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Feebly Projectable Algebraic Frames and Multiplicative Filters of Ideals

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Abstract In the article (Martinez and Zenk, *Algebra Universalis*, 50, 231–257, 2003.), the authors studied several conditions on an algebraic frame L. In particular, four properties called Reg(1), Reg(2), Reg(3), and Reg(4) were considered. There it was shown that Reg(3) is equivalent to the more familiar condition known as projectability. In this article we show that there is a nice property, which we call feebly projectable, that is between Reg(3) and Reg(4). In the main section of the article we apply our notions to the frame of multiplicative filters of ideals in a commutative ring with unit and give characterizations of several well-known classes of commutative rings.

Key words algebraic frame · feebly projectable · flatly projectable · multiplicative filter of ideals · clean ring

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1 Introduction

There is a long well-known relationship between ring theory and general topology. For example, given a commutative ring with identity there are three structure spaces associated to it: the prime spectrum, the maximal ideal space, and the minimal prime

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ideal space. It is often the case that knowing something about a ring's structure spaces allows one to know something about the ring. Over the last 25 years there has been a steady state of activity relating ring theory with the study of point-free topology, (or frame theory). Recently, in [1] and [2], Banaschewski has investigated ideal theoretic properties of a commutative ring A by imposing frame theoretic conditions on the ring's frame of radical ideals.

The theme of this article is in the same vein as [1] and [2]. We investigate ring-theoretic properties of commutative rings with identity by imposing restrictions on a specific frame associated to the given ring. In particular, we investigate $\mathfrak{M}_A$, the frame of multiplicative filters of ideals of the ring A. The important frame-theoretic tools are the ones discussed by Martinez and Zenk in their paper on algebraic frames (see [5]). We determine when $\mathfrak{M}_A$ is a zero-dimensional frame or a projectable frame (as well as a few other types of frames). Furthermore, we create a new concept called feebly projectable and classify when $\mathfrak{M}_A$ is such a frame.

In this article we shall assume the foundations of Zermelo–Fraenkel set theory as well as the Axiom of Choice. For the casual reader we do point out that in much of Banaschewski's work the Axiom of Choice is not assumed. Without the Axiom of Choice, arguments which require the existence of primes often do not work.

We begin by going over the facts and notions that will be used throughout the paper.

2 Algebraic Frames

A frame is a complete distributive lattice L which satisfies the strengthened distributive law

$$a \wedge \bigvee_{s\in S} s = \bigvee \{a \wedge s | s \in S\}$$

for all $a \in L$ and all $S \subseteq L$. This equality is known as the *frame law*. We point out that a frame is also known as a complete Brouwerian lattice. We denote the top and bottom elements of a frame by 1 and 0, respectively. A frame is necessarily a pseudocomplemented lattice (in the sense of Birkhoff). In particular, for $a \in L$ the pseudo-complement of a is given by

$$a^\perp = \bigvee \{x \in L | x \wedge a = 0\}.$$

Observe that by the frame law $a \wedge a^\perp = 0$. An element of the form $a^\perp$ is called a *polar of* L. When $a \vee a^\perp = 1$, we say that a is a *complemented element of* L. We say $a, b \in L$ are *disjoint* if $a \wedge b = 0$. If a, b are disjoint elements of L such that $a \vee b = 1$, then we say a and b form a *complementary pair*. Obviously, if a and b are a complementary pair then $b = a^\perp$. Whenever $a \leq b$ we will say that a is *below* b (or that b is *above* a).

Definition 2.1 We now recall some basic notions regarding frames. Throughout, L denotes a frame.

(1) Let $c \in L$. We call c *compact* if whenever $c \leq \bigvee_{i\in I} a_i$, then there is a finite subset of I, say $\{i_1, \cdots, i_n\}$, such that $c \leq a_{i_1} \vee \cdots \vee a_{i_n}$. If the top element of

L is compact, then we call L *compact*. Whenever every element of L is the supremum of compact elements, L is called an *algebraic frame*. We denote the set of compact elements of L by $\mathfrak{k}(L)$. This collection is closed under finite joins. When $\mathfrak{k}(L)$ is closed under nonempty finite meets, then we say L has the *finite intersection property*, or that L satisfies the FIP.

(2) L is said to be *zero-dimensional* when every element is a supremum of complemented elements. It is straightforward to check that for an algebraic frame being zero-dimensional is equivalent to having the property that every compact element is complemented.

(3) L is said to be *normal* if whenever $a, b \in L$ and $a \vee b = 1$, then there are disjoint $c, d \in L$ for which $a \vee d = b \vee c = 1$. It is straightforward to check that if such $c, d \in L$ exist, then such a pair exists with $c \leq a$ and $d \leq b$.

(4) We say L is *weakly zero-dimensional* if whenever $a, b \in L$ and $a \vee b = 1$, then there exists a complemented element $c \in L$ such that $c \leq a$ and $c^{\perp} \leq b$. It is obvious that a weakly zero-dimensional frame is normal. The definition of weakly zero-dimensional is due to Banaschewski [2]. A zero-dimensional frame need not be weakly zero-dimensional, but it is true for compact algebraic frames. The latter result can be found prior to Lemma 4 of [2]. In Example 2.2 we supply an example of a zero-dimensional frame which is not weakly zero-dimensional.

(5) Call $p < 1$ *prime* if whenever $a \wedge b \leq p$, then either $a \leq p$ or $b \leq p$. This is equivalent to saying if $a \wedge b = p$, then $a = p$ or $b = p$. The collection of prime elements of L is denoted by $Spec(L)$ and is called the *spectrum* of the frame. A typical Zorn's Lemma argument guarantees that when primes exist then so do minimal primes. We denote the collection of minimal primes of L by $Min(L)$. A similar Zorn's Lemma argument assures us that if $0 < c$ is compact, then there exists an element, say z, that is maximal with respect to $c \nleq z$. Such an element is often called a *value of c*. Due to their maximality, values are always prime.

(6) For a given frame L and $b \in L$ there is a natural quotient:

$$\downarrow b = \{x \in L : x \leq b\}$$

This set is called the *open quotient of b*.

