^{\perp\perp} = 1$. Now, $G(b) \leq G(x)^{\perp} = x^{\perp}$ implies $b \in x^{\perp}$ and $G(a) \leq G(x)^{\perp\perp} = x^{\perp\perp}$ implies $a \in x^{\perp\perp}$. Finally, $G(x)^{\perp} \vee G(x)^{\perp\perp} = 1$ implies $x^{\perp} \oplus x^{\perp\perp} = G$. Thus, G is feebly projectable.

Next, suppose G is a feebly projectable ℓ-group. Let $G(a), G(b) \in \mathcal{C}(G)$ with $G(a) \wedge G(b) = 0$. Since $a \wedge b = 0$, there exists $x \in G^{+}$ such that $a \in x^{\perp\perp}$, $b \in x^{\perp}$, and $x^{\perp} \oplus x^{\perp\perp} = G$. It follows that $G(a) \leq G(x^{\perp\perp}) = G(x)^{\perp\perp}$,

$G(b) \leq G(x^\perp) = G(x)^\perp$, and $G(x)^{\perp\perp} \vee G(x)^\perp = 1$. Therefore, $\mathcal{C}(G)$ is feebly projectable. □

The proof of the next lemma is similar to that of the previous lemma and is left to the reader.

Lemma 2.8. *Suppose G is an ℓ-group. Then G is flatly projectable if and only if $\mathcal{C}(G)$ is a flatly projectable frame.*

At this point, the proof of the next result should be apparent.

Proposition 2.9. *Consider the following properties on an ℓ-group G.*

(1) *G is projectable.*
(2) *G is feebly projectable.*
(3) *G is flatly projectable.*
(4) *G has stranded primes.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *If G possesses a weak order unit, then* (2) *and* (3) *are equivalent.*

Later, we will supply several examples of ℓ-groups which show that no two of these conditions are equivalent to each other. Presently, we recall the property that characterizes projectable ℓ-groups within the class of ℓ-groups with stranded primes.

Definition 2.10. Let G be an ℓ-group. A pair of positive elements, say x, y, of G are said to be *complementary* if $x \wedge y = 0$, and $x \vee y$ is a weak-order unit of G. We say G is a *complemented* ℓ-group if for every $x \in G^+$, there is a $y \in G^+$ such that x, y are complementary. It is known that every projectable ℓ-group is complemented and that the converse is false. In particular, $C(\mathbb{R})$ is a complemented ℓ-group that does not have stranded primes (*c.f.* [24] Example 4.6 (2)).

Theorem 2.11. [4] *Suppose G is an ℓ-group with a weak order unit. Then G is projectable if and only if G is complemented and has stranded primes.*

We use an appropriate generalization of Definition 2.10 to obtain the analogous result for feebly projectable ℓ-groups.

Definition 2.12. Let G be an ℓ-group. We call G *weakly complemented* if whenever $a, b \in G^+$ with $a \wedge b = 0$, then there is a complementary pair x, y such that $a \leq x$ and $b \leq y$. The class of weakly complemented ℓ-groups was defined in [23]. We should note that there it was assumed the ℓ-groups in question were abelian. A quick check of the proof of the next theorem from [23] demonstrates that this added condition is not necessary. Recall from [23], that the *inverse topology* on $\mathrm{Min}(G)$ is the one obtained by taking the collection $\{V_m(g) \mid g \in G\}$ as a base for the topology. This topology is always compact and satisfies the T_1 separation axiom.

Theorem 2.13. *The following are equivalent for an ℓ-group G.*

(a) *G is weakly complemented.*
(b) *The inverse topology on* $\mathrm{Min}(G)$ *is zero-dimensional.*
(c) *For each pair of distinct minimal prime subgroups, there exists a complemented element belonging to exactly one of them.*
(d) *For each* $0 < g$ *and each minimal prime subgroup* P *containing* g*, there is a complemented element* x *above* g *which is also in* P*.*

The counterpart to Theorem 2.11 is our next theorem.

Theorem 2.14 (Theorem 3.13, [17])**.** *Suppose G is an ℓ-group containing a weak order unit. Then G is feebly projectable if and only if it is weakly complemented and has stranded primes.*

The next few results, to our knowledge, do not have analogues in frame theory. They are motivated from the the theory of ℓ-groups. Our proof is modeled after the proof that a projectable ℓ-group is representable. Recall that an ℓ-group is *representable* if it is representable as a subdirect product of totally ordered groups. (The reader is urged to check section 47 of [5].)

Theorem 2.15. *Suppose G is flatly projectable. Then G is representable.*

Proof. It is known that an ℓ-group is representable if and only if every polar subgroup is a normal subgroup (Prop. 47.1, [5]). Since for any polar $P^{\perp}$, we have $P^{\perp} = \bigcap_{p\in P} |p|^{\perp}$, and since an intersection of normal subgroups is again normal, it suffices to show that every polar of the form $p^{\perp}$ with $p \geq 0$ is normal. To that end, let $0 \leq q \in p^{\perp}$ and $p \geq 0$. It follows that $p \wedge q = 0$, and since G is flatly projectable, there exists a polar $S \leq G^{+}$ such that $G = S \oplus S^{\perp}$ with $p \in S$ and $q \in S^{\perp}$. Now, since a cardinal summand is a normal subgroup, it follows that $S^{\perp}$ is a normal subgroup of G, and hence, for any $g \in G$, we have $gqg^{-1} \in S^{\perp}$. The membership $p \in S = S^{\perp\perp}$ implies that $gqg^{-1} \in S^{\perp} = S^{\perp\perp\perp} \subseteq p^{\perp}$, from which we conclude that $p^{\perp}$ is a normal subgroup of G, whence G is representable. □

Proposition 2.16. *Suppose $H \in \mathcal{C}(G)$. If G is a flatly projectable ℓ-group, then so is H.*

Proof. Recall that the collection $\mathcal{C}(G)$ is a distributive lattice. Now, suppose G is flatly projectable and let $a, b \in H^{+}$ with $a \wedge b = 0$. Then there is a polar P of G such that $a \in P$, $b \in P^{\perp}$ and $G = P \oplus P^{\perp}$. Now, $H = H \cap (P \vee P^{\perp})$, and so by distributivity, we obtain that

$$H = (H \cap P) \vee (H \cap P^{\perp}) = (H \cap P) \oplus (H \cap P^{\perp}),$$

and that $a \in H \cap P$ and $b \in H \cap P^{\perp}$, whence H is flatly projectable. □

Proposition 2.17. *Suppose $H \leq G$ is a normal convex ℓ-subgroup of the ℓ-group G. If G is a flatly projectable ℓ-group, then so is G/H. Therefore, every homomorphic image of a flatly projectable ℓ-group is flatly projectable.*

Proof. Let $a + H, b + H \in G/H$ and $a + H \wedge b + H = 0$. A simple reduction allows us to assume that our representatives are positive disjoint elements of G, that is, $a \wedge b = 0$. Since G is flatly projectable, there is a summand S such that $G = S \oplus S^{\perp}$ with $a \in S$ and $b \in S^{\perp}$. Consider $K_1 = S \vee H$ and $K_2 = S^{\perp} \vee H$. Then

$$K_1/H \cap K_2/H = (K_1 \cap K_2)/H = H/H,$$

where the last equality follows from

$$K_1 \cap K_2 = (S \vee H) \cap (S^{\perp} \vee H) = (S \cap S^{\perp}) \vee H = H.$$

Next,

$$K_1/H \vee K_2/H = (K_1 \vee K_2)/H = G/H.$$

Thus, K_1/H and K_2/H are complemented summands of G/H. Since $a \in S$, it follows that $a + H \in K_1/H$, and similarly, $b + H \in K_2/H$. Therefore, G/H is flatly projectable. □

It is known that a homomorphic image of a projectable ℓ-group need not be projectable. In the next section, we will demonstrate that it need not be feebly projectable (though it is, of course, flatly projectable). We now consider the arbitrary product of a collection of ℓ-groups.

