MAXIMAL d-SUBGROUPS AND ULTRAFILTERS

PAPIYA BHATTACHARJEE[1] AND WARREN WM. MCGOVERN[2]

Abstract. We study the space $\mathrm{Max}_d(G)$ of maximal d-subgroups of a lattice-ordered group, paying specific attention to archimedean ℓ-groups with weak order unit. For such an object (G,u), $\mathrm{Max}_d(G)$ lays at a level in between the space of minimal prime subgroups and the Yosida space of (G,u). Theorem 5.10 gives the appropriate generalization of a quasi F-space to $\mathbf{W}$-objects which avoids a discussion of o-complete ℓ-groups.

1. Introduction

It is a classical result in the theory of rings of continuous functions that, for a Tychonoff space X, the space of maximal ideals of $C(X)$ and the space of zero-set ultrafilters of X are homeomorphic; this latter space is the Stone-Čech compactification of X. This result has been generalized in the context of archimedean ℓ-groups with distinguished weak order unit: the category $\mathbf{W}$ whose objects are pairs (G,u) for an archimedean ℓ-group G and $0<u\in G$ a distinguished weak order unit, and whose morphisms between two objects (G,u) and (H,v) are ℓ-group homomorphisms $\phi : G \longrightarrow H$ for which $\phi(u)=v$.

For a compact space X, the space of ultrafilters on the Wallman lattice

$$Z^{\#}(X)=\{cl_X \operatorname{int}_X Z : Z\in Z(X)\}$$

is known as the quasi F-cover of X and is in bijective correspondence to the maximal d-ideals of $C(X)$. It is this correspondence that we shall show generalizes in the context of $\mathbf{W}$. We also generalize some results that occur for uniformly complete archimedean ℓ-groups.

We assume the reader is familiar with the fundamental results from the theory of lattice-ordered groups. In particular, we assume the reader is familiar with terms like convex ℓ-subgroups, values, prime subgroups, and weak and strong order units. The texts [9], [12], [32], and [7] are excellent sources for the material to be discussed here. For a condensed version of the background information for this article, the reader is urged to familiarize themselves with the ideas found in [8]; for which this article is a continuation.

The prime spectrum of G is denoted by $\mathrm{Spec}(G)$. The collection of the minimal prime subgroups is denoted by $\mathrm{Min}(G)$. Globally, $\mathrm{Spec}(G)$ can be topologized with the hull-kernel topology. Basic open sets are of the form $\mathcal{S}(g)=\{P\in \mathrm{Spec}(G) : g\notin P\}$, indexed over $0\neq g\in G$. Each $\mathcal{S}(g)$ is compact, but not necessarily Hausdorff. The set $\mathrm{Val}(g)$ of values of g is a subset of $\mathcal{S}(g)$. The hull-kernel topologies on $\mathrm{Min}(G)$ and $\mathrm{Val}(g)$ are precisely the subspace topologies inherited from $\mathrm{Spec}(G)$. The basic open set $\mathcal{S}(g)\cap\mathrm{Min}(G)$ of $\mathrm{Min}(G)$ will be denoted instead by $U(g)$, while a basic open set of $\mathrm{Val}(g)$ has the form $\mathcal{S}(h)\cap\mathrm{Val}(g)$. Each space $\mathrm{Val}(g)$ is a compact Hausdorff space. Since $\mathrm{Spec}(G)$ is a root system, each $P\in\mathcal{S}(g)$ is contained in a unique $\mu_g(P)\in\mathrm{Val}(g)$. The restriction of μ_g to $U(g)$ will be denoted by

$\lambda_g : U(g) \longrightarrow \mathrm{Val}(g)$. It is known that both μ_g and λ_g are continuous maps. More is now known.

For $g \in G$, let $V(g) = \{P \in \mathrm{Min}(G) : g \in V\}$ and observe that $V(g) = \mathrm{Min}(G) \setminus U(g)$, a basic closed subset of the hull-kernel topology on $\mathrm{Min}(G)$. Interestingly, the collection $\{V(g) : g \in G\}$ is closed under finite intersections (and unions) and thus is a base for an open topology on $\mathrm{Min}(G)$ called the *inverse topology*; $\mathrm{Min}(G)^{-1}$ denotes the space of minimal prime subgroups equipped with the inverse topology. The hull-kernel topology on $\mathrm{Min}(G)$ is finer than the inverse topology.

Proposition 1.1 (Theorem 3.10 [8]). *For any weak order unit $0 < u \in G$, the map*

$$\lambda_u : \mathrm{Min}(G)^{-1} \longrightarrow \mathrm{Val}(u)$$

is continuous.

In the context of **W**, it is standard to denote the set of values of u by YG and call YG the *Yosida space* of (G, u); YG is a compact Hausdorff space. A basic open set has the form

$$\mathrm{coz}(g) = \{p \in YG : g \notin p\},$$

for some $g \in G$, and which is simply the set $\mathrm{coz}(g) = YG \cap \mathcal{S}(g)$. This set is called the *cozero-set of g*. Any subset of YG of this form is known as a *G-cozero-set*; the collection of all such subsets is denoted by $\mathrm{coz}(G)$, and obviously is a base for the topology of open subsets of YG. The complement of a G-cozero-set is a *G-zero-set* and the collection of these is denoted by $Z(G)$. In the few cases where a discussion of (G, u) and (G, v) takes place with $0 < u, v$ different weak order units, we shall use the symbol $\mathrm{Yos}_G(u)$ to denote the Yosida space relative to u.

We revisit the Yosida Embedding Theorem. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ denote the two-point compactification of the real numbers with the obvious ordering. For a Tychonoff space X and a continuous function $f : X \longrightarrow \bar{\mathbb{R}}$, set $\mathrm{re}(f) = f^{-1}(\mathbb{R})$; this is known as the reality set of f.

$$D(X) = \{f : X \longrightarrow \bar{\mathbb{R}} : f \text{ is continuous and } \mathrm{re}(f) \text{ is a dense subset of } X\}.$$

In general, $D(X)$ is a lattice under the pointwise operations but not a group under (almost) pointwise addition. However, by an ℓ-subgroup of $D(X)$ is meant a subcollection H of $D(X)$ that is a sublattice and is also closed under the addition defined as follows: for $f, g \in H$ there is an $h \in H$ such that for all $x \in \mathrm{re}(f) \cap \mathrm{re}(g)$, $f(x) + g(x) = h(x)$. We now state one of the most important theorems in the context of **W**.

Theorem 1.2 (The Yosida Embedding Theorem). *Let (G, u) be a **W**-object. There is an ℓ-isomorphism of G ($g \mapsto \hat{g}$) onto an ℓ-subgroup $\hat{G} \leq D(YG)$ such that $\hat{u} = \mathbf{1}$ and $\hat{G}$ has the following separation property: for each $p \in YG$ and closed set $V \subseteq YG$ not containing p, there is some $g \in G$ for which $\hat{g}(p) = 1$ and $\hat{g}(q) = 0$ for all $q \in V$. Moreover, YG is the unique compact space, up to homeomorphism, satisfying these two properties.*

Example 1.3. The quintessential example of a **W**-object is $(C(X), \mathbf{1})$ for a Tychonoff space X. By properties of the Stone-Čech compactfication, each $f \in C(X)$ extends to an $\bar{f} : \beta X \longrightarrow \bar{\mathbb{R}}$, inducing an ℓ-isomorphism of $C(X)$ inside $D(\beta X)$ which separates points of βX.

Therefore, $YC(X) = \beta X$. Observe then that a $C(X)$-zero-set is a member of $Z(\beta X)$, and vice versa. It is standard to call a subset Z of X, a zero-set of X if $Z = \{x \in X : f(x) = 0\}$ for some $f \in C(X)$. The set of all zero-sets of X is denoted by $Z(X)$. Note that if X is not compact, then $Z(X)$ and $Z(C(X))$ are not the same, as the first is a collection of subsets of X, while the second is a collection of subsets of βX.

A second important example is $C(X, \mathbb{Z})$. Recall that this notation stands for the ℓ-group of integer-valued continuous functions on X. When studying this ℓ-group it will be assumed that X is a zero-dimensional space. A similar argument is useful in characterizing $YC(X, \mathbb{Z})$ as $\beta_0 X$, the Banaschewski compactification of X. This compactification is the Stone dual of $\mathfrak{B}(X)$, the boolean algebra of clopen subsets of X.

We do point out that, obviously, both $C(X)$ and $C(X, \mathbb{Z})$ have many weak order units. However, when we speak of $C(X)$ and $C(X, \mathbb{Z})$, unless otherwise noted, it will be assumed that $\mathbf{1}$ is the distinguished weak order unit.

We conclude this section with a recollection of some important terminology from the theory of ℓ-groups. For $S \subseteq G$, the polar of S is the set

$$S^{\perp} = \{h \in G : |g| \wedge |h| = 0 \text{ for all } g \in S\}.$$

When $S = \{g\}$ we instead write $g^{\perp}$. Notice that the symbols $S^{\perp\perp}$ and $g^{\perp\perp}$ are obvious. A central concept is the following. The ℓ-group G is said to be *projectable* if for all $g \in G$, $G = g^{\perp} + g^{\perp\perp}$. Next, the convex ℓ-subgroup generated by an element $g \in G$ is the set

$$\mathfrak{G}(g) = \{h \in G : |h| \leq n|g| \text{ for some } n \in \mathbb{N}\}.$$

The collection of all convex ℓ-subgroups of G is denoted by $\mathfrak{C}(G)$. When partially-ordered by inclusion $\mathfrak{C}(G)$ is an algebraic frame with the FIP and disjointification. For $H, K \in \mathfrak{C}(G)$, the join and the meet of H and K will be denoted by $H \bigvee K$ and $H \bigcap K$, respectively.