Example 2.2 The terminology for frames is motivated largely by topology. For a given topological space X, the collection of open sets is a frame when ordered by inclusion. The frame of open sets of X is denoted $\mathcal{O}(X)$. An arbitrary frame isomorphic to an $\mathcal{O}(X)$ for some topological space X is called *spatial*. It is a consequence of the Axiom of Choice that all algebraic frames are spatial (see [4]). X is a normal (compact) space if and only if $\mathcal{O}(X)$ is a normal (compact) frame. A topological space is *zero-dimensional* if it has a base of clopen sets. So X is a zero-dimensional space precisely when $\mathcal{O}(X)$ is a zero-dimensional frame. Furthermore, $\mathcal{O}(X)$ is weakly zero-dimensional if and only if disjoint closed subsets of X can be separated by disjoint clopen sets. It follows then that if $\mathcal{O}(X)$ is weakly zero-dimensional, then X is normal. Consequently, any space which is zero-dimensional yet not normal supplies us with an example of a frame that is zero-dimensional yet not weakly zero-dimensional. We suggest the reader consult [9] for definitions from topology as well as an example of a zero-dimensional space which is not normal.

Example 2.3 Let A be a commutative ring with identity, and let $Rad(A)$ denote the collection of radical (i.e. semiprime) ideals of A. When partially ordered by inclusion, $Rad(A)$ is a frame since the intersection of radical ideals is again a radical ideal. Some properties of $Rad(A)$ are listed here. For a more detailed reference, the reader should consult [1] or [2].

1. $Rad(A)$ is always an algebraic frame as its compact elements are precisely the radical ideals generated by a finite set.
2. For a semiprime ring A, $Rad(A)$ is a zero-dimensional frame if and only if A is a von Neumann regular ring ([2], Corollary to Lemma 1).
3. $Rad(A)$ is a normal frame precisely when A is a Gelfand ring ([2], Proposition 1). Recall that a *Gelfand ring* is a ring A for which whenever $a, b \in A$ satisfy $a + b = 1$, there exist $r, s \in A$ such that $(1 + ar)(1 + bs) = 0$. Assuming the Axiom of Choice, being a Gelfand ring is equivalent to being a *pm*-ring, that is, a ring in which every prime ideal is contained in a unique maximal ideal. The notion of a *pm*-ring is due to De Marco and Orsatti [3].
4. $Rad(A)$ is a weakly zero-dimensional frame if and only if A is an exchange ring ([2] Proposition 2). For a commutative ring with identity, being an exchange ring is equivalent to being a clean ring, where a ring is *clean* if every element can be written as the sum of a unit and an idempotent.

3 Regularity in Frames

Throughout this section L *is assumed to be an algebraic frame*.

For $a, b \in L$, we say a is *well-below* b if $a^{\perp} \vee b = 1$, in which case we write $a \preceq b$. The element $a \in L$ is called *regular* if

$$a = \bigvee \{x \in L : x \preceq a\}.$$

When every element of L is regular, then L is called a *regular* frame. In [5] and [6] the authors investigate regularity in algebraic frames. As with normality, regularity in frames comes from topology in the sense that X is a regular topological space precisely when $\mathcal{O}(X)$ is a regular frame. It is shown in [5] that a regular element $a \in L$ has the property that

$$a = \bigvee \{c^{\perp\perp} : c \leq a, c \in \mathfrak{k}(L)\}.$$

In general, an element satisfying this property is called a *d-element of L*.

One of the main theorems of [5] determines the relationship between several properties defined via regularity. Recall the following conditions:

Reg(1): L is regular.
Reg(2): Each d-element of L is regular.
Reg(3): Each polar of L is regular.
Reg(4): For each $c \in \mathfrak{k}(L)$, $c^{\perp}$ is regular.

Reg(1) implies Reg(2). Reg(2) and Reg(3) are equivalent and both imply Reg(4). Furthermore, in [5] the validity of the statements in the next lemma are demonstrated.

Lemma 3.1

(1) *Reg(1) is equivalent to L being zero-dimensional.*
(2) *Reg(2) (and hence Reg(3)) is equivalent to the statement for each $c \in \mathfrak{k}(L)$, $c^{\perp\perp}$ is complemented.*
(3) *Reg(4) is equivalent to the statement for any disjoint $a, b \in \mathfrak{k}(L)$, $1 = a^{\perp} \vee b^{\perp}$. Moreover, if L has the FIP, then these are equivalent to having $p \vee q = 1$ for all distinct $p, q \in Min(L)$.*

The algebraic frame L is called *projectable* if for every $c \in \mathfrak{k}(L)$, $c^{\perp\perp}$ is a complemented element, which by (2) of Lemma 3.1 is equivalent to both Reg(2) and Reg(3). It follows by (3) of Lemma 3.1 that a projectable frame L satisfies Reg(4).

We now will show that there are a couple of classes of algebraic frames between Reg(3) and Reg(4).

Definition 3.2 We call L *feebly projectable* if whenever $a, b \in \mathfrak{k}(L)$ and $a \wedge b = 0$, then there exists a $c \in \mathfrak{k}(L)$ such that $c^{\perp\perp}$ is complemented and $a \leq c^{\perp\perp}$, $b \leq c^{\perp}$. More generally, L is *flatly projectable* if whenever $a, b \in \mathfrak{k}(L)$ and $a \wedge b = 0$, then there exists a complemented $c \in L$ such that $a \leq c$ and $b \leq c^{\perp}$. It is evident that a projectable frame is feebly projectable, which in turn is flatly projectable. In some cases the latter two conditions coincide (see below).

The proof of the next proposition is straightforward and is left to the reader.

Proposition 3.3 *Suppose L is flatly projectable. Then L satisfies Reg(4).*

Remark 3.4 It follows that the class of feebly projectable frames lies between Reg(3) and Reg(4). We will give an example in the last section that shows that the class lies properly between Reg(3) and Reg(4).

Proposition 3.5 *A frame L is flatly projectable if and only if $\downarrow b$ is flatly projectable for all $b \in L$.*

Proof Suppose L is flatly projectable, and let $b \in L$. Suppose $a, d \in \mathfrak{k}(\downarrow b)$ with $a \wedge d = 0$. Then $a, d \in \mathfrak{k}(L)$, so there exists a complemented element $c \in L$ such that $a \leq c$ and $d \leq c^{\perp}$. Let $x = c \wedge b \in \downarrow b$. Note that $a \leq c \wedge b = x$ and $d \leq c^{\perp} \wedge b \leq x^{\perp} \wedge b$ where $x^{\perp} \wedge b$ is the complement of x in $\downarrow b$. Hence $\downarrow b$ is flatly projectable. The reverse direction holds since $L = \downarrow 1$. □

Proposition 3.6 *If a compact frame L satisfies the FIP, then L is feebly projectable if and only if $\downarrow b$ is feebly projectable for all $b \in \mathfrak{k}(L)$.*