Proposition 2.18. *Suppose $\{G_i\}_{i \in I}$ is a collection of ℓ-groups and $G = \Pi_{i \in I} G_i$. Then G is feebly (flatly) projectable if and only if each G_i is feebly (flatly) projectable.*

Proof. First suppose G is flatly projectable; then each G_i is flatly projectable by Proposition 2.17. Conversely, suppose each G_i is flatly projectable. Let $a = (a_i), b = (b_i) \in G$ with $a \wedge b = 0$. Since $a_i \wedge b_i = 0$ for all $i \in I$, there exist polars S_i of G_i satisfying $a_i \in S_i$, $b_i \in S_i^{\perp}$, and $G_i = S_i \oplus S_i^{\perp}$. Let $S = \prod_{i \in I} S_i$; then S is a polar of G. Clearly, $a \in S$, $b \in S^{\perp}$, and $G = S \oplus S^{\perp}$. Hence, G is flatly projectable.

Now suppose G is feebly projectable. Fix $n \in I$, and let $a_n, b_n \in G_n$ be such that $a_n \wedge b_n = 0$. Let $a_i = a_n$ if $i = n$ and $a_i = 0$ otherwise; let $b_i = b_n$ if $i = n$ and $b_i = 0$ otherwise. Then $a = (a_i)$ and $b = (b_i)$ are elements of G with $a \wedge b = 0$. It follows that there exists $x \in G^+$ such that $a \in x^{\perp\perp}$, $b \in x^{\perp}$, and $G = x^{\perp} \oplus x^{\perp\perp}$. It is easy to see that $G_n = x_n^{\perp} \oplus x_n^{\perp\perp}$, where $a_n \in x_n^{\perp\perp}$ and $b_n \in x^{\perp}$.

Finally, suppose each G_i is feebly projectable. Let $a = (a_i), b = (b_i) \in G$ with $a \wedge b = 0$. For each $i \in I$, there exists $x_i \in G_i^+$ such that $a_i \in x_i^{\perp\perp}$, $b_i \in x_i^{\perp}$, and $G_i = x_i^{\perp} \oplus x_i^{\perp\perp}$. Let $x = (x_i)$; then $G = x^{\perp} \oplus x^{\perp\perp}$, where $a \in x^{\perp\perp}$ and $b \in x^{\perp}$. Therefore, G is feebly projectable. □

Question 2.19. Within an ℓ-group, when is the join of a set of flatly projectable convex ℓ-subgroups again flatly projectable? If so, then the collection of all flatly projectable ℓ-groups forms a torsion class.

3. Examples

In this section, we will show that no two of the conditions of Proposition 2.9 are equivalent to each other. It will be useful to observe the following facts:

(1) $C(X)$ is a projectable ℓ-group if and only if X is a basically disconnected space, that is, the closure of every cozero set is clopen. This theorem is known as a Stone-Nakano theorem; see [11] for more details.

(2) $C(X)$ is feebly projectable if and only if X is a strongly zero-dimensional F-space. This will be proved, and in more generality, in Theorem 4.12; also see Commentary 4.16.

(3) $C(X)$ has stranded primes if and only if X is an F-space. This is a classical result that can be found in Chapter 14 of [7].

Example 3.1. (1) Let $X_1 = \beta\mathbb{N} \setminus \mathbb{N}$. Then X_1 is an example of a strongly zero-dimensional F-space which is not basically disconnected, and so $C(X_1)$ is feebly projectable but not projectable.

(2) Let $X_2 = \beta\mathbb{R} \setminus \mathbb{R}$. Then X_2 is an example of a connected F-space. Therefore, $C(X_2)$ is an example of an ℓ-group with stranded primes that is not feebly projectable.

We now construct an example of a flatly projectable ℓ-group which is not feebly projectable. Our motivating example is the example $C(\mathbb{N})/C^*(\mathbb{N})$ of Kenny [16]. Recall from [1, Example 36], that $C(\mathbb{N})/C^*(\mathbb{N})$ is an example of a homomorphic image of a projectable ℓ-group which is not projectable. Since the image of a projectable ℓ-group is flatly projectable, and it is also straightforward to show $C(\mathbb{N})/C^*(\mathbb{N})$ possesses a weak-order unit, it follows that $C(\mathbb{N})/C^*(\mathbb{N})$ is a feebly projectable ℓ-group. Therefore, $C(\mathbb{N})/C^*(\mathbb{N})$ is not the example we have in mind, but it does give us a place to look. In particular, we use the rest of this section to flesh out the theory of ℓ-groups of the form $C(X)/C^*(X)$ (for X strongly zero-dimensional). For the ease of the reader, we let $G = C(X)/C^*(X)$ and $B = C^*(X)$.

Example 3.2. Let X be a strongly zero-dimensional space which is not pseudocompact, and let $f \in C(X)^+$ be a fixed element. Consider the zerosets $Z_1 = f^{-1}([0,1])$ and $Z_2 = f^{-1}([2,\infty))$. Since X is strongly zero-dimensional, there exists a clopen set S_1 of X such that $Z_1 \subseteq S_1$ and $S_1 \cap Z_2 = \emptyset$. Notice that $f(x) \leq 2$ for all $x \in S_1$. Next, working inside $X \setminus S_1$, we can find a clopen set $S_2 \subseteq X \setminus S_1$ which separates $(X \setminus S_1) \cap f^{-1}([1,2])$ and $f^{-1}([3,\infty))$. Also, $f(x) \leq 3$ for all $x \in S_2$. Recursively, we can construct a sequence of disjoint clopen sets $S_f = \{S_n\}_{n\in\mathbb{N}}$ which cover X and have the property that $f(x) \leq n+1$ for any $x \in S_n$.

Throughout the rest of this section, X denotes a strongly zero-dimensional space which is not pseudocompact, $f \in C(X)^+$ is a fixed element, and S_f is defined as above.