For $(G, u) \in \mathbf{W}$, the convex ℓ-subgroup of G generated by u is denoted by G^*. Observe that $(G^*, u) \in \mathbf{W}$. The Yosida Embedding Theorem represents elements of G^* as bounded elements of $D(YG)$, so that $G^* \subseteq C(YG)$ and $YG^* = YG$.

2. Spaces of Ultrafilters

Throughout this section we assume that $(G, u) \in \mathbf{W}$.

As mentioned in the first section there is a nice correspondence between YG and the collection of $Z(G)$-ultrafilters. This correspondence is obtained as follows. Start with a convex ℓ-subgroup $H \leq G$ and form

$$Z[H] = \{Z(h) \in Z(G) : h \in H\}.$$

It is straightforward to check that $Z[H]$ is a $Z(G)$-filter. Also, $\emptyset \in Z[H]$ if and only if $H = G$.

Next, let $\mathcal{F}$ be a $Z(G)$-filter and form

$$\overleftarrow{\mathcal{F}} = \{h \in G : Z(h) \in \mathcal{F}\}.$$

Then $\overleftarrow{\mathcal{F}}$ is a convex ℓ-subgroup, and is proper if and only if $\mathcal{F}$ is a proper filter. The main result is that $Z[H]$ is a $Z(G)$-ultrafilter if and only if H is a value of u. Next, the space of $Z(G)$-ultrafilters can be topologized with the Wallman topology. As this topology is central to our discussion, we elaborate.

Definition 2.1. Let $(L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice, and let $\mathrm{Ult}(L)$ denote the collection of L-ultrafilters. For $a \in L$, denote the set of ultrafilters containing a by $\mathcal{V}(a)$. The operator $\mathcal{V}(\cdot)$ has the following properties: [3].

(i) For each $a, b \in L$, $\mathcal{V}(a) \cup \mathcal{V}(b) = \mathcal{V}(a \vee b)$ and $\mathcal{V}(a) \cap \mathcal{V}(b) = \mathcal{V}(a \wedge b)$.
(ii) The collection $\{\mathcal{V}(a) : a \in L\}$ forms a base for a topology of closed sets on $\mathrm{Ult}(L)$. This is called the *Wallman topology* on $\mathrm{Ult}(L)$.
(iii) For each $a < 1$, there is a $0 < c \in L$ such that $a \wedge c = 0$ if and only if the map $a \longrightarrow \mathcal{V}(a)$ is injective. (A lattice satisfying either of these equivalent conditions is called a *Wallman* lattice.)
(iv) If L is a Wallman lattice, then $\mathrm{Ult}(L)$ is a compact T_1-space.
(v) The space $\mathrm{Ult}(L)$ is a Hausdorff space if and only if for any $a, b \in L$ such that $a \wedge b = 0$ there exists $x, y \in L$ such that $x \vee y = 1$ and $a \wedge y = 0 = b \wedge x$. (When $\mathrm{Ult}(L)$ is a Hausdorff space, we shall say L is a normal lattice.)

Example 2.2. A boolean algebra $\mathcal{B}$ is easily seen to be a normal Wallman lattice. Its space of ultrafilters is known to be isomorphic to its Stone dual, i.e. the space of maximal ideals of $\mathcal{B}$. Therefore, $\mathrm{Ult}(\mathcal{B})$ is a compact zero-dimensional Hausdorff space.

Now, $Z(G)$ is a bounded distributive lattice. It is straightforward to check that $Z(G)$ is a normal Wallman lattice. Therefore, the space of $Z(G)$-ultrafilters is a compact Hausdorff space. Furthermore, YG and $\mathrm{Ult}(Z(G))$ are homeomorphic via the restriction of the map $Z[\cdot]$ to YG.

Another example where this type of construction has been useful is in the construction of the essential closure of a **W**-object (see [10]). Starting with a compact Hausdorff space X one forms $\mathcal{R}(X)$, the collection of regular closed subsets of X. (Recall that $V \subseteq X$ is called *regular closed* if $V = \mathrm{cl}_X \, \mathrm{int}_X \, V$.) It is well-known that $\mathcal{R}(X)$ is a (complete) boolean algebra when partially ordered by inclusion. The meet, join, and complement are given as follows: for $V_1, V_2 \in \mathcal{R}(X)$

(i) $V_1 \cup' V_2 = V_1 \cup V_2$;
(ii) $V_1 \cap' V_2 = \mathrm{cl}_X \, \mathrm{int}_X (V_1 \cap V_2)$;
(iii) $V_1' = \mathrm{cl}_X (X \smallsetminus V_1)$.

Since $\mathcal{R}(X)$ is a boolean algebra, it is a normal Wallman lattice, and thus one can speak of its space of ultrafilters $\mathrm{Ult}(\mathcal{R}(X))$. It is customary to denote the space of $\mathcal{R}(X)$-ultrafilters by $\mathcal{E}(X)$ and call $\mathcal{E}(X)$ the *absolute of* X. It is known that $\mathcal{E}(X)$ is the extremally disconnected cover of X, as well as the projective cover constructed by Gleason. The covering map is defined by $e_X : \mathcal{E}(X) \longrightarrow X$:

$$e_X(\mathcal{U}) = \bigcap \mathcal{U}$$

where $\bigcap \mathcal{U} = \{p\}$, a unique point in this set $e_X(\mathcal{U})$, since X is compact. (For a detailed discussion on covers and covering maps, we point the reader to [35].)

We now turn to another construction that has been developed and the one that we are interested in generalizing for **W**-objects (see [25]). Recall that

$$Z^\sharp(X) = \{\mathrm{cl}_X \, \mathrm{int}_X \, Z : Z \in Z(X)\}.$$

This collection is a sub-lattice of $R(X)$ and is a normal Wallman lattice. The space of ultrafilters of $Z^\sharp(X)$ is denoted by $QF(X)$ and it is well-known that $QF(X)$ is a compact quasi F-space which covers X with the covering map Ψ_X defined in the analogous way.

For $(G,u) \in \mathbf{W}$, define

$$Z^\sharp(G) = \{\mathrm{cl}_{YG}\,\mathrm{int}_{YG}\, Z(f) : f \in G\}.$$

Observe that each member of $Z^\sharp(G)$ belongs to $Z^\sharp(YG)$ and so is a regular closed set. Moreover, it follows from Lemma 2.2 of [25] that $Z^\sharp(G)$ is a sublattice of $\mathcal{R}(YG)$. Furthermore, for all $Z_1, Z_2 \in Z(G)$

$$\mathrm{cl}_{YG}\,\mathrm{int}_{YG}\, Z_1 \cap' \mathrm{cl}_{YG}\,\mathrm{int}_{YG}\, Z_2 = \mathrm{cl}_{YG}\,\mathrm{int}_{YG}(Z_1 \cap Z_2)$$

and

$$\mathrm{cl}_{YG}\,\mathrm{int}_{YG}\, Z_1 \cup' \mathrm{cl}_{YG}\,\mathrm{int}_{YG}\, Z_2 = \mathrm{cl}_{YG}\,\mathrm{int}_{YG}(Z_1 \cup Z_2).$$

It will be shown later that $\mathrm{Ult}(Z^\sharp(G))$ is a Hausdorff space, by an indirect route. We leave it to the interested reader to show that $Z^\sharp(G)$ is a normal Wallman lattice; the Yosida Embedding Theorem is pivotal.

Example 2.3. As pointed out in the previous section, for a compact space X, $Z(C(X)) = Z(X)$. There are many examples of $\mathbf{W}$-objects (G,u) such that $Z(G) = Z(YG)$. Some examples of this include i) uniformly complete $\mathbf{W}$-objects, ii) convex $\mathbf{W}$-objects, that is, whenever $f \in D(YG)$ and there are $g_1, g_2 \in G$ such that $g_1 \le f \le g_2$, then $f \in G$. In this case, the construction of $\mathrm{Ult}(Z^\sharp(G))$ produces the quasi F-cover of $X = YG$.

For a general $\mathbf{W}$-object (G,u), it is possible that $Z^\sharp(G)$ is actually nothing more than the boolean algebra of clopen subsets of YG; it is always the case that $\mathfrak{B}(YG) \subseteq Z^\sharp(G)$. For example, if YG is basically disconnected, then $\mathfrak{B}(YG) = Z^\sharp(G)$. This equality leads us to consider the coincidence of the sets $Z(G), Z^\sharp(G)$, and $\mathfrak{B}(YG)$. Recall from [18] that (G,u) is said to be *bounded away* if for every $g \in G^+$, there is some $\epsilon > 0$ such that for all $p \in \mathrm{coz}(g)$, $\epsilon \le g(p)$.

Proposition 2.4 (Theorem 2.3 [18])**.** *Let (G,u) be a $\mathbf{W}$-object. The following statements are equivalent.*

1. $Z(G) = \mathfrak{B}(YG)$.
2. $Z(G) = \mathfrak{B}(YG) = Z^\sharp(G)$.
3. (G,u) *is a bounded away ℓ-group.*
4. (G^*,u) *is hyper-archimedean.*
5. (G^*,u) *is bounded away.*
6. *Every $\mathbf{W}$-homomorphic image is bounded away.*
7. *Every value of u is a minimal prime subgroup of G.*
8. $\mathrm{Min}(G) = YG$.

Proof. The equivalencies of the conditions (3) through (8) are shown in [18, Theorem 2.3].

(1) is equivalent to (2). Clearly, if $Z(G) = \mathfrak{B}(YG)$, then $\mathfrak{B}(YG) = Z^\sharp(G)$. The converse is obvious.