Proof Let $b \in \mathfrak{k}(L)$, and suppose $a, d \in \mathfrak{k}(\downarrow b)$ with $a \wedge d = 0$. Then $a, d \in \mathfrak{k}(L)$, so there exists $c \in \mathfrak{k}(L)$ such that c is complemented, $a \leq c^{\perp\perp}$, and $d \leq c^{\perp}$. Let $x = c \wedge b$, then $x \in \mathfrak{k}(\downarrow b)$ because L satisfies the FIP. Observe that $x^{\perp\perp} \vee x^{\perp} = b$, so $x^{\perp\perp}$ is a complemented element of $\downarrow b$. Furthermore, note that $a = a \wedge b \leq c^{\perp\perp} \wedge b = x^{\perp\perp}$ and $d = d \wedge b \leq c^{\perp} \wedge b \leq x^{\perp} \wedge b = x^{\perp}$. Hence $\downarrow b$ is feebly projectable. The reverse direction is obvious. □

Proposition 3.7 *Let $\{F_i\}_{i\in I}$ be a collection of frames, and let $F = \prod F_i$.*

(1) *F is flatly projectable if and only if each F_i is flatly projectable.*
(2) *F is feebly projectable if and only if each F_i is feebly projectable.*

Proof We will prove (1) and leave (2) to the reader. Assume F is flatly projectable, and fix $j \in I$. Suppose $a_j, b_j \in \mathfrak{k}(F_j)$ with $a_j \wedge b_j = 0$. Let $a = (a_i)$ where $a_i = a_j$ if $i = j$ and 0 otherwise, and define $b = (b_i)$ similarly. Then we see that $a, b \in \mathfrak{k}(F)$ with $a \wedge b = 0$. Since F is flatly projectable, there exists a complemented element $c = (c_i) \in F$ satisfying $a \leq c$ and $b \leq c^{\perp}$. The element c_j is complemented in F_j such that $a_j \leq c_j$ and $b_j \leq c_j^{\perp}$.

Conversely, assume each F_i is flatly projectable. Let $a, b \in \mathfrak{k}(F)$ with $a \wedge b = 0$. For each $i \in I$, we have that $a_i, b_i \in \mathfrak{k}(F_i)$ such that $a_i \wedge b_i = 0$. So there exists a complemented element $c_i \in F_i$ satisfying $a_i \leq c_i$ and $b_i \leq c_i^{\perp}$ for each i. Let $c = (c_i)$, then it is easy to see that c is a complemented element of F with $a \leq c$ and $b \leq c^{\perp}$. □

Definition 3.8 If $u \in \mathfrak{k}(L)$ has the property that $u^{\perp} = 0$, then we call u a unit and say that the frame L *possesses a unit*. An element $c \in L$ for which $c^{\perp} = 0$ is usually called dense. Our use of the word unit is to signify that the element is also compact. A particular example of a frame possessing a unit is a compact frame.

The following proposition is well-known. We partially generalize it in the subsequent lemma.

Proposition 3.9 *Suppose L is a compact algebraic frame. Then every complemented element is compact.*

Lemma 3.10 *Suppose L is an algebraic frame possessing a unit, say $u \in L$. If L satisfies Reg(4), then every complemented element is of the form $a^{\perp\perp}$ for some $a \in \mathfrak{k}(L)$.*

Proof Let $e \in L$ be a complemented element and let $f = e^{\perp}$. Now, $u = (e \wedge u) \vee (f \wedge u)$ and since L is algebraic we can write each of the components of u as a supremum of compact elements. Since u is a unit it is compact and thus we can write $u = s \vee t$ where $s \leq e$, $t \leq f$, and $s, t \in \mathfrak{k}(L)$. We claim that $s^{\perp\perp} = e$. Clearly, $s^{\perp\perp} \leq e$. Since $s \wedge t = 0$ it follows that $t \leq s^{\perp}$, whence $t^{\perp\perp} \leq s^{\perp}$. Next,

$$(s^{\perp} \wedge t^{\perp}) \wedge u = (s^{\perp} \wedge t^{\perp} \wedge s) \vee (s^{\perp} \wedge t^{\perp} \wedge t) = 0,$$

from which it follows that $s^{\perp} \wedge t^{\perp} = 0$, whence the reverse inequality holds $s^{\perp} \leq t^{\perp\perp}$. We conclude that $s^{\perp} = t^{\perp\perp}$. By hypothesis, L satisfies Reg(4) and since both s and t

are compact it follows that $s^{\perp} \vee t^{\perp} = 1$, whence $s^{\perp\perp}, t^{\perp\perp}$ is a complementary pair. It follows that $s^{\perp\perp} = e$. □

Theorem 3.11 *Suppose L is an algebraic frame which possesses a unit, say $u \in L$. Consider the following statements.*

(1) *L is weakly zero-dimensional and satisfies Reg(4).*
(2) *L is a normal, feebly projectable frame.*
(3) *L is a normal, flatly projectable frame.*

Statements (2) *and* (3) *are equivalent. Statement* (1) *implies* (2)*, and if L is compact, then all three are equivalent.*

Proof We begin by showing that a frame possessing a unit is feebly projectable if and only if it is flatly projectable. From this it follows that (2) and (3) are equivalent.

Suppose that L is flatly projectable and let $a, b \in \mathfrak{k}(L)$ be disjoint. Then there is a complementary pair e, f such that $a \leq e$ and $b \leq f$. By the Lemma 3.10, since a flatly projectable frame satisfies Reg(4), $e = s^{\perp\perp}$ and $f = t^{\perp\perp}$ for some $s, t \in \mathfrak{k}(L)$. Therefore L is feebly projectable.

Suppose L is weakly zero-dimensional and satisfies Reg(4). Clearly L is normal. Let $a, b \in \mathfrak{k}(L)$ be disjoint. By Reg(4), we know that $a^{\perp} \vee b^{\perp} = 1$. The frame is weakly zero-dimensional, so there exist $c, d \in L$ such that $c \leq a^{\perp}$, $d \leq b^{\perp}$, $c \wedge d = 0$, and $c \vee d = 1$. It follows from Lemma 3.10 that $c = x^{\perp\perp}$ and $d = y^{\perp\perp}$ for $x, y \in \mathfrak{k}(L)$. Furthermore, $x^{\perp} = c^{\perp} = d = y^{\perp\perp}$. Since $x^{\perp\perp} \leq a^{\perp}$ and $x^{\perp} \leq b^{\perp}$, it follows that $a \leq a^{\perp\perp} \leq x^{\perp}$ and $b \leq b^{\perp\perp} \leq x^{\perp\perp}$. Therefore L is feebly projectable and so (1) implies (2).