Proposition 3.3. *In $C(X)/C^*(X)$, the polar of $f+B$ is the set*

$$H = \{g + B \mid \exists N \in \mathbb{N} \text{ such that the restriction of } g \text{ to } \bigcup_{n>N} S_n \text{ is bounded}\}.$$

Proof. Suppose $g + B \in (f + B)^{\perp}$. Without loss of generality, we suppose that $g \in C(X)^+$. Since $(f \wedge g) + B = f + B \wedge g + B = B$, it follows that $f \wedge g \in B$. Thus, there exists a natural number N such that $(f \wedge g)(x) \leq N$ for all $x \in X$. Now, for any $x \in \bigcup_{n>N} S_n$, $f(x) > N$, it follows that $g(x) \leq N$ for all $x \in \bigcup_{n>N} S_n$, whence $g + B \in H$.

Conversely, let $g + B \in H$. Without loss of generality, we can assume that $g \in C(X)^+$. Suppose there exists an $N \in \mathbb{N}$ such that the restriction of g to $\bigcup_{n>N} S_n$ is bounded. This means there is a natural number $M \in \mathbb{N}$ for which $g(x) \leq M$ for all $x \in \bigcup_{n>N} S_n$. For all $c \in S_1 \cup \cdots \cup S_N$ and $f(x) \leq N$, it follows that $f \wedge g \in B$, and so, $g + B \in (f + B)^{\perp}$. □

Corollary 3.4. *We have that $f + B$ is a weak order unit of G if and only if S_n is pseudocompact for each $n \in \mathbb{N}$.*

Proof. Suppose $f + B$ is a weak order unit of G, yet S_n is not pseudocompact for some $n \in \mathbb{N}$. Let $h \in C(S_n)$ be any unbounded continuous function on S_n, and extend h to all of X by defining h to be 0 on all S_m for $m \neq n$. Then by Proposition 3.3, $h + B$ is a nonzero element of $(f + B)^{\perp}$, contradicting that $f + B$ is a weak order unit of G.

Conversely, suppose S_n is pseudocompact for all $n \in \mathbb{N}$. Let $g + B \in (f + B)^{\perp}$. By Proposition 3.3, there is some $N \in \mathbb{N}$ such that the restriction of g to $\bigcup_{n>N} S_n$ is bounded. But the restriction of g to $S_1 \cup \cdots \cup S_N$ is also bounded, and therefore, g is a bounded function. It follows that $g + B = B$, whence $(f + B)^{\perp} = N$. Consequently, $f + B$ is a weak order unit of G. □

Proposition 3.5. *We have that $(f + B)^{\perp\perp}$ is the set*

$$K = \{h + B \in G \mid \text{the restriction of } h \text{ to } S_n \text{ is bounded } \forall n \in \mathbb{N}\}.$$

Proof. Suppose $h + B \in (f + B)^{\perp\perp}$ and that $h \in C(X)^+$. Suppose by means of contradiction, that there exists an $n \in \mathbb{N}$ such that h is unbounded on the clopen set S_n. Choose a sequence $T = \{x_i\}_{i \in \mathbb{N}} \subseteq S_n$ such that $\{h(x_i)\}$ is an unbounded increasing sequence of real numbers. By a method similar to the one used above in the creation of the S_is, we can construct a sequence of disjoint clopen subsets $\{D_j\}_{j \in \mathbb{N}}$ whose union is all of S_n, and so that $x_j \in D_j$.

Define $g \in C(X)$ as follows:

$$g(x) = \begin{cases} 0, & \text{if } x \notin S_n \\ j, & \text{if } x \in D_j. \end{cases}$$

Observe that by Proposition 3.3, $g + B \in (f + B)^{\perp}$. Therefore, $g + B \wedge h + B = B$, i.e., $g \wedge h \in B$. But since both g and h are unbounded on the set T, this is a contradiction. We must conclude that for every $n \in \mathbb{N}$, h is unbounded on S_n. In other words, $h + B \in K$.

Conversely, suppose $h + B \in K$ with $h \in C(X)^+$. We need to show that $h + B \wedge g + B = B$ for every $g + B \in (f + B)^\perp$ with $g \in C(X)^+$. To that end, let $g \in C(X)^+$ have the property that there exists an $N \in \mathbb{N}$ such that the restriction of g to $\bigcup_{n>N} S_n$ is bounded. Since $h + B \in K$, it follows that h is bounded on each $S_1, S_2, \dots, S_N$. But then $h \wedge g$ is bounded on all of X. $\square$

Proposition 3.6. *For a fixed $f \in C(X)$, $G = (f + B)^{\perp\perp} \oplus (f + B)^\perp$ if and only if S_n is pseudocompact for all but a finite number of n.*

Proof. Assume that $G = (f + B)^{\perp\perp} \oplus (f + B)^\perp$. Suppose by way of contradiction, that $\{S_n\}_{n\in\mathbb{N}}$ has a subsequence $\{S_{n_j}\}_{j\in\mathbb{N}}$, all of whose elements are not pseudocompact. For each $j \in \mathbb{N}$, let $t_{n_j} \in C(S_{n_j})^+ \smallsetminus C^*(S_{n_j})$. Construct $k \in C(X)$ as follows:

$$k(x) = \begin{cases} t_{n_j}(x) & \text{if } x \in S_{n_j}, \\ 0 & \text{otherwise.} \end{cases}$$

By hypothesis, $k = h+g$ for some $h \in (f+B)^{\perp\perp}$ and $g \in (f+B)^\perp$. By Proposition 3.5, the restriction of h to S_n is bounded for all $n \in \mathbb{N}$. It follows then that the restriction of g to each S_{n_j} is unbounded. But this contradicts that $g \in (f+B)^\perp$. Therefore, at most a finite number of the S_n are pseudocompact.

Conversely, let $S_{n_1}, \dots, S_{n_k}$ be the collection of those S_n which are not pseudocompact, and set $T = S_{n_1} \cup \dots \cup S_{n_k}$. Let $k \in G^+$. Define

$$h(x) = \begin{cases} 0 & \text{if } x \in T, \\ k(x) & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} k(x) & \text{if } x \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $k = h + g$. Furthermore, by Proposition 3.3, $g + B \in (f + B)^\perp$. Since S_n is pseudocompact for all $n \neq n_1, \dots, n_k$, we can apply Proposition 3.5 and gather that $h+B \in (f+B)^{\perp\perp}$. Thus, $G = (f+B)^{\perp\perp} \oplus (f+B)^\perp$. $\square$

Corollary 3.7. *Suppose X is a strongly zero-dimensional space. Then X is pseudocompact if and only if G is projectable.*