(1) is equivalent to (3). If every G-zero-set of G is clopen, then so is every G-cozero-set. Since YG is compact it follows that the image of $g(\mathrm{coz}(g))$ is a compact subset of $(0,\infty]$

and so has a nonzero minimum. Therefore, (G, u) is bounded away. Conversely, if (G, u) is bounded away then for each $g \in G^+$, $\operatorname{coz}(g) = g^{-1}([\epsilon, \infty])$ a closed subset of YG. Therefore, each G-cozero-set is clopen. □

A more general class of **W**-objects that is of interest here is the class of *weakly projectable* **W**-objects. (G, u) is said to be *weakly projectable* if for every $g \in G$, $\operatorname{cl}\operatorname{coz}(g) \in \mathfrak{B}(YG)$. Observe that this is equivalent to saying that $\operatorname{int} Z(g) \in \mathfrak{B}(YG)$ for all $g \in G$. Obviously, in this case, $Z^\sharp(G) = \mathfrak{B}(G)$. For more information on weakly projectable **W**-objects we suggest the reader check [19] and the more recent carnation [21]. The concept of a weakly projectable ℓ-group does indeed generalize the concept of a projectable ℓ-group.

Theorem 2.5. *Let (G, u) be a **W**-object. The following statements are equivalent.*
1. $Z^\sharp(G) = \mathfrak{B}(YG)$.
2. (G, u) *is weakly projectable.*
3. (G^*, u) *is weakly projectable.*

Proof. As was pointed out above, a weakly projectable **W**-object (G, u) satisfies $Z^\sharp(G) = \mathfrak{B}(G)$. Conversely, suppose that $Z^\sharp(G) = \mathfrak{B}(G)$ and let $g \in G$. Then $\operatorname{cl}\operatorname{int} Z(g)$ is a clopen subset of YG. By taking complements, this means that $\operatorname{int}\operatorname{cl}\operatorname{coz}(g)$ is also clopen. Therefore,

$$\operatorname{int}\operatorname{cl}\operatorname{coz}(g) = \operatorname{cl}\operatorname{int}\operatorname{cl}\operatorname{coz}(g) = \operatorname{cl}\operatorname{coz}(g)$$

is clopen. Consequently, (G, u) is weakly projectable. □

Example 2.6. For a Tychonoff space X, $C(X)$ is bounded away if and only if X is finite. On the other hand $C(X, \mathbb{Z})$ is always bounded away. Turning to the concept of weakly projectable, it is true that $C(X)$ is weakly projectable if and only if it is projectable if and only if X is basically disconnected. $C(X, \mathbb{Z})$ is always projectable.

What is left to discuss is the situation when the equality $Z(G) = Z^\sharp(G)$ holds. However, we leave this to Theorem 5.3 in order to be able to expand on the discussion.

3. $\operatorname{cl}\operatorname{coz}(G)$

It ought to be apparent by looking at the proof of Theorem 2.5, that the collection $cl\operatorname{coz}(G) = \{\operatorname{cl}\operatorname{coz}(g) : g \in G\}$ is of high importance. The collection has some nice properties which we aim to discuss in this section.

Lemma 3.1. *For a **W**-object (G, u), $\operatorname{cl}\operatorname{coz}(G)$ is a Wallman sublattice of $\mathcal{R}(YG)$.*

Proof.

$$\begin{aligned}
\operatorname{cl}\operatorname{coz}(g_1) \cap' \operatorname{cl}\operatorname{coz}(g_2) &= \operatorname{cl}\operatorname{int}(\operatorname{cl}\operatorname{coz}(g_1) \cap \operatorname{cl}\operatorname{coz}(g_2)) \\
&= \operatorname{cl}(\operatorname{coz}(g_1) \cap \operatorname{coz}(g_2)) \\
&= \operatorname{cl}\operatorname{coz}(|g_1| \wedge |g_2|)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{cl}\operatorname{coz}(g_1) \cup' \operatorname{cl}\operatorname{coz}(g_2) &= \operatorname{cl}\operatorname{coz}(g_1) \cup \operatorname{cl}\operatorname{coz}(g_2) \\
&= \operatorname{cl}(\operatorname{coz}(g_1) \cup \operatorname{coz}(g_2)) \\
&= \operatorname{cl}\operatorname{coz}(|g_1| \vee |g_2|).
\end{aligned}$$

This shows that $cl\,\mathrm{coz}(G)$ is a sublattice of $\mathcal{R}(YG)$. The Yosida Embedding Theorem is used to show that it is a Wallman lattice. □

In general, the lattice $cl\,\mathrm{coz}(G)$ need not be a normal lattice. Example 2.6 of [25] shows that there is a compact Hausdorff space X, for which the space of ultrafilters of $cl\,\mathrm{coz}(X)$ is not Hausdorff. We use the rest of this section to discuss $\mathrm{Ult}(cl\,\mathrm{coz}(G))$, concluding with a characterization of when $\mathrm{Ult}(\mathrm{coz}(G))$ is Hausdorff.

Lemma 3.2. *Let (G,u) be a* **W***-object.*

(a) *For $\mathcal{U} \in \mathrm{Ult}(\mathrm{coz}(G))$, the collection $\overline{\mathcal{U}} = \{\mathrm{cl}\,C \in \mathrm{cl}\,\mathrm{coz}(G) : C \in \mathcal{U}\}$ is a $\mathrm{cl}\,\mathrm{coz}(G)$-ultrafilter.*

(b) *For $\mathcal{F} \in \mathrm{Ult}(\mathrm{cl}\,\mathrm{coz}(G))$, the collection $\underline{\mathcal{F}} = \{C \in \mathrm{coz}(G) : \mathrm{cl}\,C \in \mathcal{F}\}$ is a $\mathrm{coz}(G)$-ultrafilter.*

(c) *The map $\overline{(\cdot)} : \mathrm{Ult}(\mathrm{coz}(G)) \longrightarrow \mathrm{Ult}(\mathrm{cl}\,\mathrm{coz}(G))$ is a bijection.*

(d) *Moreover, the map $\overline{(\cdot)}$ is a homeomorphism with respect to the Wallman topologies.*

Proof. (a) Let $\mathrm{cl}\,C, \mathrm{cl}\,D \in \overline{\mathcal{U}}$ for $C, D \in \mathrm{coz}(G)$. Then $\mathrm{cl}\,C \cap' \mathrm{cl}\,D = \mathrm{cl}(C \cap D)$ which must also belong to $\overline{\mathcal{U}}$ since $C \cap D \in \mathcal{U}$.

Next, let $\mathrm{cl}\,C \in \overline{\mathcal{U}}$ with $C \in \mathcal{U}$. Let $D \in \mathrm{coz}(G)$ satisfy $\mathrm{cl}\,C \subseteq \mathrm{cl}\,D$. Then

$$\mathrm{cl}\,D = \mathrm{cl}\,C \cup' \mathrm{cl}\,D = \mathrm{cl}(C \cup D)$$

which belongs to $\overline{\mathcal{U}}$ since $C \cup D \in \mathcal{U}$.

Lastly, to show that $\overline{\mathcal{U}}$ is an ultrafilter, let $\mathrm{cl}\,D \notin \overline{\mathcal{U}}$ with $D \in \mathrm{coz}(G)$. This means that $D \notin \mathcal{U}$ and so there is some $C \in \mathcal{U}$ such that $C \cap D = \emptyset$. Then, $\mathrm{cl}\,C \in \overline{\mathcal{U}}$ and

$$\mathrm{cl}\,C \cap' \mathrm{cl}\,D = \mathrm{cl}(C \cap D) = \emptyset,$$

whence we gather that $\overline{\mathcal{U}}$ is a $cl\,\mathrm{coz}(G)$-ultrafilter.

(b) Let $\mathcal{F} \in \mathrm{cl}\,\mathrm{coz}(G)$ and $\underline{\mathcal{F}}$ defined as in the statement of the lemma. Let $C, D \in \underline{\mathcal{F}}$, which means that $\mathrm{cl}\,C, \mathrm{cl}\,D \in \mathcal{F}$. Since $\mathcal{F}$ is a filter, then $\mathrm{cl}(C \cap D) \in \mathcal{F}$. Therefore, $C \cap D \in \underline{\mathcal{F}}$. Similarly, as above, if $C \in \underline{\mathcal{F}}$ and $C \subseteq D$, then $\mathrm{cl}\,C \subseteq \mathrm{cl}\,D$ which means that $\mathrm{cl}\,D \in \mathcal{F}$, whence $D \in \underline{\mathcal{F}}$.

Finally, suppose $D \in \mathrm{coz}(G)$ and $D \notin \underline{\mathcal{F}}$. Then $\mathrm{cl}\,D \notin \mathcal{F}$ and so there is some $\mathrm{cl}\,C \in \mathcal{F}$ for which $\mathrm{cl}\,C \cap' \mathrm{cl}\,D = \emptyset$. Then $C \cap D = \emptyset$ with $C \in \underline{\mathcal{F}}$.

(c) Let $\mathcal{U} \in \mathrm{Ult}(\mathrm{coz}(G))$. Observe that $\mathcal{U} \subseteq \underline{(\overline{\mathcal{U}})}$. Since $\mathcal{U}$ is an ultrafilter it follows that they are equal. Conversely, given $\mathcal{V} \in \mathrm{Ult}(\mathrm{cl}\,\mathrm{coz}(G))$. Then since $\mathcal{V} \subseteq \overline{(\underline{\mathcal{V}})}$, we again conclude that these two sets are equal. It follows that the identifications given above are inverse functions of each other.