Suppose (2) holds and that L is a compact frame. We show that L is weakly zero-dimensional. Observe that L satisfies Reg(4) by Proposition 3.3. Let $a, b \in L$ with $a \vee b = 1$. Since L is algebraic and compact there are compact elements c and d satisfying $c \leq a$, $d \leq b$, and $c \vee d = 1$. By normality of L, there exist disjoint $x, y \in L$ such that $x \leq c$, $y \leq d$, and $x \vee d = y \vee c = 1$. Without loss of generality we assume that x and y are compact. Since L is feebly projectable there exists $t \in \mathfrak{k}(L)$ with $x \leq t^{\perp\perp}$, $y \leq t^{\perp}$, and $t^{\perp} \vee t^{\perp\perp} = 1$.

Now let $w = t^{\perp\perp} \wedge c$ and $z = t^{\perp} \wedge d$. Note that $w \leq c \leq a$ and $z \leq d \leq b$. We have

$$\begin{aligned} w \vee z &= (t^{\perp\perp} \wedge c) \vee (t^{\perp} \wedge d) = (t^{\perp\perp} \vee t^{\perp}) \wedge (c \vee t^{\perp}) \wedge (t^{\perp\perp} \vee d) \wedge (c \vee d) \\ &= (c \vee t^{\perp}) \wedge (t^{\perp\perp} \vee d) = 1. \end{aligned}$$

Clearly $w \wedge z = 0$, so w and z form a complementary pair. It follows that $w \leq a$ and $z \leq b$, hence L is weakly zero-dimensional. □

At this point we are unable to decide whether the above three statements are equivalent under the weaker hypothesis that L possesses a unit. We conclude this section by giving an alternate characterization of feebly projectable algebraic frames.

Definition 3.12 We call the frame L *weakly complemented* if whenever $a, b \in \mathfrak{k}(L)$ are disjoint, then there exists $x, y \in \mathfrak{k}(L)$ such that $a \leq x$, $b \leq y$, $x \wedge y = 0$, and $x \vee y$ is a unit.

Theorem 3.13 *Suppose L is an algebraic frame that possesses a unit. L is feebly projectable if and only if L is weakly complemented and satisfies Reg(4).*

Proof Suppose L is feebly projectable. It suffices to show that L is weakly complemented. To that end let $a, b \in \mathfrak{k}(L)$ and $a \wedge b = 0$. Since L is feebly projectable we can choose $t \in \mathfrak{k}(L)$ such that $t^{\perp\perp}$ is complemented, $a \leq t^{\perp\perp}$, and $b \leq t^{\perp}$. Since L satisfies Reg(4) and $t^{\perp}$ is complemented, we know that there is some compact element $z \in \mathfrak{k}(L)$ such that $z^{\perp\perp} = t^{\perp}$. Let $x = a \vee t$ and $y = b \vee z$ so that $x, y \in \mathfrak{k}(L)$. Using distributivity it is straightforward to check that $x \wedge y = 0$. Futhermore, $(x \vee y)^{\perp\perp} \geq t^{\perp\perp} \vee t^{\perp} = 1$ so that $x \vee y$ is a unit. Therefore L is weakly complemented.

Conversely, suppose L is weakly complemented and satisfies Reg(4). Let $a, b \in \mathfrak{k}(L)$ with $a \wedge b = 0$. Choose $x, y \in \mathfrak{k}(L)$ such that $a \leq x$, $b \leq y$, $x \wedge y = 0$, and $x \vee y$ is a unit. Since L satisfies Reg(4), $x^{\perp} \vee y^{\perp} = 1$. Furthermore, since $x \vee y$ is a unit we have $0 = (x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$. Therefore, $x^{\perp}$ is a complemented element with complement $y^{\perp}$. Thus, $a \leq x^{\perp\perp}$, $b \leq y^{\perp\perp} = x^{\perp}$ where $x \in \mathfrak{k}(L)$ and $x^{\perp}$ is complemented. So L is feebly projectable. □

4 The Frame of Multiplicative Filters of Ideals

Throughout this section A denotes a commutative ring with identity and $\mathcal{L}(A)$ denotes the collection of ideals of A. $\mathcal{L}(A)$ is a lattice when ordered by inclusion, but it is not a distributive lattive. For $I, J \in \mathcal{L}(A)$, we write $I \leq K$ to mean $I \subseteq K$. A *multiplicative filter of ideals* is a nonempty collection of ideals, say $\mathcal{F}$, satisfying the following:

(1) If $I \in \mathcal{F}$ and $I \leq K$, then $K \in \mathcal{F}$
(2) Whenever $I, J \in \mathcal{F}$, then so is IJ

We denote the collection of multiplicative filters of ideals of A by $\mathfrak{M}_A$. We will drop the subscript when it is clear from the context.

Notice that $\mathfrak{M}_A$ is a subset of the power set of $\mathcal{L}(A)$, and so we can partially order it under inclusion. It is straightforward to check that the intersection of an arbitrary collection of multiplicative filters of ideals is again a multiplicative filter of ideals. It follows that $\mathfrak{M}_A$ is a complete lattice. In particular, for any $\mathcal{F}_i \in \mathfrak{M}_A$ we have

$$\bigvee_{i \in I} \mathcal{F}_i = \{K \in \mathcal{L}(A) : K \geq J_{i_1} ... J_{i_n} \text{ for some } i_1, ..., i_n \in I, J_{i_k} \in \mathcal{F}_{i_k}\}.$$

It is also straightforward to check that $\mathcal{F} \wedge \bigvee_{i \in I} \mathcal{G}_i = \bigvee_{i \in I}(\mathcal{F} \wedge \mathcal{G}_i)$. Observe that the top element of $\mathcal{M}_A$ is $\mathcal{L}(A)$ and that $0 = \{A\}$. Next, for any $I \in \mathcal{L}(A)$ there is a least multiplicative filter of ideals which contains I. We denote this element by $\mathcal{M}_I$ and say this is the multiplicative filter of ideals generated by I. Observe that

$$\mathcal{M}_I = \{J \in \mathcal{L}(A) : I^n \leq J \text{ for some } n \in \mathbb{N}\}.$$

Definition 4.1 The *Jacobson radical* is denoted by $\mathfrak{J}(A)$. The ideal of nilpotent elements of A, i.e., the *nilradical* of A, is denoted by $n(A)$. By a *local ring* we mean a ring with a unique maximal ideal.

The proof of the next lemma is straightforward and is left to the interested reader.