Proof. If X is pseudocompact, then G is trivially projectable. So suppose X is not pseudocompact. Let $f \in C(X) \smallsetminus C^*(X)$ be a fixed element, and let $S_f = \{S_n\}_{n\in\mathbb{N}}$ be defined as above. Then S_f is a collection of disjoint clopen sets where infinitely many S_n are nonempty. Let $\mathbb{J} = \{n_j \mid j \in \mathbb{N}\}$ be the subsequence of $\mathbb{N}$ consisting of those natural numbers n_j for which $S_{n_j} \neq \emptyset$. Observe that $X = \bigcup_{m\in\mathbb{J}} S_m$. Since $\mathbb{J}$ is countably infinite, we can write $\mathbb{J}$ as a disjoint union of nonempty countably infinite sets, say $\mathbb{J} = \bigcup_{m=1}^{\infty} \mathbb{J}_m$. For each $m \in \mathbb{J}$, let f_m be a bijection from J_m to $\mathbb{N}$. Now define $g \in C(X)$ by $g(x) = f_m(x)$ when $x \in S_{n_j}$ and $n_j \in J_m$. Let $S_g = \{S_n^g\}$ be as above. Observe that $g^{-1}(\{n\}) = \bigcup_{m=1}^{\infty} f_m^{-1}(\{n\}) \subseteq S_n^g$ for each $n \in \mathbb{J}$. It follows that no S_n^g is pseudocompact. According to Proposition 3.6, $G \neq (g + B)^{\perp\perp} \oplus (g + B)^\perp$, and thus, G is not projectable. $\square$

Example 3.8. Let X be any strongly zero-dimensional F-space which has no clopen subsets that are pseudocompact, e.g., a P-space with no isolated points. Then G is flatly projectable as it is the homomorphic image of a feebly projectable ℓ-group. For all $f \in C(X) \smallsetminus C^*(X)$,

$$(f + B)^{\perp\perp} \oplus (f + B)^{\perp} \neq G.$$

This is because if $G = (f+B)^{\perp\perp} \oplus (f+B)^{\perp}$ for some $f \in C(X) \smallsetminus C^*(X)$, then we would know that X has a pseudocompact clopen subset, contradicting that X is specifically chosen not to have this property. Consequently, G cannot be feebly projectable.

Remark 3.9. If $\mathcal{FP}$ denotes the class of feebly projectable ℓ-groups, then $\mathcal{FP}$ is not a radical class. Recall that a class of ℓ-groups is said to be a *radical class* if it satisfies (1) it is closed under convex ℓ-subgroups, (2) closed under isomorphic images, and (3) closed under the join of objects in the class. Let G be any ℓ-group which is flatly projectable but not feebly projectable. Note that G is the join of its principally generated convex ℓ-subgroups. Since a convex ℓ-subgroup of a flatly projectable ℓ-group is flatly projectable and each $G(g)$ has a strong order unit, it follows that for every $g \in G$, $G(g)$ is feebly projectable. Therefore, the join of the set of ℓ-groups $\{G(g)\}_{g \in G}$, each of which is feebly projectable, is not feebly projectable.

4. Archimedean ℓ-groups with weak order unit

An ℓ-group is said to be *archimedean* if whenever $g, h \in G^+$ and $ng \leq h$ for all $n \in \mathbb{N}$, then $g = 0$. Examples of archimedean ℓ-groups include $C(X, \mathbb{G})$ for any $\mathbb{G} \leq \mathbb{R}$. It is well known that an archimedean ℓ-group is necessarily abelian.

Throughout this section, G will denote an archimedean ℓ-group possessing a distinguished weak order unit e_G. A morphism between two such ℓ-groups is an ℓ-group homomorphism $\psi\colon G \to H$ such that $\psi(e_G) = e_H$. This creates a category, denoted $\mathbf{W}$, whose objects are of the form (G, e_G). We shall often say G belongs to $\mathbf{W}$ in the sense that G has a weak order unit, say e_G, such that $(G, e_G) \in \mathbf{W}$. Moreover, we let YG denote the Yosida space of e_G.

We assume some familiarity with the Yosida Representation Theorem. We use $\overline{\mathbb{R}}$ to denote the two-point compactification of the real numbers: $\mathbb{R} \cup \{\pm\infty\}$. For a topological space X, $D(X)$ denotes the collection of almost real-valued continued functions:

$$D(X) = \{f\colon X \to \overline{R} \mid f^{-1}(\mathbb{R}) \text{ is a dense subset of } X\}.$$

In general, $D(X)$ is a lattice under the pointwise operations, but it need not be closed under the pointwise operation of addition (nor multiplication). Yet we can speak of sublattices of $D(X)$ which are in fact closed under the pointwise operations of addition as an ℓ-subgroup of $D(X)$. The Yosida Representation Theorem states for each $\mathbf{W}$-object (G, e_G), there is an ℓ-isomorphism $\phi\colon G \to \hat{G}$ where $\hat{G}$ is an ℓ-subgroup of $D(YG)$ and $\phi(e_G) = \mathbf{1}$. The theorem also states that $\hat{G}$ separates the points from closed subsets of YG, that is, for any closed set $K \subseteq YG$ and $x \in YG \setminus K$, there is a $g \in G^+$ such that $\mathbf{0} \leq \phi(g) \leq \mathbf{1}$ and $\phi(g)(x) = 1$ and $\phi(g)(p) = 0$ for all $p \in K$. Also, YG, up to homeomorphism, is the unique compact Hausdorff space X for which G is represented as an ℓ-group of almost continuous real-valued functions on X separating the points of X. (See [12] for more details.)

For each $g \in G$, the set $\operatorname{coz}(g) = \{V \in YG \mid \phi(g)(V) \neq 0\}$ is called the *G-cozero set of g*. We denote the collection of G-cozero sets of G by $\operatorname{coz} G$. It is a fact that the G-cozero set of g is none other than the set $U(g) = \{V \in YG \mid g \notin V\}$, and thus, $\operatorname{coz} G$ forms a base for the topology on YG. From this point on, we identify G with its ℓ-isomorphic image $\hat{G}$ and omit any mention of the function ϕ.

Example 4.1. Consider the **W**-object $(C(X), \mathbf{1})$. $YC(X)$ is none other than βX, the Stone-Čech compactification of X. On the other hand, if we consider the **W**-object $(C(X, \mathbb{G}), \mathbf{1})$ (for any proper subgroup $\mathbb{G} < \mathbb{R}$) and assume that X is zero-dimensional, then $YC(X, \mathbb{G}) = \beta_0 X$, the Banaschewski zero-dimensional compactification of X (see [25]).

For any space X (not necessarily compact), by a *cozero set of X* we mean a subset of the form $\operatorname{coz}(f) = \{x \in X \mid f(x) \neq 0\}$ for some $f \in C(X)$. We let the collection of cozero sets of X be denoted by $\operatorname{coz} X$. Since we are assuming that our spaces are Tychonoff, it follows that $\operatorname{coz} X$ is a base for the topology on X. Finally, notice that for any **W**-object (G, e_G), we have $\operatorname{coz} G \subseteq \operatorname{coz} YG$. That is, every G-cozero set is a cozero set of YG.

We investigate the class of **W**-objects which are feebly projectable. Since **W**-objects possess a unit, it follows by Proposition 2.9 that flatly projectable and feebly projectable are equivalent notions. Consider the following theorem.