(d) Recall that a basic closed subset of the Wallman topology on $\mathrm{Ult}(\mathrm{coz}(G))$ is the collection of ultrafilters that contain a fixed cozero-set: $\mathcal{V}(C)$ for $C \in \mathrm{coz}(G)$. We leave it to the interested reader to check that

$$\overline{\mathcal{V}(C)} = \mathcal{V}(\mathrm{cl}\,C) \text{ and } \underline{\mathcal{V}(\mathrm{cl}\,C)} = \mathcal{V}(C).$$

□

Remark 3.3. In [4], the authors show that for an arbitrary ℓ-group G, the space of ultrafilters of the (bounded below) lattice G^+ is homeomorphic to $\mathrm{Min}(G)^{-1}$. This uses the well-known Lemma on Ultrafilters.

For a $\mathbf{W}$-object (G, u), the space of ultrafilters of $\mathrm{coz}(G)$ is also homeomorphic to the space of ultrafilters of G^+. Therefore, it is obvious that the space of ultrafilters of $\mathrm{coz}(G)$, and hence of $\mathrm{cl}\,\mathrm{coz}(G)$, has to do more with the structure of G, rather than of the Yosida space itself. We state this formally in our next two results.

Definition 3.4. Recall from [8], that an ℓ-group G is called *lamron* if whenever $a, b \in G^+$ such that $a \wedge b = 0$, then there are $x, y \in G^+$ such that $a \leq x$, $b \leq y$, $a \wedge y = 0 = b \wedge x$, and $x \vee y$ is a weak order unit.

Theorem 3.5. *For a* $\mathbf{W}$*-object* (G, u)*, the spaces* $\mathrm{Ult}(cl\,\mathrm{coz}(G))$ *and* $\mathrm{Min}(G)^{-1}$ *are homeomorphic. Consequently, the following statements are equivalent.*

(1) $\mathrm{Ult}(cl\,\mathrm{coz}(G))$ *is a Hausdorff space.*
(2) $\mathrm{Min}(G)^{-1}$ *is a Hausdorff space.*
(3) G *is a lamron* ℓ*-group.*
(4) *For each pair of disjoint* G*-cozero-sets* C_1, C_2*, there exists* G*-zero-sets* Z_1, Z_2 *such that* $C_1 \subseteq Z_1, C_2 \subseteq Z_2$*, and* $\mathrm{int}\, Z_1 \cap \mathrm{int}\, Z_2 = \emptyset$*.*

Proof. [4, Theorem 4.8] states that $\mathrm{Ult}(\mathrm{coz}(G))$ and $\mathrm{Min}(G)^{-1}$ are homeomorphic. That (2) and (3) are equivalent follows from [8, Theorem 2.7], while [8, Theorem 3.15] states and proves that (3) and (4) are equivalent. □

Corollary 3.6. *Let* G *be an archimedean* ℓ*-group and* $0 < u, v \in G$ *be weak order units. Let* $G = G_1 = G_2$ *and consider the* $\mathbf{W}$*-objects* (G_1, u) *and* (G_2, v)*. Then the space of* $\mathrm{cl}\,\mathrm{coz}(G_1)$*-ultrafilters and the space of* $\mathrm{cl}\,\mathrm{coz}(G_2)$*-ultrafilters are homeomorphic.*

Example 3.7. Let D be an uncountable discrete space and αD its one-point compactificaton. Interestingly, the $\mathbf{W}$-object $C(\alpha D, \mathbb{Z})$ satisfies the property

$$\mathrm{coz}(C(\alpha D, \mathbb{Z})) = \mathfrak{B}(\alpha D) = \mathrm{cl}\,\mathrm{coz}(C(\alpha D, \mathbb{Z})).$$

$C(\alpha D, \mathbb{Z})$ is a lamron ℓ-group, and hence $\mathrm{Ult}(cl\,\mathrm{coz}(C(\alpha D, \mathbb{Z}))$ is Hausdorff. However, $C(\alpha D)$ is not a lamron ℓ-group. Hence, $\mathrm{cl}\,\mathrm{coz}(\alpha D)$ is not a normal lattice. Of course, this works for any compact zero-dimensional Hausdorff space X for which $C(X)$ is not lamron.

4. d-SUBGROUPS

Definition 4.1. Let G be a ℓ-group and $K \in \mathfrak{C}(G)$[1]. K is called a *d-subgroup* if whenever $g \in K$, then $g^{\perp\perp} \subseteq K$.

Examples of d-subgroups include polar subgroups and minimal prime subgroups. There has been much work on the study of d-ideals of $C(X)$ and other types of archimedean f-rings. From a different vantage point, Martinez and Zenk [33] studied d-elements in algebraic frames with FIP. The work of Huisjmans and de Pagter [27] is particulary influential in that they studied maximal d-ideals in uniformly complete vector lattices.

[1] $\mathfrak{C}(G)$ is the frame of convex ℓ-subgroups of G.

Denote the set of d-subgroups of G by $\mathfrak{C}_d(G)$. An intersection of d-subgroups is again a d-subgroup. Therefore, $\mathfrak{C}_d(G)$ forms a complete lattice. In fact, it is an algebraic frame with FIP. Each convex ℓ-subgroup is contained in a smallest d-subgroup; denote this operator by $\mathfrak{G}_d(\cdot)$.

Lemma 4.2. *Let $K \in \mathfrak{C}(G)$. The smallest d-subgroup containing K is*

$$\mathfrak{G}_d(K) = \bigvee_{k \in K} k^{\perp\perp}.$$

Consequently, $K \in \mathfrak{C}(G)$ is a d-subgroup if and only if $K = \bigvee_{k \in K} k^{\perp\perp}$. Furthermore, $\mathfrak{G}_d(g) = g^{\perp\perp}$.

The only d-subgroup of G that contains a weak order unit is G itself (in the case it has one). Furthermore, a union of a chain of d-subgroups of G is again a d-subgroup, and if G possesses a weak order unit, then a union of a chain of proper d-subgroups is again proper. Therefore, one may speak of maximal d-subgroups when G has a weak order unit. Let $\mathrm{Max}_d(G)$ denote the set of maximal d-subgroups. When studying $\mathrm{Max}_d(G)$ it will be assumed that G possesses a weak order unit to ensure that $\mathrm{Max}_d(G) \neq \emptyset$. However, there is no need to assume that G is even abelian. Of course, we will focus on **W**-objects.

Proposition 4.3. *Let G possess a weak order unit.*

(a) *Let $H \in \mathfrak{C}(G)$. If H does not contain any weak order units, then neither does $\mathfrak{G}_d(H)$.*
(b) *If $K \in \mathrm{Max}_d(G)$, then K is maximal with respect to not containing a weak order unit.*
(c) *If H is maximal with respect to not containing any weak order unit, then $H \in \mathrm{Max}_d(G)$.*
(d) *If $K \in \mathrm{Max}_d(G)$, then $K \in \mathrm{Spec}(G)$.*

Proof. (a) If $0 < u \in G^+$ belongs to $\mathfrak{G}_d(H)$, then there is some $0 < h \in H^+$ such that $u \in h^{\perp\perp}$. If u is a weak order unit, then so is h.

(b) Let $K \in \mathrm{Max}_d(G)$. Indeed, K does not contain any weak order unit. Let $K \leq H$ and suppose that H does not contain any weak order unit. By (a), $\mathfrak{G}_d(H)$ does not contain any weak order unit and is a d-subgroup. By maximality, $K = H = \mathfrak{G}_d(H)$.

(c) Suppose H is maximal with respect to not containing any weak order unit (such things exist by Zorn's Lemma). By (a), neither does $\mathfrak{G}_d(H)$, and so by maximality, $H = \mathfrak{G}_d(H)$ is a d-subgroup. Any proper d-subgroup containing H will not contain any weak order units, so that $H \in \mathrm{Max}_d(G)$.

(d) Suppose that $a \wedge b = 0$ and $a \notin K$. Then $\mathfrak{C}_d(a, K) = G$. So, there is some $0 < k \in K^+$ such that $b \in a^{\perp\perp} \vee k^{\perp\perp}$. Applying, the Riesz Decomposition Theorem, there is some $0 < b_1 \in a^{\perp\perp}$ and $0 < b_2 \in k^{\perp\perp}$ such that $b = b_1 + b_2$.

$$b = b \wedge b = b \wedge (b_1 + b_2) \leq (b \wedge b_1) + (b \wedge b_2) = b \wedge b_2.$$

Thus, $0 \leq b \leq b_2 \in K$. □

The set $\mathrm{Max}_d(G)$ can be equipped with the hull-kernel topology. For $g \in G$, let $U_d(g) = \{M \in \mathrm{Max}_d(G) : g \notin M\}$. Then, similar to what occurs for $\mathrm{Spec}(G)$, the following hold (see [8, Proposition 2.1]). Clearly, $U_d(g) = U_d(|g|)$.

Lemma 4.4. *The following hold for all $g, h \in G^+$.*
(a) $U_d(g) = \mathrm{Max}_d(G)$ *if and only if g is a weak order unit.*
(b) $U_d(g) \cup U_d(h) = U_d(g \vee h)$.
(c) $U_d(g) \cap U_d(h) = U_d(g \wedge h)$.
(d) *The subset $T \subseteq \mathrm{Max}_d(G)$ is open in the hull kernel topology if and only if there is some d-subgroup H for which $T = U_d(H)$.*
(e) *If $(G, u) \in \mathbf{W}$, then $U_d(g) = \emptyset$ if and only if $g = 0$.*

Theorem 4.5. *Let G possess a weak order unit. The space $\mathrm{Max}_d(G)$ is a compact Hausdorff space.*

Proof. Since G possesses a weak order unit it is clear that $\mathrm{Max}_d(G)$ is nonempty. In [33] it is pointed out that, and in a more general setting, the set of d-subgroups of G, denoted $\mathfrak{C}_d(G)$, is an algebraic frame with FIP. This yields that $\mathrm{Max}_d(G)$ is a compact space. Furthermore, disjointification in G can be used to show that $\mathrm{Max}_d(G)$ is Hausdorff as we now demonstrate.