Lemma 4.2 *Suppose $\mathcal{F}, \mathcal{G} \in \mathfrak{M}_A$. Then $\mathcal{F} \vee \mathcal{G} = 1$ if and only if there are ideals $I \in \mathcal{F}$ and $J \in \mathcal{G}$ such that $IJ = \{0\}$.*

Lemma 4.3 *The compact elements of $\mathfrak{M}_A$ are precisely the multiplicative filters of the form $\mathcal{M}_I$ for some $I \in \mathcal{L}(A)$. Moreover, $\mathfrak{M}_A$ is a compact, algebraic frame with the FIP.*

Proof Suppose $\mathcal{F}$ is a compact multiplicative filter. Notice that for every $I \in \mathcal{F}$, $\mathcal{M}_I \leq \mathcal{F}$. It follows that $\bigvee_{I \in \mathcal{F}} \mathcal{M}_I = \mathcal{F}$. By compactness of $\mathcal{F}$, there exist $I_1, ..., I_n \in \mathcal{F}$ such that $\mathcal{F} = \mathcal{M}_{I_1} \vee ... \vee \mathcal{M}_{I_n} = \mathcal{M}_{I_1 I_2 ... I_n}$ since $\mathcal{F}$ is multiplicative. Conversely, suppose $\mathcal{M}_I \leq \bigvee_{j \in J} \mathcal{F}_j$. For some $j_1, ..., j_n \in J$ there exist $K_{j_k} \in \mathcal{F}_k$ with $K_{j_1} ... K_{j_n} \leq I$. Therefore $\mathcal{M}_I \leq \mathcal{F}_{j_1} \vee ... \vee \mathcal{F}_{j_n}$ which shows compactness of $\mathcal{M}_I$.

It is easy to see that $\mathfrak{M}_A$ has the FIP because $\mathcal{M}_I \wedge \mathcal{M}_J = \mathcal{M}_{I+J}$ for any ideals I, J of A. □

Lemma 4.4 *The complemented elements of $\mathfrak{M}_A$ are precisely the elements of the form $\mathcal{M}_I$ where $I = Ae$ for some idempotent element e of A.*

Proof Suppose $\mathcal{F} \vee \mathcal{G} = \mathcal{L}(A)$ and $\mathcal{F} \wedge \mathcal{G} = 0$. Then there exist $I \in \mathcal{F}, J \in \mathcal{G}$ with $IJ = \{0\}$. Since $\mathcal{F} \wedge \mathcal{G} = 0$, we see that $H + K = A$ for all $H \in \mathcal{F}, K \in \mathcal{G}$. Therefore $I + J = A$. Together with $IJ = \{0\}$, it follows that I is a direct summand of A, and so $I = eA$ for some idempotent element $e \in A$. Clearly $\mathcal{M}_I \leq \mathcal{F}$. To show equality, let $H \in \mathcal{F}$. Then $H \cap I \in \mathcal{F}$. Note that $(H \cap I)J = \{0\}$ and $(H \cap I) + J = A$. Hence $(H \cap I) \supseteq I$, that is, $H \supseteq I$. Thus $\mathcal{M}_I = \mathcal{F}$.

Conversely, suppose $\mathcal{F} = \mathcal{M}_I$ for some $I = Ae$ where e is idempotent in A. Let $\mathcal{G} = \mathcal{M}_{A(1-e)}$, then it is straightforward to check that $\mathcal{F} \vee \mathcal{G} = \mathcal{L}$ and $\mathcal{F} \wedge \mathcal{G} = 0$. □

Lemma 4.5 *For any $I \in \mathcal{L}(A)$, $\mathcal{M}_I^{\perp} = \{J \in \mathcal{L}(A) : J + I = A\}$.*

Proof Observe that it is sufficient to show that the set in question is a multiplicative filter. Clearly it is a filter. If $J + I = K + I = A$, then $j + i_1 = k + i_2 = 1$ for appropriate $i_1, i_2 \in I$, $j \in J$, $k \in K$. Multiplying yields $1 = jk + (ji_2 + i_1 k + i_1 i_2)$ where $ji_2 + i_1 k + i_1 i_2 \in I$. Whence $JK + I = A$. □

Now is a good time to recall the structure spaces of a commutative ring (with unit). Let $Max(A)$ denote the collection of maximal ideals of A topologized via the Zariski topology. Basic open sets are of the form $U(a) = \{M \in Max(A) : a \notin M\}$. Closed sets are of the form $V(I) = \{M \in Max(A) : I \leq M\}$ for some $I \in \mathcal{L}(A)$. For any commutative ring with identity A, $Max(A)$ is a compact T_1 space.

Theorem 4.6 *Suppose A is a commutative ring with identity. The following are equivalent:*

(1) *$\mathfrak{M}_A$ is a projectable frame.*
(2) *For every $I \in \mathcal{L}(A)$, there exists an idempotent $e \in A$ such that $I + J = A$ if and only if $J \geq Ae$.*

(3) *For every $I \in \mathcal{L}(A)$, there exists an idempotent $e \in A$ such that $V(I) = U(e)$.*
(4) *A is a finite product of local rings.*

Proof We start by showing that (1) and (2) are equivalent.

Suppose (1) holds. Let $I \in \mathcal{L}(A)$ so that $\mathcal{M}_I$ is compact and thus $\mathcal{M}_I^{\perp}$ is complemented, i.e. $\mathcal{M}_I^{\perp} = \mathcal{M}_{Ae}$ for some idempotent $e \in A$. This means $I + Ae = A$. Moreover, if $I + J = A$, then $J \in \mathcal{M}_I^{\perp} = \mathcal{M}_{Ae}$, so $J \geq Ae$. Conversely, if $J \geq Ae$, then $I + J = A$. Therefore (2) is satisfied.

If (2) holds then it is straightforward to check that for each ideal I of A, $M_I^{\perp} = \mathcal{M}_{Ae}$. Thus, each compact element $\mathcal{M}_I$ of $\mathfrak{M}_A$ satisfies that $\mathcal{M}_I^{\perp}$ is complemented. Consequently, $\mathfrak{M}_A$ is a projectable frame, i.e. (1) holds.

Next assume (2) holds, and we will prove (3). Let $I \in \mathcal{L}(A)$, and choose an idempotent $e \in A$ satisfying the condition of (2). Since $I + Ae = A$, we see that $V(I) \cap V(Ae) = \emptyset$, that is, $V(I) \subseteq U(e)$. If $M \notin V(I)$, then $M + I = A$, and thus $M \geq Ae$. This forces $M \in V(I)$, proving that $V(I) = U(e)$.