Theorem 4.2 (2.2 of [11])**.** *Let $(G, e_G) \in$ **W**. Then G is a projectable ℓ-group if and only if (G, e_G) is local and $\operatorname{cl}_{YG} \operatorname{coz}(a)$ is a clopen subset of YG for all $a \in G$.*

It is our aim to characterize feebly projectable **W**-objects in a similar fashion; see Proposition 4.7. First, we recall the notion of a local **W**-object.

Definition 4.3. Let $(G, e_G) \in$ **W**. The function $f \in D(YG)$ is said to be *locally in G* if for each $p \in YG$ there is a neighbourhood U_p of p, and a $g_p \in G$ such that $g_p(q) = f(q)$ for all $q \in U_p$. We denote the collection of functions that are locally in G by $\operatorname{loc}G$. It is straightforward to check that $\operatorname{loc}G$ is an ℓ-subgroup of $D(YG)$ containing G. This last fact, together with the uniqueness of YG, implies that $YG = Y\operatorname{loc}G$. It is not too difficult to show that $\operatorname{coz} G = \operatorname{coz}(\operatorname{loc}G)$.

We call (G, e_G) *local* if $G = \operatorname{loc}G$; **Loc** denotes the full subcategory of **W** consisting of local **W**-subobjects. (For more information on local **W**-objects, the reader is urged to read [12].)

Definition 4.4. Let $(G, e_G) \in$ **W**. We call (G, e_G) *weakly projectable* if for each $a \in G$, $\operatorname{cl}_{YG} \operatorname{coz}(a)$ is a clopen subset of YG. (Note: this definition has nothing to do with the stranded primes property. See [9] for more information on weakly projectable ℓ-groups.) It follows that an archimedean ℓ-group G containing any weak order unit is projectable if and only if G has a weak order unit e_G such that (G, e_G) is local and weakly projectable. In this case it also follows that for any other weak order unit, say f_G, (G, f_G) is local and weakly projectable.

Proposition 4.5. *Let $(G, e_G) \in \mathbf{W}$. If G is feebly projectable, then YG is zero-dimensional.*

Proof. Since YG is a compact Hausdorff space, it is sufficient to show that YG is totally disconnected. To that end, let p and q be distinct points in YG. Since $\operatorname{coz} G$ is a base for the topology on YG, it follows that we can find $a, b \in G^+$ such that $p \in \operatorname{coz}(a)$, $q \in \operatorname{coz}(b)$, and $\operatorname{coz}(a) \cap \operatorname{coz}(b) = \emptyset$. Furthermore, $a \wedge b = 0$, and so there is some $g \in G^+$ such that $G = g^{\perp\perp} \oplus g^\perp$, $a \in g^{\perp\perp}$, and $b \in g^\perp$. Write $e_G = x_1 + x_2$ where $0 \le x_1 \in g^{\perp\perp}$ and $0 \le x_2 \in g^\perp$. Since $x_1 \wedge x_2 = 0$, it follows that x_1 is the characteristic function of some clopen subset of YG; call it K. Therefore, $p \in \operatorname{coz}(a) \subseteq K$ and $q \in \operatorname{coz}(b) \subseteq YG \smallsetminus K$, whence YG is totally disconnected. □

Remark 4.6. For any $\mathbf{W}$-object (G, e_G) and any clopen subset $K \subseteq YG$, the characteristic function χ_K belongs to G. This is a consequence of the Yosida embedding, and a proof can be found in Lemma 1.2 of [11]. It is known (see Remark 2.3(c) of [11]) that when YG is zero-dimensional, then (G, e_G) is local if and only if for each $a \in G^+$ and clopen subset $K \subseteq YG$, the function $a\chi_K \in G$.

Proposition 4.7. *Let $(G, e_G) \in \mathbf{W}$. If G is a feebly projectable ℓ-group, then (G, e_G) is local.*

Proof. Let $g \in G^+$, and let $K \subseteq YG$ be a clopen subset. Now, by Remark 4.6, it follows that $a = \chi_K \in G$ and $b = \chi_{YG \smallsetminus K} \in G$. Since $a \wedge b = 0$, we can find a polar $S \le G^+$ such that $G = S \oplus S^\perp$ with $a \in S$ and $b \in S^\perp$. Let $g = t_1 + t_2$ with $t_1 \in S$ and $t_2 \in S^\perp$. We claim that $g\chi_K = t_1$, from which it follows that $g\chi_K \in G$, thus proving that (G, e_G) is local.

Let $p \in YG$. If $p \in Z(g)$, then $(g\chi_K)(p) = g(p) = t_1(p)$. So assume that $p \in \operatorname{coz}(g)$. Since $\operatorname{coz}(g)$ is a disjoint union of $\operatorname{coz}(t_1)$ and $\operatorname{coz}(t_2)$, we first consider the case where $p \in \operatorname{coz}(t_1)$. Note that $0 \neq t_1(p) = g(p)$. If $p \notin K$, then $t_1 \wedge b \neq 0$, a contradiction. In the second case, $p \in \operatorname{coz}(t_2)$. If $p \in K$, then $t_2 \wedge a \neq 0$. This contradiction implies that $p \notin K$, and so $t_1(p) = 0 = (g\chi_K)(p)$. □

Corollary 4.8. *Let $(G, e_G) \in \mathbf{W}$ be weakly projectable. Then G is projectable if and only if it is feebly projectable.*

Remark 4.9. Suppose G is an ℓ-group and $s \in G^+$. We call s *singular* if whenever $0 \le a \le s$, then $a \wedge (s - a) = 0$. A *singular* $\mathbf{W}$*-object* is a $\mathbf{W}$-object (G, e_G) so that e_G is a singular element of G. The most common example of a singular **W**-object is $(C(X,\mathbb{Z}),\mathbf{1})$. For more information on the full subcategory of **W** consisting of singular **W**-objects, the reader is referred to [10]. Surprisingly, our next proposition is not stated anywhere, even in [9].

Proposition 4.10. *Suppose (G,e_G) is a singular* **W***-object. Then (G,e_G) is weakly projectable.*

Proof. If (G,e_G) is a singular **W**-object, then $loc(G)$ is a projectable ℓ-group by Proposition 5.5 of [10]. But then Proposition 4.1 of [9] states that $loc(G)$ is projectable if and only if G is weakly projectable. Therefore, a singular **W**-object is weakly projectable. □

Corollary 4.11. *Let $(G,e_G)\in$* **W** *be a singular* **W***-object. Then G is projectable if and only if it is feebly projectable.*

We end this section with our main theorem on feebly projectable **W**-objects.

Theorem 4.12. *For any* **W***-object (G,e_G), G is feebly projectable if and only if every pair of disjoint G-cozero sets can be separated by a clopen set. Furthermore, if (G,e_G) satisfies $\operatorname{coz} G=\operatorname{coz} YG$, then G is feebly projectable if and only if YG is a zero-dimensional F-space.*

Proof. The second statement follows from the first when $\operatorname{coz} G=\operatorname{coz} YG$ because any description of G-cozero sets is a description of the cozero sets of YG. For a compact Hausdorff space, being a zero-dimensional F-space is equivalent to saying that any two disjoint cozero sets can be separated by a clopen set.