Let $M, N \in \mathrm{Max}_d(G)$ be distinct maximal d-subgroups. Since M and N are incomparable, there are $p \in M^+ \smallsetminus N$ and $q \in N^+ \smallsetminus M$. By disjointification, it follows that there are $0 < p \in M^+ \smallsetminus N$ and $0 < q \in N^+ \smallsetminus M$ such that $p \wedge q = 0$. Then, $U_d(p) \cap U_d(q) = U_d(p \wedge q) = \emptyset$ and $N \in U_d(p)$, $M \in U_d(q)$. Consequently, the hull-kernel topology on $\mathrm{Max}_d(G)$ is Hausdorff. □

Now is the time to connect the concept of d-subgroups to the collection $Z^\sharp(G)$ for a $\mathbf{W}$-object (G, u). Let H be any convex ℓ-subgroup that does not contain any weak order unit of G, and define

$$Z^\sharp[H] = \{\mathrm{cl}\,\mathrm{int}\, Z(g) : g \in H\}.$$

Then $Z^\sharp[H]$ is a proper filter on $Z^\sharp(G)$ since, in the first place, for all $f, g \in H$,

$$\begin{aligned} \mathrm{cl}\,\mathrm{int}\, Z(f) \cap' \mathrm{cl}\,\mathrm{int}\, Z(g) &= \mathrm{cl}\,\mathrm{int}(Z(f) \cap Z(g)) \\ &= \mathrm{cl}\,\mathrm{int}\, Z(|f| \vee |g|), \end{aligned}$$

with $|f| \vee |g| \in H$. In the second place, if $\mathrm{cl}\,\mathrm{int}\, Z(f) \subseteq \mathrm{cl}\,\mathrm{int}\, Z(g)$ and $f \in H$, then

$$\begin{aligned} \mathrm{cl}\,\mathrm{int}\, Z(g) &= \mathrm{cl}\,\mathrm{int}\, Z(f) \cup \mathrm{cl}\,\mathrm{int}\, Z(g) \\ &= \mathrm{cl}\,\mathrm{int}(Z(f) \cup Z(g)) \\ &= \mathrm{cl}\,\mathrm{int}(Z(|f| \wedge |g|) \end{aligned}$$

By convexity, $|f| \wedge |g| \in H$. In the third place, notice that since $\mathrm{cl}\,\mathrm{int}\, Z(f) = \emptyset$ if and only if $\mathrm{int}\, Z(f) = \emptyset$ if and only if f is a weak order unit, $Z^\sharp[H]$ is a proper $Z^\sharp(G)$-filter. Therefore, the map $Z^\sharp[\cdot]$ takes convex ℓ-subgroups which do not contain weak order units to (proper) $Z^\sharp(G)$-filters.

Inversely, let $\mathcal{F}$ be a proper $Z^\sharp(G)$-filter and set

$$\overleftarrow{Z^\sharp}[\mathcal{F}] = \{f \in G : \mathrm{cl}\,\mathrm{int}\, Z(f) \in \mathcal{F}\}.$$

Then a similar argument as just provided demonstrated that $\overleftarrow{Z^\sharp}[\mathcal{F}]$ is a convex ℓ-subgroup that does not contain a weak order unit. More can be said.

Lemma 4.6. *The following hold for the* $\mathbf{W}$*-object* (G,u)*.*

(a) *For any proper* $Z^\sharp(G)$*-filter, say* $\mathcal{F}$*,* $\overleftarrow{Z^\sharp}[\mathcal{F}]$ *is a* d*-subgroup.*
(b) *For any* $g \in G$*,* $g^{\perp\perp} = \{k \in G : \operatorname{cl}\operatorname{int} Z(g) \subseteq \operatorname{cl}\operatorname{int} Z(k)\}$*.*

Proof. (a) Let $0 < k \in \overleftarrow{Z^\sharp}[\mathcal{F}]^+$. This means that $\operatorname{cl}\operatorname{int} Z(k) \in \mathcal{F}$. Let $g \in k^{\perp\perp}$. Then

$$\operatorname{coz}(g) \subseteq \operatorname{cl}\operatorname{coz}(k).$$

Complementation yields that $\operatorname{int} Z(k) \subseteq Z(g)$, and thus, $\operatorname{int} Z(k) \subseteq \operatorname{int} Z(g)$. Taking closures of both sides results in

$$\operatorname{cl}\operatorname{int} Z(k) \subseteq \operatorname{cl}\operatorname{int} Z(g).$$

By hypothesis, $\mathcal{F}$ is a $Z^\sharp(G)$-filter and so $\operatorname{cl}\operatorname{int} Z(g) \in \mathcal{F}$, and therefore $g \in \overleftarrow{Z^\sharp}[\mathcal{F}]$.

(b) Observe that

$$\begin{aligned} g^{\perp\perp} &= \{k \in G : \operatorname{coz}(k) \subseteq \operatorname{cl}\operatorname{coz}(g)\} \\ &= \{k \in G : \operatorname{int} Z(g) \subseteq Z(k)\} \\ &\subseteq \{k \in G : \operatorname{cl}\operatorname{int} Z(g) \subseteq \operatorname{cl}\operatorname{int} Z(k)\}. \end{aligned}$$

The lemma states that the first and last sets are equal. Let $k \in G$ satisfy

$$\operatorname{cl}\operatorname{int} Z(g) \subseteq \operatorname{cl}\operatorname{int} Z(k).$$

By way of contradiction, assume that $\operatorname{int} Z(g) \nsubseteq \operatorname{int} Z(k)$. Let $x \in \operatorname{int} Z(g) \smallsetminus \operatorname{int} Z(k)$. That $x \notin \operatorname{int} Z(k)$ means that $x \in \operatorname{cl}\operatorname{coz}(k)$. That $x \in \operatorname{int} Z(g)$ means there is an open set O such that $x \in O \subseteq \operatorname{int} Z(g)$. Combining these two, forces the existence of a $t \in O \cap \operatorname{coz}(k)$. On the one hand,

$$t \in O \subseteq \operatorname{int} Z(g) \subseteq \operatorname{cl}\operatorname{int} Z(k),$$

by hypothesis. So since $t \in O \cap \operatorname{coz}(k)$ there is a $y \in \operatorname{int} Z(k) \cap (O \cap \operatorname{coz}(k))$. So $y \in Z(k)$ and $y \in \operatorname{coz}(k)$, the desired contradiction. □

For emphasis, the map $\overleftarrow{Z^\sharp}[\cdot]$ takes proper $Z^\sharp(G)$-filters and converts them to proper d-subgroups of G. Evidently, $H \subseteq \overleftarrow{Z^\sharp}[Z^\sharp[H]]$ and $Z^\sharp[\overleftarrow{Z^\sharp}[\mathcal{F}]] = \mathcal{F}$. Consequently, there is a bijection between $\operatorname{Max}_d(G)$ and $Z^\sharp(G)$-ultrafilters.

Corollary 4.7. *Let* $K \in \mathfrak{C}_d(G)$ *be a* d*-subgroup. Then* $g \in K$ *if and only if* $\operatorname{cl}\operatorname{int} Z(g) \in Z^\sharp[K]$*.*

Proof. The forward direction is clear by definition. Conversely, let $\operatorname{cl}\operatorname{int} Z(g) \in Z^\sharp[K]$. This means that there is some $0 < k \in K^+$ such that

$$\operatorname{cl}\operatorname{int} Z(k) = \operatorname{cl}\operatorname{int} Z(g).$$

Consequently, $g \in k^{\perp\perp}$. Therefore, since K is a d-subgroup, $g \in K$. □

Theorem 4.8. *For a* $\mathbf{W}$*-object* (G,u) *the space of* $Z^\sharp(G)$*-ultrafilters is homeomorphic to* $\mathrm{Max}_d(G)$*.*

Proof. The bijection in question is $Z^\sharp[\cdot] : \mathrm{Max}_d(G) \longrightarrow \mathrm{Ult}(Z^\sharp(G))$. A basic closed subset of $Ult(Z^\sharp(G))$ has the form

$$\mathcal{V}(\mathrm{cl}\,\mathrm{int}\,Z) = \{\mathcal{U} \in \mathrm{Ult}(Z^\sharp(G)) : \mathrm{cl}\,\mathrm{int}\,Z \in \mathcal{U}\}$$

for $Z \in Z(G)$. Let $Z = Z(f)$ for $0 < f \in G^+$. Now,

$$\begin{aligned}
\overleftarrow{Z^\sharp}(\mathcal{V}(\mathrm{cl}\,\mathrm{int}\,Z(f)) &= \{M \in \mathrm{Max}_d(G) : Z^\sharp[M] \in \mathcal{V}(\mathrm{cl}\,\mathrm{int}\,Z(f))\\
&= \{M \in \mathrm{Max}_d(G) : \mathrm{cl}\,\mathrm{int}\,Z(f) \in Z^\sharp[M]\}\\
&= \{M \in \mathrm{Max}_d(G) : f \in M\}\\
&= \mathrm{Max}_d(G) \smallsetminus U_d(f)
\end{aligned}$$

It follows that $Z^\sharp[\cdot]$ is a continuous bijection between the compact Hausdorff spaces $\mathrm{Max}_d(G)$ and $\mathrm{Ult}(Z^\sharp(G))$. □

5. Coincidence and Bijections

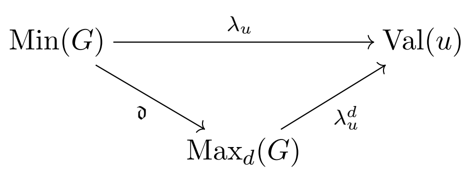
Definition 5.1. Suppose G is an ℓ-group and possesses a weak order unit, say $0 < u \in G$. Then since each minimal prime subgroup is a d-subgroup and each member of $\mathrm{Max}_d(G)$ is a prime subgroup, to each $P \in \mathrm{Min}(G)$, there is a unique maximal d-subgroup $\mathfrak{d}(P)$ containing P. This defines a map

$$\mathfrak{d} : \mathrm{Min}(G) \longrightarrow \mathrm{Max}_d(G).$$

Moreover, since each $M \in \mathrm{Max}_d(G)$ does not contain u, it follows that each $M \in \mathrm{Max}_d(G)$ is contained in a unique value of u, denoted $\lambda_u^d(M)$. This defines a map

$$\lambda_u^d : \mathrm{Max}_d(G) \longrightarrow \mathrm{Val}(u).$$

Observe that $\lambda_u = \lambda_u^d \circ \mathfrak{d}$.