Assume (3) holds, and we will prove that A is a finite product of local rings. Consider $V(M)$ where $M \in Max(A)$. By hypothesis, $V(M) = U(e)$ for some idempotent $e \in A$. It follows that every point of $Max(A)$ is isolated, and so $Max(A)$ is discrete. $Max(A)$ is compact, from which we gather that $Max(A)$ is finite, say $Max(A) = \{M_1, ..., M_n\}$. Choose idempotent elements $e_i \in A$ for which $V(M) = U(e_i)$ for each $i = 1, ..., n$. We may assume that whenever $i \neq j$, $e_i e_j = 0$ while still maintaining that $V(M_i) = U(e_i)$. It follows that $A = Ae_1 \times ... \times Ae_n$ and Ae_i is a local ring for each $i = 1, ..., n$. So (4) holds.

Finally we will show that (4) implies (2). Let I be an ideal of A. Suppose $A = A_1 \times ... \times A_n$ where each A_i is a local ring with maximal ideal M_i. For each $1 \leq i \leq n$, let

$$e_i(j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

so that e_i is a local idempotent and $1 = e_1 + ... + e_n$. Next, for each $I \in \mathcal{L}(A)$, define $I_k = Ie_k$ so that $I = I_1 \times \cdots \times I_n$. Define

$$a_j = \begin{cases} 1 & \text{if } I_j \leq M_j \\ 0 & \text{if } I_j = A_j \end{cases}$$

and set $a = (a_1, ..., a_n)$. Then $a^2 = a$ and we have

$$1 - a = \begin{cases} 0 & \text{if } I_j \leq M_j \\ 1 & \text{if } I_j = A_j \end{cases}.$$

It follows that $1 - a \in I$ and thus $I + aA = A$. Clearly, if $J \geq aA$, then $I + J = A$.

Suppose $I + K = A$, then $1 = i + k$ for some $i \in I, k \in K$. Then $I + Ak = A$ with $Ak \leq K$. Write $i = (i_a, ..., i_n)$ and $k = (k_1, ..., k_n)$. If $I_j \neq A_j$, then i_j is not a unit of A_j and so k_j is a unit of A_j with inverse $p_j \in A_j$. Define

$$p(j) = \begin{cases} p_j & \text{if } I_j \neq A_j \\ 0 & \text{otherwise} \end{cases},$$

then

$$kp(j) = \begin{cases} 1 & \text{if } I_j \neq A_j \\ 0 & \text{otherwise} \end{cases}.$$

Hence $a_j = kp(j)$ and we get that $a \in Ak$. Therefore $Aa \leq Ak \leq K$, which proves (2). □

Theorem 4.7 *Consider the following statements.*

(1) $\mathfrak{M}_A$ *is zero-dimensional.*
(2) *For all* $I \in \mathcal{L}(A)$, $I^n = Ae$ *for some* $n \in \mathbb{N}$ *and some idempotent* $e \in A$.
(3) A *is a finite product of fields.*
(4) *Every ideal of* A *is generated by an idempotent of* A.

(1) *and* (2) *are equivalent, and* (3) *and* (4) *are equivalent. If* A *is a semiprime ring, then all four statements are equivalent. In this case,* A *is a von Neumann regular ring.*

Proof Suppose (1) is true. If $\mathfrak{M}_A$ is zero-dimensional, then every compact element is complemented. As a result, for all $I \in \mathcal{L}(A)$ there exists an idempotent $e \in A$ such that $\mathcal{M}_I = \mathcal{M}_{Ae}$. This implies $Ae \supseteq I^n$ and $I \supseteq (Ae)^n$ for some n. Since Ae is an idempotent ideal, we gather that $I^n = Ae$. Hence (1) implies (2).

The proof of (2) implies (1) follows similarly. Therefore (1) and (2) are equivalent.

That (3) implies (4) is patent. Suppose (4) holds. Then A is a von Neumann regular ring. Moreover, (2) and, thus, (1) are also true. Since $\mathfrak{M}_A$ is algebraic and zero-dimensional, it is projectable. So A is a finite product of local rings. However, each of the factor (local) rings is also regular, so each factor is a field. Thus (3) holds and we have demonstrated that (3) and (4) are equivalent.

Next, we assume that A is a semiprime ring. Since (4) always implies (2) it suffices to show that (1) and (2) imply (3). By Theorem 4.6 A is a finite product of local rings, say $A = A_1 \times \cdots \times A_n$ where each A_i is a local ring with maximal ideal M_i. By (2) there is some $m \in \mathbb{N}$ such that M_i^m is generated by an idempotent. Since A_i is local it is indecomposable and so $M_i^n = \{0\}$. But A is semiprime and hence so is A_i. It follows that $M_i = \{0\}$, whence each A_i is a field. Consequently, (3) holds. □

Example 4.8 Let $A = \mathbb{Z}/4\mathbb{Z}$. Then A is a local ring and $\mathfrak{M}_A$ is simply the two element frame $\{0, 1\}$. It follows that $\mathfrak{M}_A$ is zero-dimensional yet A is not semiprime. Therefore it is necessary that we include the condition that A is semiprime to obtain the equivalences of all four statements in Theorem 4.7.

Example 4.9 Suppose A is a local ring which is not a field. Then A is not regular, so $\mathfrak{M}_A$ is not zero-dimensional. However, $\mathfrak{M}_A$ is trivially projectable by (2). Observe that $J(A) \neq \{0\}$. This gives us an example of a ring for which $\mathfrak{M}_A$ is projectable but not zero-dimensional.

Definition 4.10 A subset of $Max(A)$ is called an *idempotent clopen* subset if it is of the form $U(e)$ where $e^2 = e$. In [8], the author calls a ring *clean* if every element can be written as the sum of a unit and an idempotent. It is shown in [7] that a ring A is clean if and only if the collection of idempotent clopen subsets of $Max(A)$ is a base for the topology on $Max(A)$. In particular, when A is clean then $Max(A)$ is a compact zero-dimensional Hausdorff space (see Example 2.2 for definitions).