So suppose G is feebly projectable, and let C_1, C_2 be a pair of disjoint G-cozero sets. This means that there are $g_1,g_2\in G^+$ such that $\operatorname{coz}(g_i)=C_i$ for $i=1,2$. Since C_1 and C_2 are disjoint, it follows that $g_1\wedge g_2=0$. Since G is feebly projectable, there is a $h\in G^+$ such that $G=h^{\perp}\oplus h^{\perp\perp}$ and $g_1\in h^{\perp}$, while $g_2\in h^{\perp\perp}$. Write $e=h_1+h_2$ for appropriate $h_1\in h^{\perp}$ and $h_2\in h^{\perp\perp}$, and observe that both h_1,h_2 are characteristic functions on YG, and so correspond to disjoint complementary clopen subsets of YG. Moreover, it is straightforward to check that $C_1=\operatorname{coz}(g_1)\subseteq\operatorname{coz}(h_1)$ and $C_2=\operatorname{coz}(g_2)\subseteq\operatorname{coz}(h_2)$ Therefore, disjoint G-cozero sets can be separated by a clopen subset of YG.

Conversely, suppose $g_1\wedge g_2=0$ for $g_1,g_2\in G^+$. Then $\operatorname{coz}(g_1)\cap\operatorname{coz}(g_2)=\emptyset$, so by hypothesis there is a clopen subset of YG, say C, for which $\operatorname{coz}(g_1)\subseteq C$ and $C\cap\operatorname{coz}(g_2)=\emptyset$. Since G is saturated, it follows that the characteristic function $\chi_C\in G^+$. We leave it to the interested reader to show that $G=\chi_C^{\perp\perp}\oplus\chi_C^{\perp}$ with $g_1\in\chi_C^{\perp\perp}$ and $g_2\in\chi_C^{\perp}$, when G is feebly projectable. □

Example 4.13. Notice that for any zero-dimensional space X, $C(X,\mathbb{Z})$ is always projectable and hence, feebly projectable. On the other hand, there are examples of zero-dimensional spaces X which are not F-spaces (e.g., $\mathbb{Q}$). Therefore, in the last statement of Theorem 4.12, it is necessary that we suppose that $\operatorname{coz} G=\operatorname{coz} YG$.

Definition 4.14. Let $(G, e_G) \in \mathbf{W}$. We let G^* denote the convex ℓ-subgroup of G generated by e_G. Observe that via the Yosida Representation, G^* is precisely the collection of bounded functions on YG belonging to G.

Corollary 4.15. *Let $(G, e_G) \in \mathbf{W}$. Then G is feebly projectable if and only if G^* is feebly projectable.*

Proof. This follows from Theorem 4.12 once we observe that $\operatorname{coz} G = \operatorname{coz} G^*$. □

Commentary 4.16. We say that the $\mathbf{W}$-object (G, e_G) is *convex* if G is a convex subset of $D(YG)$, that is, if $g_1, g_2 \in G$, $f \in D(YG)$, and $g_1 \leq f \leq g_2$, then $f \in G$. Observe that $(C(X), \mathbf{1})$ is the motivating example of a convex $\mathbf{W}$-object. Convex $\mathbf{W}$-objects have the property that G-cozero sets of YG are precisely the cozero sets of YG, i.e., $\operatorname{coz} G = \operatorname{coz} YG$. It follows then that if (G, e_G) is a convex ℓ-group, then G is feebly projectable if and only if YG is a zero-dimensional F-space. In particular, $C(X)$ is a feebly projectable ℓ-group if and only if X is a strongly zero-dimensional F-space. We now consider the case for $C(X, \mathbb{Q})$.

Definition 4.17. For a space X, we call a subset $K \subseteq X$ a *σ-clopen subset of X* if K is the union of countably many clopen subsets of X. Clearly, every clopen subset is a σ-clopen subset, and every σ-clopen subset is a cozero set of X. It is well-known that X is strongly zero-dimensional precisely when every cozero set is a σ-clopen subset of X, e.g., Theorem 4.7(j) of [25].

Proposition 4.18. *Consider the $\mathbf{W}$-object $(C(X, \mathbb{Q}), \mathbf{1})$. Then $C(X, \mathbb{Q})$ is feebly projectable if and only if every pair of disjoint σ-clopen subsets of X can be separated by a clopen subset of X.*

Proof. The only thing we need to point out is that when $G = C(X, \mathbb{Q})$, then a G-cozero set is a σ-clopen subset of X, but that is straightforward (or one can check [18] for a more detailed discussion). □

Question 4.19. Is the collection of feebly projectable $\mathbf{W}$-objects a hull class? For more information on hull classes, the reader is urged to read [21].

5. Commutative semiprime f-rings

In this section, we will investigate when a commutative semiprime f-ring is feebly projectable. By semiprime, we mean that the ring has no nonzero nilpotent elements. By an f-ring, we mean a lattice-ordered ring A for which whenever $a, b, c \in A$ and $a \wedge b = 0$ and $c \geq 0$, then $ca \wedge b = 0$. We denote the set of units of A by $\mathcal{U}(A)$.

Lemma 5.1. *Let A be a semiprime f-ring. Then $a \wedge b = 0$ if and only if $ab = 0$. In particular, the set of minimal prime (ring) ideals equals the set of minimal prime subgroups.*

One nice consequence of this lemma is that in a semiprime f-ring, a positive element is a weak-order unit precisely when it is a regular element, that is, a non zero-divisor.

We let $\mathrm{Max}(A)$ denote the space of maximal ideals topologized under the Zariski topology. We recall that $\mathrm{Max}(A)$ is a compact T_1-space. (This is true for any commutative ring; see § 2.5 of [19].) In [6], the authors define a *pm-ring* as a ring for which every prime ideal is contained in a unique maximal ideal. There it is shown that if A is a *pm*-ring, then $\mathrm{Max}(A)$ is a Hausdorff space. Since we shall have several occasions to compare the structure spaces $\mathrm{Max}(A)$ and $\mathrm{Min}(A)$, we should stress that the bases for these two topological spaces with respect to the Zariski topology are given by the collections of sets of the form

$$U_M(a) = \{M \in \mathrm{Max}(A) \mid a \notin M\} \text{ and } U_m(a) = \{P \in \mathrm{Min}(A) \mid a \notin P\},$$

respectively. The complements of these are denoted by $V_M(a)$ and $V_m(a)$, respectively. We define the inverse topology on $\mathrm{Max}(A)$ in a similar fashion as has already been defined for $\mathrm{Min}(A)$ in Definition 2.12. The topology on $\mathrm{Max}(A)$ formed by taking as basic open sets the collection $\{V_M(a) \mid a \in A\}$ is called the *inverse topology* and is denoted by $\mathrm{Max}(A)^{-1}$. (See [23] and [24] for more information on the inverse topology).