When the situation warrants it, the subscript of u will be dropped.

Proposition 5.2. *Let* G *be an* ℓ*-group with a weak order unit. The following hold.*

(a) *The map* $\mathfrak{d} : \mathrm{Min}(G)^{-1} \longrightarrow \mathrm{Max}_d(G)$ *is continuous.*

(b) *For a* $\mathbf{W}$*-object* (G,u)*, the map* $\lambda_u^d : \mathrm{Max}_d(G) \longrightarrow YG$ *is continuous.*

Proof. For the purposes of this proof let $V_d(x) = \mathrm{Max}_d(G) \smallsetminus U_d(x)$.

(a) Let $P \in \mathrm{Min}(G)$ so that

$$\mathfrak{d}(P) \in U_d(h),$$

a basic open subset of $\mathrm{Max}_d(G)$. To each $Q \in V_d(h)$, $\mathfrak{d}(P) \neq Q$ and so there are disjoint basic open subsets of $\mathrm{Max}_d(G)$, say $U_d(t_Q)$ and $U_d(g_Q)$, for $0 \leq t_Q, g_Q$, such that

$$Q \in U_d(t_Q) \text{ and } \mathfrak{d}(P) \in U_d(g_Q).$$

The collection $\{U_d(t_Q)\}$ is an open cover of the basic closed set (and hence compact) $V_d(h)$. Therefore, there is a finite subcover, say $\{U_d(t_{Q_1}), \ldots, U_d(t_{Q_n})\}$. Set

$$t = t_{Q_1} \vee \ldots \vee t_{Q_n} \text{ and } g = g_{Q_1} \wedge \ldots \wedge g_{Q_n}.$$

Note that since $\emptyset = U_d(t_Q) \cap U_d(g_Q) = U_d(t_Q \wedge g_Q)$, it follows that $t_Q \wedge g_Q = 0$ and hence, $t \wedge g = 0$. Now, $\mathfrak{d}(P) \in U_d(g_{Q_1}) \cap \ldots \cap U_d(q_{Q_n}) = U_d(g_{Q_1} \wedge \ldots \wedge g_{Q_n}) = U_d(g)$. This forces $g \notin P$, and so $t \in P$. i.e. $P \in V(t)$, a basic open subset of $\mathrm{Min}(G)^{-1}$.

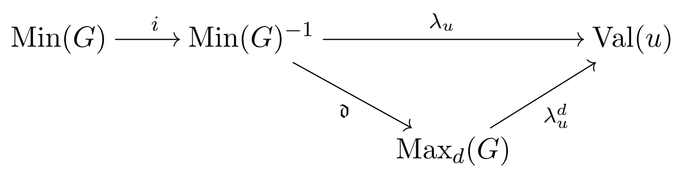
Now, let $R \in \mathrm{Min}(G)$ so that $R \in V(t)$. If it were the case that $\mathfrak{d}(R) \in V_d(h)$, then $\mathfrak{d}(R) \in U_d(t_{Q_i})$ for some i. But $0 \leq t_{Q_i} \leq t \in R$, yields a contradiction. So $\mathfrak{d}(R) \in U_d(h)$.

What has been demonstrated is that the $P \in V(t) \subseteq \mathfrak{d}^{-1}(U_d(h))$. This means that $\mathfrak{d}$ is a continuous map from the inverse topology on $\mathrm{Min}(G)$ to the hull-kernel topology on $\mathrm{Max}_d(G)$.

(b) Observe that since each maximal d-subgroup contains no weak order unit, $\mathrm{Max}_d(G) \subseteq \mathcal{S}(u)$. Thus, the restriction of the continuous map $\mu_u : \mathcal{S}(u) \longrightarrow \mathrm{Val}(u)$ to the set $\mathrm{Max}_d(G)$ is also continuous. This map is λ_d^u.

□

The commutative triangle at the end of Definition 5.1 can be expanded to include the identity map $i : \mathrm{Min}(G) \longrightarrow \mathrm{Min}(G)^{-1}$.



Next, our aim is to classify coincidence of the three sets $\mathrm{Min}(G), \mathrm{Max}_d(G)$, and $\mathrm{Val}(g)$. Specifically, we consider the case for a **W**-object. We answer this in the next result. Notice that the last condition of the next theorem answers the question of when $Z(G) = Z^\sharp(G)$.

Theorem 5.3. *The following hold for a* **W***-object* (G, u).

(a) $\mathrm{Min}(G) = YG$ *if and only if* (G, u) *is bounded away.*

(b) $\mathrm{Min}(G) = \mathrm{Max}_d(G)$ *if and only if* G *is complemented.*

(c) *The following statements are equivalent.*

(I) $\mathrm{Max}_d(G) = YG$.

(II) *Every value of* u *contains no weak order units.*

(III) *There are no proper dense* G*-cozero-sets of* YG.

(IV) $Z(G) = Z^\sharp(G)$.

Proof. (a) This is part of Theorem 2.4.

(b) This is from [33, Remark 5.6 (d)] , where the ℓ-groups for which $\mathrm{Min}(G) = \mathrm{Max}_d(G)$ are termed *d-regular*. For vector lattices, Theorem 9.5 and Remark 9.6 of [28] are useful for comparison.

(c) Clearly, (I), by Proposition 4.3, is the same as saying each value of u is maximal with respect to not containing any weak order unit. Therefore, (I) and (II) are equivalent.

(II) and (III) are equivalent. If $\operatorname{coz}(g)$ is dense in YG for some $g \in G^+$, then g is a weak order unit. Using (II), no value of u contains g. Hence $\operatorname{coz}(g) = YG$. Contrapositively, suppose $P \in YG$ and there is a weak order unit g such that $g \in P$. Therefore, $P \in Z(g)$. Consequently, $\operatorname{coz}(g)$ is a proper dense subset of YG.

(II) implies (IV). Let $0 < g \in G$ and let $p \in Z(g)$. Suppose that $p \notin \operatorname{cl}\operatorname{int} Z(g)$. By the Yosida Embdding Theorem, there is an $0 < f \in G$ such that $f(p) = 0$ and $f(q) = 1$ for all $q \in \operatorname{cl}\operatorname{int} Z(g)$. Consider $Z(f \wedge g) = Z(f) \cap Z(g)$. If there is some $p' \in \operatorname{int} Z(f \wedge g)$, then $f(p') = 0$ and $p' \in \operatorname{int} Z(g)$. The latter implies that $f(p') = 1$, a contradiction. Therefore, $\operatorname{int} Z(f \wedge g) = \emptyset$, i.e. $f \wedge g$ is a weak order unit. However, by convexity of $p \in YG$, $f \wedge g \in p$, a contradiction. It follows that $Z(g) = \operatorname{cl}\operatorname{int} Z(g)$, whence $Z(G) \subseteq Z^\sharp(G)$. Now, for any $\operatorname{cl}\operatorname{int} Z(g) \in Z^\sharp(G)$, we have that $\operatorname{cl}\operatorname{int} Z(g) = Z(g) \in Z(G)$, demonstrating the reverse containment.

(IV) implies (III). Suppose $Z(G) = Z^\sharp(G)$ and let $C = \operatorname{coz}(g)$ be a dense cozero-set. Then, by hypothesis, $Z(g) = \operatorname{cl}\operatorname{int} Z(f)$ for some $0 \leq f \in G$. Taking complements means that $\operatorname{coz}(g) = \operatorname{cl}\operatorname{coz}(f)$ so that $\operatorname{coz}(g) = YG$.

□

Remark 5.4. Theorem 5.3 (b) is true for any ℓ-group. For a **W**-object (G, u) we can also add that G is complemented if and only if $Z^\sharp(G)$ is a boolean sub-algebra of $\mathcal{R}(YG)$. We leave it to the interested reader to check this.

We now turn to classifying when the maps λ_u, $\mathfrak{d}$, and λ_u^d are bijections. The first and third maps will be considered in the case of **W**-objects while for the map $\mathfrak{d}$ we can generalize to arbitrary ℓ-groups with weak order units.

Proposition 5.5. *Let (G, u) be a **W**-object. The map λ_u is a bijection if and only if (G, u) has **W**-stranded primes.*

Remark 5.6. For some more equivalent conditions on what it means for a **W**-object to have **W**-stranded primes we point out [8, Theorem 3.7]. A **W**-object (G, u) for which G has stranded primes certainly has **W**-stranded primes. However, the converse is not true. Let $H = C^*(\mathbb{N})$, the set of bounded sequences, and let $G =< H, i >$ where i is the sequence $i(n) = n$. Then $(G, \mathbf{1})$ has **W**-stranded primes, while (G, i) does not. In particular, G does not have stranded primes.