Proposition 4.11 *Suppose A is a commutative ring with identity. The following are equivalent:*

(1) $\mathfrak{M}_A$ *is feebly projectable.*
(2) $\mathfrak{M}_A$ *is flatly projectable.*
(3) *A is a clean ring.*

Proof Since $\mathfrak{M}_A$ is a compact, algebraic frame, (1) and (2) are equivalent by Theorem 3.11. Suppose $\mathfrak{M}_A$ is feebly projectable, and let $N \in U(a)$ for $a \in A$. Then $N + Aa = A$, and so $\mathcal{M}_N \wedge \mathcal{M}_{Aa} = 0$. By hypothesis there is a complemented pair of compact elements of $\mathfrak{M}_A$, say $\mathcal{M}_{Ae}$ and $\mathcal{M}_{A(1-e)}$ with $e^2 = e$ such that $\mathcal{M}_N \leq \mathcal{M}_{Ae}$ and $\mathcal{M}_{Aa} \leq \mathcal{M}_{A(1-e)}$. As a result, $Ae \leq N$ and $1 - e \in Aa$. Notice that $1 - e \notin N$, so $N \in U(1-e)$. Let $P \in U(1-e)$. If $P \notin U(a)$, then $a \in P$ and hence $1 - e \in P$, a contradiction. Thus $N \in U(1-e) \subseteq U(a)$, and we conclude that A is a clean ring.

Conversely, assume A is a clean ring and $\mathcal{M}_I \wedge \mathcal{M}_J = 0$. This means that $I + J = A$, and so $i + j = 1$ for some $i \in I$, $j \in J$. It follows that $V(i) \cap V(j) = \emptyset$. Since A is clean, the idempotent clopen subsets of $Max(A)$ form a base. So we can find an idempotent clopen set, say $U(e)$ with $e^2 = e$, such that $V(j) \subseteq U(e)$ and $V(i) \cap U(e) = \emptyset$. Let $f = 1 - e$. Notice that $V(ie) = V(i) \cup V(e) = V(e)$ and $V(fj) = V(f) \cup V(j) = V(f)$. We know that $V(e) \cap V(f) = \emptyset$, thus $A(fj) + A(ei) = A$. Also, because $(fj)(ei) = 0$, we can say that $\mathcal{M}_{A(fj)}$ and $\mathcal{M}_{A(ei)}$ form a complementary pair with $\mathcal{M}_{Ai} \leq \mathcal{M}_{A(ei)}$ and $\mathcal{M}_{Aj} \leq \mathcal{M}_{A(fj)}$. Therefore $\mathfrak{M}_A$ is feebly projectable. □

Proposition 4.12 $\mathfrak{M}_A$ *satisfies Reg(4) if and only if A is a Gelfand ring, that is, whenever $a, b \in A$ satisfy $a + b = 1$, there exist $r, s \in A$ such that $(1 + ar)(1 + bs) = 0$.*

Proof Let $a, b \in A$ with $a + b = 1$, then it follows that $Aa + Ab = A$. As a result, $\mathcal{M}_{Aa} \wedge \mathcal{M}_{Ab} = 0$ and so $\mathcal{M}_{Aa}^{\perp} \vee \mathcal{M}_{Ab}^{\perp} = 1$ by Reg(4). Then there exist $I \in \mathcal{M}_{Aa}^{\perp}$ and $J \in \mathcal{M}_{Ab}^{\perp}$ such that $IJ = \{0\}$. Notice that $I + Aa = A$ and $J + Ab = A$, so there exist $r, s \in A$ satisfying $1 = i - ar$ and $1 = j - bs$. Now, $(1 + ar)(1 + bs) = ij = 0$. Hence A is a Gelfand ring.

Conversely, suppose A is a Gelfand ring. Let $\mathcal{M}_I$ and $\mathcal{M}_J$ be disjoint elements of $\mathfrak{M}_A$. Since $I + J = A$, there exist $a \in I$ and $b \in J$ with $a + b = 1$. There exist $r, s \in A$ such that $(1 + ar)(1 + bs) = 0$. Let $u = 1 + ar$ and $v = 1 + bs$. Observe that $Au \in \mathcal{M}_I^{\perp}$ and $Av \in \mathcal{M}_J^{\perp}$. We see that $AuAv = \{0\}$, thus $\mathcal{M}_I^{\perp} \vee \mathcal{M}_J^{\perp} = 1$. Therefore $\mathfrak{M}_A$ satisfies Reg(4). □

Proposition 4.13 *Consider the following statements regarding A, a commutative ring with identity.*

1. $\mathfrak{M}_A$ *is normal.*
2. *Whenever* $I, J \in \mathcal{L}(A)$ *such that* $IJ = \{0\}$ *there exist* $P, Q \in \mathcal{L}(A)$ *and* $n \in \mathbb{N}$ *so that* $I \leq Q$, $J \leq P$, $QJ^n = PI^n = \{0\}$, *and* $Q + P = A$.
3. *Whenever* $I, J \in \mathcal{L}(A)$ *such that* $IJ = \{0\}$ *there exist* $P, Q \in \mathcal{L}(A)$ *with* $I \leq Q$, $J \leq P$, $QJ = PI = \{0\}$, *and* $Q + P = A$.

1 *and* 2 *are equivalent. If A is semiprime, then all three are equivalent.*

Proof We begin by showing that 1 and 2 are equivalent.

Suppose 1. Let $I, J \in \mathcal{L}(A)$ with $IJ = \{0\}$. Then $\mathcal{M}_I \vee \mathcal{M}_J = 1$. Because $\mathfrak{M}_A$ is normal, there exist $S, T \in \mathcal{L}(A)$ satisfying $\mathcal{M}_S \wedge \mathcal{M}_T = 0$, $\mathcal{M}_S \leq \mathcal{M}_J$, $\mathcal{M}_T \leq \mathcal{M}_I$, and $\mathcal{M}_I \vee \mathcal{M}_S = \mathcal{M}_J \vee \mathcal{M}_T = 1$. There are $n, m \in \mathbb{N}$ so that $I^n \leq T$ and $J^m \leq S$. Furthermore, there are $p, q \in \mathbb{N}$ so that $T^q J^q = S^p I^p = \{0\}$. Let $s = \max\{n, m, p, q\}$ and $t = s^2$. Next, let $P = S^s + J$ and $Q = T^s + I$ so that $I \leq Q$, $J \leq P$. Also, $QJ^t = PI^t = \{0\}$. Furthermore, $Q + P = A$. Therefore, condition 2 is satisfied.

Suppose 2 holds. Let $\mathcal{F}, \mathcal{G} \in \mathfrak{M}_A$ with $\mathcal{F} \vee \mathcal{G} = 1$. Then there exist $I \in \mathcal{F}$ and $J \in \mathcal{G}$ with $IJ = \{0\}$. By hypothesis, there exist $P, Q \in \mathcal{L}(A)$ and $n \in N$ satisfying $I \leq Q$, $J \leq P$, $QJ^n = PI^n = \{0\}$ and $Q + P = A$. Clearly $\mathcal{M}_Q \leq \mathcal{M}_I$ and $\mathcal{M}_P \leq \mathcal{M}_J$. Observe that $\mathcal{F} \vee \mathcal{M}_P \geq \mathcal{M}_I \vee \mathcal{M}_P = 1$, which implies that $\mathcal{F} \vee \mathcal{M}_P = 1$. Similarly, $\mathcal{G} \vee \mathcal{M}_Q = 1$. Also, we see that $\mathcal{M}_P \wedge \mathcal{M}_Q = 0$ because $Q + P = A$. Consequently, $\mathfrak{M}_A$ is normal.