As the title of this section suggests, we assume that all rings are semiprime, commutative, and possess an identity.

Lemma 5.2. *Suppose that for $f \in A$, there exists $u \in A$ such that $f = u|f|$. Let P be any minimal prime ideal not containing f. If $f^+ \notin P$, then $u + P = 1 + P$. If $f^- \notin P$, then $u + P = -1 + P$.*

Proof. Let P be any minimal prime ideal not containing f. Now, since $f^+ \wedge f^- = 0$ and P is a prime subgroup, it follows that exactly one of the two elements belongs to P. Therefore, $f + P = f^+ + P$ or $f + P = f^- + P$. Suppose that $f^+ \notin P$, and so $f + P = f^+ + P$. In this case, it readily follows that $f + P = |f| + P$, and so $u + P = 1 + P$. Similarly, if $f^- \notin P$, then $f + P = -|f| + P$, and so $u + P = -1 + P$. □

Definition 5.3. A ring A is called *clean* if every element can be written as a sum of a unit and an idempotent. In the commutative case, these rings are also known as *exchange rings*.

Johnstone's Theorem (see [15]) states that a commutative ring with identity is clean if and only if A is a *pm*-ring and $\mathrm{Max}(A)$ is a zero-dimensional space with respect to the Zariski topology. Furthermore, in [23], it is shown that A is a clean ring if and only if for each $a \in A$, there is an idempotent $e \in A$ such that $V_M(a) \subseteq U_M(e)$ while $V_M(1-a) \subseteq V_M(e)$.

Remark 5.4. In the literature, f-rings with the property that their ℓ-group structure have stranded primes, have been called *normal f-rings*. We will comply with the usage of this terminology. Notice that the property of being a Bézout ring (that is, every finitely generated ideal is principal) is stronger than being normal, and for some classes of rings (e.g., uniformly complete f-rings), the two notions coincide (see [14]). Our next result generalizes parts of Proposition 3.1 of [20]. Condition (b) below is new. Recall that an f-ring A is called a *U-ring* if it satisfies condition (d) below.

Theorem 5.5. *Let A be a semiprime f-ring and consider the following statements.*

(a) *A is clean, and for each $f \in A$, there is an $r \in A$ such that $f = r|f|$.*
(b) *A is a clean normal f-ring, and the natural bijection between $\mathrm{Max}(A)$ and $\mathrm{Min}(A)^{-1}$ is a homeomorphism.*
(c) *A is feebly projectable.*
(d) *For each $f \in A$, there exists a unit $u \in A$ such that $f = u|f|$.*

Statements (a) *and* (b) *are equivalent, while statements* (c) *and* (d) *are equivalent. Statement* (a) *always implies* (c)*, and the converse is true when A is a pm-ring.*

Proof. We begin by demonstrating that statements (c) and (d) are equivalent.

Suppose (c) is valid, and let $f \in A$. Then $f^+ \wedge f^- = 0$, and so there is an $x \in A^+$ such that $A = x^\perp \oplus x^{\perp\perp}$ while $f^+ \in x^\perp$ and $f^- \in x^{\perp\perp}$. It follows that $x^\perp = eA$ for some idempotent $e^2 = e \in A$. Set $e' = 1 - e$ and $u = e - e'$, and observe that $u \in U(A)$. One can check that for each minimal prime ideal P of A, that either $u + P = 1 + P$ or $u + P = -1 + P$, and therefore, $f = u|f|$, which shows that (c) implies (d).

Now, suppose (d), and let $a, b \in A^+$ be disjoint elements. Let $f = a - b$, and choose a unit $u \in A$ for which $f = u|f|$. Observe that $f^+ = a$ and $f^- = b$. Next, let v be the inverse of u, and set $x = u^+v$ and $y = -u^-v$. Observe that $1 = x + y$ and $xy = 0$. It follows that x is an idempotent of A, and so $A = x^\perp \oplus x^{\perp\perp}$. Furthermore, $a \in x^{\perp\perp}$ and $b \in x^\perp$. Therefore, A is feebly projectable, whence (c) and (d) are equivalent.

Next, we demonstrate that if A is a clean ring such that for each $f \in A$, there is an $r \in A$ such that $f = r|f|$, then A is feebly projectable. To that end, let $f \in A$. Choose $r \in A$ such that $f = r|f|$. Since A is clean, and therefore, $\mathrm{Max}(A)$ is zero-dimensional, there is an idempotent $e \in A$ such that $V_M(r-1) \subseteq U_M(e)$ and $V_M(r+1) \subseteq V_M(e)$. Letting $e' = 1 - e$, it is straightforward to check that $u = e - e' \in \mathcal{U}(A)$ and $f = u|f|$ (for the latter, use that A is semiprime). It follows that (a) implies (d) and hence (c).

Suppose (a) is true. By what we have just proved, A is feebly projectable, and therefore, A has stranded primes. This means that A is a normal f-ring, and so, every maximal ideal contains a unique minimal prime ideal. Since A is a clean ring and hence, a *pm*-ring, we can conclude that there is an obvious bijection between the sets of maximal ideals and the set of minimal prime ideals. Explicitly, consider the map $\mathcal{O}\colon \mathrm{Max}(A) \to \mathrm{Min}(A)^{-1}$ defined by

$$\mathcal{O}(M) = \{a \in A \mid \text{there is some } b \notin M \text{ such that } ab = 0\},$$

which in an arbitrary semiprime ring is the intersection of all of the minimal primes of A contained in M. In our case, it means that $\mathcal{O}(M)$ is the unique minimal prime ideal contained in M. We will show that $\mathcal{O}$ is a homeomorphism.

Let $a \in A$, and consider $\mathcal{O}^{-1}(V_m(a))$. We will demonstrate that this set is an open subset of $\mathrm{Max}(A)$ with respect to the Zariski topology. Let $M \in \mathcal{O}^{-1}(V_m(a))$. This means that $a \in \mathcal{O}(M)$, and so, there is some $b \notin M$ such that $ab = 0$. It follows that $M \in U_m(b)$. We claim that $U_m(b) \subseteq \mathcal{O}^{-1}(V_m(a))$. But this is clear, since if $b \notin N$, then $ab = 0$ implies that $a \in \mathcal{O}(N)$. Thus, $\mathcal{O}^{-1}(V_m(a))$ is an open subset of the Zariski topology, whence $\mathcal{O}\colon \mathrm{Max}(A) \to \mathrm{Min}(A)^{-1}$ is a continuous bijection. Finally, since both $\mathrm{Max}(A)$ and $\mathrm{Min}(A)^{-1}$ are compact Hausdorff spaces, we conclude by basic topological considerations, that $\mathcal{O}$ is a homeomorphism.