Theorem 5.7. *Let G be an ℓ-group with a weak order unit. The following are equivalent.*

1. *The map $\mathfrak{d} : \operatorname{Min}(G) \longrightarrow \operatorname{Max}_d(G)$ is a bijection.*
2. *G is a lamron ℓ-group.*
3. *The map $\mathfrak{d}$ is a homeomorphism between $\operatorname{Min}(G)^{-1}$ and $\operatorname{Max}_d(G)$.*

Proof. (1) implies (2). Suppose $\mathfrak{d}$ is a bijection, and let $P, Q \in \operatorname{Min}(G)$ be minimal primes. If the ℓ-subgroup generated by P and Q, $P \bigvee Q$, does not contain a weak order unit, then $P \bigvee Q$ is contained in some $M \in \operatorname{Max}_d(G)$ and so $\mathfrak{d}(P) = M = \mathfrak{d}(Q)$. By hypothesis, $P = Q$. Therefore, the convex ℓ-subgroup generated by distinct minimal primes contains a weak order unit. Consequently, G is a lamron ℓ-group.

(2) implies (1). Suppose G is a lamron ℓ-group and let $\mathfrak{d}(P) = \mathfrak{d}(Q)$ for $P, Q \in \operatorname{Min}(G)$. Then
$$P \bigvee Q \subseteq \mathfrak{d}(P) \bigvee \mathfrak{d}(Q) = \mathfrak{d}(P).$$

Since $\mathfrak{d}(P)$ does not contain any weak order units and G is a lamron ℓ-group, $P = Q$.

(2) implies (3). In the case that G is a lamron ℓ-group, $\mathrm{Min}(G)^{-1}$ is a compact Hausdorff space. The map $\mathfrak{d} : \mathrm{Min}(G)^{-1} \longrightarrow \mathrm{Max}_d(G)$ is a continuous bijection between compact Hausdorff spaces, therefore a homeomorphism.

(3) implies (1). This is obvious. □

Remark 5.8. To our knowledge, Theorem 5.7 is new in the the theory of ℓ-groups. For archimedean uniformly complete vector lattices, this was proved in Theorem 5.1 of [27]. The authors were not aware of the importance of the inverse topology, but instead were interested in properties such as the σ-interpolation property. They also show that in their set-up, the map $\mathfrak{d}$ is a bijection if and only if $\mathrm{Max}_d(G)$ is an F-space. This is not true in general. For example, for a compact zero-dimensional Hausdorff space X, the $\mathbf{W}$-object $G = C(X, \mathbb{Z})$ has the property that $\mathfrak{d}$ is the identity map, yet $\mathrm{Max}_d(G) \cong X$ need not be an F-space.

Next, a classification of when the map λ_u^d is a bijection is in order. To state this theorem, one must have a deeper understanding of the set $\mathrm{Max}_d(G)$, or equivalently, the set of $Z^\sharp(G)$-ultrafilters. Let $p \in YG$ and define

$$\mathcal{F}_p = \{\mathrm{cl}\,\mathrm{int}\, Z : Z \in Z(G) \text{ and } p \in \mathrm{int}\, Z\}.$$

Since, if $p \in \mathrm{int}\, Z_1$ and $p \in \mathrm{int}\, Z_2$ implies

$$p \in \mathrm{int}\, Z_1 \cap \mathrm{int}\, Z_2 = \mathrm{int}(Z_1 \cap Z_2),$$

it follows that for any $B_1, B_2 \in \mathcal{F}_p$, then $B_1 \cap' B_2 \in \mathcal{F}_p$. Therefore, $\mathcal{F}_p$ is a filter base for a filter on $Z^\sharp(G)$.

Lemma 5.9. *Let $\mathcal{U} \in \mathrm{Ult}(Z^\sharp(G))$. Then $\mathcal{F}_p \subseteq \mathcal{U}$ if and only if $p \in \cap\, \mathcal{U}$.*

Proof. First, suppose that $p \in \cap\, \mathcal{U}$, and let $Z' \in Z(G)$ such that $\mathrm{cl}\,\mathrm{int}\, Z' \in \mathcal{U}$. Let $Z \in Z(G)$ such that $p \in \mathrm{int}\, Z$. Now, $p \in \mathrm{cl}\,\mathrm{int}\, Z'$. For any open subset of YG containing p, say O, then $p \in O \cap \mathrm{int}\, Z$ so that $O \cap (\mathrm{int}\, Z \cap \mathrm{int}\, Z') = (O \cap \mathrm{int}\, Z) \cap \mathrm{int}\, Z' \neq \emptyset$. It follows that

$$p \in \mathrm{cl}\,\mathrm{int}(Z \cap Z') = \mathrm{cl}\,\mathrm{int}\, Z \cap' \mathrm{cl}\,\mathrm{int}\, Z'.$$

Therefore, each element of $\mathcal{U}$ meets each element of $\mathcal{F}_p$ in a non-empty set. Since $\mathcal{U}$ is a $Z^\sharp(G)$-ultrafilter, the conclusion is that $\mathcal{F}_p \subseteq \mathcal{U}$.

Second, suppose that $\mathcal{F}_p \subseteq \mathcal{U}$. If $p \notin \cap\, \mathcal{U}$, then there is some $Z \in Z(G)$ such that $p \notin \mathrm{cl}\,\mathrm{int}\, Z$. By the Yosida Embedding Theorem, there is some $0 < f \in G^+$ such that $p \in \mathrm{int}\, Z(f)$ and $Z(f) \cap \mathrm{cl}\,\mathrm{int}\, Z = \emptyset$. But then

$$\emptyset = \mathrm{cl}\,\mathrm{int}\, Z \cap' \mathrm{cl}\,\mathrm{int}\, Z(f) \in \mathcal{U},$$

a contradiction. □

Theorem 5.10. *Let (G, u) be a $\mathbf{W}$-object. The following are equivalent.*

1. *The map $\lambda^d : \mathrm{Max}_d(G) \longrightarrow YG$ is a bijection.*
2. *The map $\lambda^d : \mathrm{Max}_d(G) \longrightarrow YG$ is a homeomorphism.*
3. *For each $p \in YG$, there is a unique $Z^\sharp(G)$-ultrafilter containing $\mathcal{F}_p$.*
4. *For all $f, g \in G$,*

$$\mathrm{cl}\,\mathrm{int}(Z(f) \cap Z(g)) = \mathrm{cl}\,\mathrm{int}\, Z(f) \cap \mathrm{cl}\,\mathrm{int}\, Z(g).$$

(5) *For all* $f, g \in G$, *if* $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$, *then*

$$\operatorname{cl}\operatorname{int} Z(f) \cap \operatorname{cl}\operatorname{int} Z(g) = \emptyset.$$

(6) *The collection* $\mathcal{S}_p = \{\operatorname{cl}\operatorname{int} Z(f) : p \in \operatorname{cl}\operatorname{int} Z(f)\}$ *is a filter.*

Proof. That (1) and (2) are equivalent uses that λ_d is a continuous map between two compact Hausdorff spaces.

(1) is equivalent to (3). This is obvious since, on the one hand, $\lambda^d(\mathcal{U}) = p$ if and only if $p \in \cap\,\mathcal{U}$, and on the other hand, a $Z^\sharp(G)$-ultrafilter $\mathcal{U}$ satisfies $p \in \cap\,\mathcal{U}$ if and only if $\mathcal{F}_p \subseteq \mathcal{U}$.

(4) implies (5). Recall that

$$\operatorname{cl}\operatorname{int} Z(f) \cap' \operatorname{cl}\operatorname{int} Z(g) = \operatorname{cl}\operatorname{int}(Z(f) \cap Z(g)).$$

Thus, if $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$, then

$$\begin{aligned} \operatorname{cl}\operatorname{int} Z(f) \cap \operatorname{cl}\operatorname{int} Z(g) &= \operatorname{cl}\operatorname{int} Z(f) \cap' \operatorname{cl}\operatorname{int} Z(g) \\ &= \operatorname{cl}\operatorname{int}(Z(f) \cap Z(g)) \\ &= \operatorname{cl}\ \emptyset \\ &= \emptyset \end{aligned}$$

(5) implies (4). Clearly, $\operatorname{cl}\operatorname{int} Z(f) \cap' \operatorname{cl}\operatorname{int} Z(g) \subseteq \operatorname{cl}\operatorname{int} Z(f) \cap \operatorname{cl}\operatorname{int} Z(g)$. Suppose

$$p \in \operatorname{cl}\operatorname{int} Z(f) \cap \operatorname{cl}\operatorname{int} Z(g) \text{ and } p \notin \operatorname{cl}\operatorname{int}(Z(f) \cap Z(g)).$$

By the Yosida Embedding Theorem, there is some $Z \in Z(G)$ such that $p \in \operatorname{int} Z$ and $Z \cap \operatorname{cl}\operatorname{int}(Z(f) \cap Z(g)) = \emptyset$. So in particular, we are now in position to apply the hypothesis of (5). Now,

$$p \in \operatorname{cl}\operatorname{int}(Z \cap Z(f)) \text{ and } p \in \operatorname{cl}\operatorname{int}(Z \cap Z(g)).$$

Applying the hypothesis yields, $p \in \operatorname{cl}\operatorname{int}(Z \cap Z(f)) \cap \operatorname{cl}\operatorname{int}(Z \cap Z(g)) = \operatorname{cl}\operatorname{int}(Z \cap Z(f) \cap Z \cap Z(g))$. However, $\operatorname{cl}\operatorname{int}(Z \cap Z(f) \cap Z(g)) \subseteq \operatorname{cl}\operatorname{int} Z \cap \operatorname{cl}\operatorname{int}(Z(f) \cap Z(g)) = \emptyset$, a contradiction.