Observe that 3 is a stronger condition than 2. If A is semiprime and $QJ^n = \{0\}$ then $(QJ)^n = \{0\}$, whence $QJ = \{0\}$. Similarly, $PI = \{0\}$. It follows that if 2 holds then 3 is satisfied. □

Remark 4.14 We do not have an example showing that it is necessary that A be semiprime for the conditions in the previous proposition to be equivalent.

Proposition 4.15 $\mathfrak{M}_A$ *is weakly zero-dimensional if and only if whenever* $I, J \in \mathcal{L}(A)$ *such that* $IJ = \{0\}$ *there exist* $P, Q \in \mathcal{L}(A)$ *with* $I \leq Q$, $J \leq P$, $PQ = \{0\}$, *and* $Q + P = A$.

Proof Suppose $\mathfrak{M}_A$ is weakly zero-dimensional. If $I, J \in \mathcal{L}(A)$ with $IJ = \{0\}$, then $\mathcal{M}_I \vee \mathcal{M}_J = 1$. Using the fact that $\mathfrak{M}_A$ is weakly zero-dimensional, there exists a complemented element $\mathcal{M}_{Ae}$ (where $e \in A$ satisfies $e^2 = e$) such that $\mathcal{M}_{Ae} \leq \mathcal{M}_I$ and $\mathcal{M}_{A(1-e)} \leq \mathcal{M}_J$. Let $P = Ae$ and $Q = A(1 - e)$. Observe that P and Q are both radical ideals. Thus, if $I^n \leq Q$, then $I \leq Q$. Similarly, $J \leq P$. $PQ = \{0\}$ and $P + Q = A$.

Conversely, suppose $\mathcal{F}, \mathcal{G} \in \mathfrak{M}_A$ with $\mathcal{F} \vee \mathcal{G} = 1$. Then there exist $I \in \mathcal{F}$ and $J \in \mathcal{G}$ with $IJ = \{0\}$. By hypothesis, there exist $P, Q \in \mathcal{L}(A)$ satisfying $I \leq Q$, $J \leq P$, $PQ = \{0\}$ and $Q + P = A$. Clearly $\mathcal{M}_Q \leq \mathcal{M}_I$ and $\mathcal{M}_P \leq \mathcal{M}_J$. We have that $\mathcal{M}_P \wedge \mathcal{M}_Q = 0$ because $Q + P = A$, and $\mathcal{M}_P \vee \mathcal{M}_Q = 1$ since $PQ = \{0\}$. Hence $\mathcal{M}_P, \mathcal{M}_Q$ form a complementary pair, and we conclude that $\mathfrak{M}_A$ is weakly zero-dimensional.

□

Theorem 4.16 *If $\mathfrak{J}(A) = \{0\}$, then the following are equivalent:*

(1) *$\mathfrak{M}_A$ is normal.*
(2) *$\mathfrak{M}_A$ is weakly zero-dimensional.*
(3) *$Max(A)$ is extremally disconnected.*

Proof The implication (2) implies (1) is always true. Assume (1) holds, and we will show (3). In particular, we will show that disjoint open sets have disjoint closures. Suppose $I, J \in \mathcal{L}(A)$ such that $U(I) \cap U(J) = \emptyset$. Since $\mathfrak{J}(A) = \{0\}$, it follows that $IJ = \{0\}$. By Proposition 4.13, there exist ideals P and Q of A with $I \leq Q$, $J \leq P$, $QJ = IP = \{0\}$, and $Q + P = A$. So $U(I) \subseteq V(P)$ and $U(J) \subseteq V(Q)$ where $V(P) \cap V(Q) = \emptyset$. Hence $clU(I) \cap clU(J) \subseteq V(P) \cap V(Q)$ implies $clU(I) \cap clU(J) = \emptyset$, which proves (3).

Finally, we will prove (3) implies (2) by applying Proposition 4.15. Suppose $I, J \in \mathcal{L}(A)$ such that $IJ = \{0\}$. Since $Max(A)$ is extremally disconnected, $clU(I)$ is clopen in $Max(A)$. So there exist $P, Q \in \mathcal{L}(A)$ such that $clU(I) = V(P)$, $clU(J) \subseteq V(Q)$, $V(P) \cap V(Q) = \emptyset$, and $V(P) \cup V(Q) = Max(A)$. These ideals P and Q satisfy the conditions of Proposition 4.15. □

Proposition 4.17 *Suppose $\mathfrak{J}(A) = \{0\}$. Then $\mathfrak{M}_A$ is zero-dimensional if and only if $\mathfrak{M}_A$ is projectable.*

Proof The forward direction is clear. If $\mathfrak{M}_A$ is projectable, then A is a finite product of local rings by Theorem 4.6. But $\mathfrak{J}(A) = \{0\}$ implies that each local ring is in fact a field. □

Example 4.18 Consider the case where $A = C(X)$ for some Tychonoff space X. Since $\mathfrak{J}(A) = \{0\}$ the previous theorem tells us that $\mathfrak{M}_A$ is zero-dimensional if and only if $\mathfrak{M}_A$ is projectable. It is easy to see that these two conditions are also equivalent to X being finite.

Next, let X be a compact zero-dimensional Hausdorff space which is not extremally disconnected, e.g., the Cantor space. If $A = C(X)$, then $C(X)$ is a clean ring, whence $\mathfrak{M}_A$ is feebly projectable. However, $\mathfrak{M}_A$ is not weakly zero-dimensional because $Max(A) = X$ is not extremally disconnected. Also note that $\mathfrak{M}_A$ is not projectable because X is infinite.

For an example where $\mathfrak{M}_A$ is weakly zero-dimensional but not zero-dimensional, simply take an infinite extremally disconnected space X and let $A = C(X)$. Then $\mathfrak{M}_A$ is weakly zero-dimensional by Theorem 4.16, but it is not zero-dimensional since X is infinite.

At this point, we do not have examples where $\mathfrak{M}_A$ is normal but not weakly zero-dimensional and where $\mathfrak{M}_A$ is weakly zero-dimensional but not projectable.

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