Suppose (b) is true. Since $\mathrm{Max}(A)$ and $\mathrm{Min}(A)^{-1}$ are homeomorphic, it follows by Johnstone's Theorem, that $\mathrm{Min}(A)^{-1}$ is zero-dimensional. Whence, A is a weakly complemented ℓ-group (again, we are using that A is semiprime). Since A is a normal f-ring, we conclude that A is feebly projectable by Theorem 2.14. Therefore, since we already showed that (c) and (d) are equivalent, we conclude that for each $f \in A$, there is a unit $u \in U(A)$ such that $f = u|f|$. This shows that (a) and (b) are equivalent.

We only need to show that when A is a *pm*-ring, then (c) implies (d). To that end, suppose A is a feebly projectable *pm*-ring. It suffices to show by Johnstone's Theorem, that $\mathrm{Max}(A)$ is a zero-dimensional space. Since $\mathrm{Max}(A)$ is a compact Hausdorff space, we need only show that $\mathrm{Max}(A)$ is totally-disconnected. So, let M and N be distinct maximal ideals of A. There exist disjoint basic open sets $U_M(a), U_M(b)$ such that $M \in U_M(a)$ and $N \in U_M(b)$. Without loss of generality (by 4.), we assume that $0 \leq a, b$. Furthermore, since $U_M(ab) = U_M(a) \cap U_M(b) = \emptyset$, it is straightforward to check that $M \in U_M(a') \subseteq U_M(a)$ and $N \in U_M(b') \subseteq U_M(b)$, where

$$a' = a - a \wedge b \qquad \text{and} \qquad b' = b - a \wedge b,$$

and thus, $a' \wedge b' = 0$. Since A is feebly projectable, there is a polar P of A such that $A = P \oplus P^{\perp}$, $a' \in P$, and $b' \in P^{\perp}$. But polars are ideals, so that $P = eA$ for some idempotent $e \in A$. Let $f = 1 - e$. Thus, $U_M(e)$ is a clopen subset of $\mathrm{Max}(A)$ whose complement is $U_M(f)$. It is straight forward to check that that $U_M(a') \subseteq U_M(e)$ and $U_M(b') \subseteq U_M(f)$, whence $\mathrm{Max}(A)$ is a totally disconnected space. Therefore, in the class of *pm*-rings, all four conditions are equivalent. □

Example 5.6. In general, a feebly projectable f-ring need not be clean, e.g., $C(X, \mathbb{Z})$. This shows that in the previous theorem, it is necessary that we assume A is a *pm*-ring to get that (c) implies (a).

We now turn our attention to studying the relationship between a commutative semiprime f-ring and its classical ring of quotients. In particular, we investigate the passage of the property of being feebly projectable. We present a brief account of the classical ring of quotients of an f-ring.

Definition 5.7. Suppose A is a commutative semiprime f-ring, and let $q(A)$ denote its classical ring of quotients, i.e.,

$$q(A) = \Big\{ \frac{a}{b} \;\Big|\; a, b, \in A, \text{ and } b \text{ is not a zero-divisor of } A \Big\}.$$

The lattice order on A can be extended to $q(A)$ in the following manner. First of all, since in an f-ring squares are positive, every element of $q(A)$ can be written with a positive denominator. So let $\frac{a}{b} \in q(A)$ with $b \geq 0$. We define $0 \leq \frac{a}{b}$ precisely when $a \geq 0$. This is a lattice order on $q(A)$ which extends the order on A and makes $q(A)$ into a semiprime f-ring.

Notice that if $1 \leq \frac{a}{b}$ (with $b \geq 0$,), then $b \leq a$, and since b is a regular element of A, we gather that a is a regular element of A. Thus, $\frac{a}{b}$ is an invertible element of $q(A)$, whence $q(A)$ has bounded inversion.

If A is a ring for which $A = q(A)$, we say A is *classical.* This is equivalent to A having the property that every regular element is a unit.

We now state our main result relating the structure of A and $q(A)$.

Proposition 5.8. *Suppose A is a semiprime f-ring. The following statements are equivalent.*

(a) *A is a weakly complemented f-ring.*
(b) *$q(A)$ is a weakly complemented f-ring.*
(c) *$q(A)$ is a feebly projectable f-ring.*

Proof. Suppose (a), that is, A is weakly complemented, and let $d, e \in q(A)^+$ be disjoint. Without loss of generality, we assume that $d = \frac{a}{b}$ and $e = \frac{c}{b}$, where $b \geq 0$. It follows then that $a \wedge c = 0$, and so there are $x, y \in A^+$ such that $a \leq x, b \leq y$, $x \wedge y = 0$, and $x \vee y$ is a weak order unit of A. This last condition means that $x \vee y$ is a regular element of A. It follows that $\frac{x}{b} \wedge \frac{y}{b} = 0$, $d \leq \frac{x}{b}$, and $e \leq \frac{y}{b}$. Since $\frac{x}{b} \vee \frac{y}{b} = \frac{x \vee y}{b}$ is an invertible element of $q(A)$, it is a weak-order unit. Therefore, $q(A)$ is weakly complemented.

Conversely, suppose that $q(A)$ is a weakly complemented ℓ-group, and let $a, b \in A^+$ be disjoint. Then they are disjoint in $q(A)$, which means there are disjoint $\frac{x}{s}, \frac{y}{s} \in q(A)^+$ for which $a \leq \frac{x}{s}, b \leq \frac{y}{s}$, and $\frac{x}{s} \vee \frac{y}{s}$ is a weak-order unit of $q(A)$. But then $x \vee y$ is a weak order unit of A and thus, a regular element of A. Note that $x \wedge y = 0$.

Next, $0 \leq a \leq a \vee x$ and $0 \leq b \leq b \vee y$. Since the lattice structure of an ℓ-group is distributive, it follows that $(a \vee x) \wedge (b \vee y) = 0$. Finally, $x \vee y \leq (a \vee x) \vee (b \vee y)$, so that $(a \vee x) \vee (b \vee y)$ is also a regular element of A, whence A is weakly complemented. It follows that (a) and (b) are equivalent.

Since (c) implies (b), all that is left to be done is to prove that (b) implies (c). To that end, let $a, b \in q(A)^+$ be disjoint. Since we are assuming that $q(A)$ is weakly complemented, there are disjoint $x, y \in q(A)^+$ such that $a \leq x, b \leq y$, and $x \vee y$ is a regular element of $q(A)$. It follows that $x + y = x \vee y$ is a unit, and therefore, $q(A) = xq(A) \oplus yq(A)$. Thus, there is some idempotent $e = e^2 \in q(A)$ such that $eq(A) = xq(A)$. So, $xq(A)$ is a polar subgroup and therefore, a convex ℓ-subgroup. Consequently, $a \in xq(A)$ and, similarly, $b \in yq(A)$. Thus, $q(A)$ is feebly projectable. □

Corollary 5.9. *If A is weakly complemented semiprime f-ring, then $q(A)$ is a clean normal f-ring.*

Question 5.10. Does there exists a classical semiprime f-ring which is clean and normal but not Bézout? If such a ring exists, it can not be uniformly complete.

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