(4) implies (3). Let $p \in YG$ and suppose that $\mathcal{F}_p \subseteq \mathcal{U}_1$ and $\mathcal{F}_p \subseteq \mathcal{U}_2$ for $\mathcal{U}_1, \mathcal{U}_2 \in \operatorname{Ult}(Z^\sharp(G))$. If $\mathcal{U}_1 \neq \mathcal{U}_2$, then choose $Z_1 \in Z(G)$ such that $\operatorname{cl}\operatorname{int} Z_1 \in \mathcal{U}_1 \smallsetminus \mathcal{U}_2$. Then there is a $Z_2 \in Z(G)$ such that $\operatorname{cl}\operatorname{int} Z_2 \in \mathcal{U}_2$ and

$$\operatorname{cl}\operatorname{int} Z_1 \cap' \operatorname{cl}\operatorname{int} Z_2 = \emptyset.$$

Applying Lemma 5.9, we gather that $p \in \bigcap \mathcal{U}_1$ and $p \in \bigcap \mathcal{U}_2$. Therefore,

$$\begin{aligned} p &\in \operatorname{cl}\operatorname{int} Z_1 \cap \operatorname{cl}\operatorname{int} Z_2 \\ &= \operatorname{cl}\operatorname{int} Z_1 \cap' \operatorname{cl}\operatorname{int} Z_2 \\ &= \emptyset, \end{aligned}$$

where the first equality stems from (4). This contradiction means that $\mathcal{U}_1 = \mathcal{U}_2$, and hence (3) is true.

(3) implies (5). Let $f, g \in G$ satisfy $\operatorname{int} Z(f) \cap \operatorname{int} Z(g) = \emptyset$, and suppose that there is a $p \in \operatorname{cl}\operatorname{int} Z(f) \cap \operatorname{cl}\operatorname{int} Z(g)$. Then

$$\begin{aligned}\operatorname{cl}\operatorname{int} Z(f) \cap' \operatorname{cl}\operatorname{int} Z(g) &= \operatorname{cl}\operatorname{int}(Z(f) \cap Z(g)) \\ &= \operatorname{cl}(\operatorname{int}(Z(f) \cap \operatorname{int} Z(g)) \\ &= \emptyset\end{aligned}$$

Let $\mathcal{U} \in \operatorname{Ult}(Z^\sharp(G))$ be the unique ultrafilter containing $\mathcal{F}_p$. Take an element of $\mathcal{F}_p$, say $\operatorname{cl}\operatorname{int} Z$ with $p \in \operatorname{int} Z$, then since $p \in \operatorname{cl}\operatorname{int} Z(f)$ it follows that $\operatorname{int} Z \cap \operatorname{int} Z(f) \neq \emptyset$ and so

$$\begin{aligned}\operatorname{cl}\operatorname{int} Z(f) \cap' \operatorname{cl}\operatorname{int} Z &= \operatorname{cl}\operatorname{int}(Z(f) \cap Z) \\ &= \operatorname{cl}(\operatorname{int} Z(f) \cap \operatorname{int} Z) \\ &\neq \emptyset\end{aligned}$$

This means that the $Z^\sharp(G)$-filter generated by $\mathcal{F}_p$ and $\operatorname{cl}\operatorname{int} Z(f)$ is proper, and thus contained in a $Z^\sharp(G)$-ultrafilter. This means that $\operatorname{cl}\operatorname{int} Z(f) \in \mathcal{U}$. A similar argument yields that $\operatorname{cl}\operatorname{int} Z(g) \in \mathcal{U}$. However, this cannot be since these two elements meet at $\emptyset$. Consequently,

$$\operatorname{cl}\operatorname{int} Z(f) \cap \operatorname{cl}\operatorname{int} Z(g) = \emptyset.$$

(3) implies (6). Clearly, and in general, any $Z^\sharp(G)$-ultrafilter containing $\mathcal{F}_p$ must contain only elements (of the form $\operatorname{cl}\operatorname{int} Z(f)$) which contain p. Now, let $\mathcal{U}$ be the unique $Z^\sharp(G)$-ultrafilter so that $\{p\} = \bigcap \mathcal{U}$. Thus, $\mathcal{U} \subseteq \mathcal{S}_p$. As was just pointed out in the proof of (3) implies (5), if $p \in \operatorname{cl}\operatorname{int} Z(f)$, then there is some $Z^\sharp(G)$-ultrafilter, say $\mathcal{V}$, such that $\mathcal{F}_p \subseteq \mathcal{V}$. By uniqueness, $\operatorname{cl}\operatorname{int} Z(f) \in \mathcal{V} = \mathcal{U}$. Therefore, $\mathcal{U} = \mathcal{S}_p$.

(6) implies (3). If $\mathcal{S}_p$ is an filter, then it must be a $Z^\sharp(G)$-ultrafilter and the unique one containing $\mathcal{F}_p$.

□

Remark 5.11. It is known that $\operatorname{Max}_d(C(X))$ is always a quasi F-space, and that λ_d is a bijection if and only if X is a quasi F-space. Condition (4) of Theorem 5.10 is saying that for a **W**-object, the finite infimum in $Z^\sharp(G)$ is, in fact, intersection. This appears to us to be the best possible generalization of a quasi F-space to the Yosida space of an arbitrary **W**-object. This characterization of quasi F-spaces is given in [25] Theorem 2.14 (b) (ii).

6. Applications to $C(X)$

We consider the map $\mathfrak{d} : \operatorname{Max}_d(C(X)) \longrightarrow \beta X$, which of course is the quasi F-cover of βX. There are some classical types of topological spaces and covers that arise in the study of $C(X)$ and we investigate when $\operatorname{Max}_d(C(X))$ is of one of these kinds of spaces.

Definition 6.1. Recall the following classification for a Tychonoff space X.

(ED) X is called extremely disconnected if the closure of every open subset of X is clopen.
(BD) X is called basically disconnected if the closure of every cozero-set of X is clopen.
(U) X is called a U-space if it is a strongly zero-dimensional F-space.

Every ED-space is BD, and every BD-space is a U-space. A compact Hausdorff space which is extremelly (basically) disconnected is known as a (σ-)Stone space.

Theorem 6.2. [20, Propositions 2.1 and 2.4] *Let X be a Tychonoff space. The following are equivalent.*

(1) $\mathrm{Max}_d(C(X))$ *is a Stone space.*
(2) $\mathrm{Min}(C(X))^{-1}$ *is a Stone space.*
(3) $\mathrm{Min}(C(X))$ *is a Stone space.*
(4) X *is fraction dense.*
(5) *Every regular closed set is the closure of cozero-set.*
(6) $\mathcal{R}(X) = \mathrm{cl}\,\mathrm{coz}(X) = Z^\sharp(X)$.
(7) βX *is fraction dense.*
(8) $QF\beta X = \mathcal{E}(\beta X)$.

Proof. Proofs for the items (4), (5), and (7) are in [20]. Clearly, (5) and (6) are equivalent. That items (2), (3), and (4) are equivalent can be found in [34, Theorem 7.10]. The following reference should also be mentioned: [25, Lemma 3.20]

So assume that $\mathrm{Max}_d(C(X))$ is a Stone space. Then in particular, $C(X)$ is lamron and so $\mathrm{Min}(C(X))^{-1}$ and $\mathrm{Max}_d(C(X))$ are homeomorphic. Therefore, $\mathrm{Min}(C(X))^{-1}$ is a Stone space, i.e. (2) is true. Conversely, a fraction dense space is complemented and hence lamron so that $\mathrm{Min}(C(X))^{-1}$ and $\mathrm{Max}_d(C(X))$ are homeomorphic, whence $\mathrm{Max}_d(C(X))$ is a Stone space. □

Theorem 6.3. *Let X be a Tychonoff space. The following are equivalent.*

(1) $\mathrm{Max}_d(C(X))$ *is basically disconnected.*
(2) $\mathrm{Min}(C(X))^{-1}$ *is a σ-Stone space.*
(3) $\mathrm{Min}(C(X))$ *is a σ-Stone space.*
(4) X *is cozero-complemented.*
(5) $C(X)$ *is a complemented ℓ-group.*
(6) $Z^\sharp(X)$ *is a boolean subalgebra of* $\mathcal{R}(X)$.
(7) βX *is cozero-complemented.*
(8) $BD(\beta X)^2 = QF(\beta X)$.

Proof. The proof of theorem is similar to the previous proof. In either case of item (1), (2), or (3) $C(X)$ is complemented and thus $\mathrm{Max}_d(C(X))$ and $\mathrm{Min}(C(X))^{-1}$ are homeomorphic. If (4) holds, then that $\mathrm{Min}(C(X))$ is a Stone space is an application of [23, Theorem 4.5]. See [25, Theorem 2.16] for a proof that (4) and (8) are equivalent. Two other important references are [31] and [26]. □

Remark 6.4. For our final result recall that in [34] the author classified when $\mathrm{Min}(C(X))^{-1}$ is a boolean space, that is a compact zero-dimensional Hausdorff space. The underlying ℓ-group theoretic condition is that of a weakly cozero-complemented ℓ-group: if whenever $a, b \in G^+$ with $a \wedge b = 0$, then there is a complementary pair $0 \leq x, y$ such that $a \leq x$ and $b \leq y$. This was first looked at in [34] for $C(X)$ and then for general ℓ-groups in [30], specifically Theorem 2.13.

Theorem 6.5. *Let X be a Tychonoff space. The following are equivalent.*

(1) $\mathrm{Max}_d(C(X))$ *is a U-space.*
(2) $\mathrm{Min}(C(X))^{-1}$ *is a U-space.*

[2] $BD(X)$ is the basically disconnected cover of Vermeer [36]

(3) $\mathrm{Min}(C(X))^{-1}$ *is a boolean space.*
(4) X *is weakly cozero-complemented.*
(5) βX *is weakly cozero-complemented.*

Proof. In all three cases (1), (2), and (3), $C(X)$ is lamron and so $\mathrm{Max}_d(C(X))$ and $\mathrm{Min}(C(X))^{-1}$ are homeomorphic. Thus, if (1), then (2). Clearly, if (2), then (3). If (3), then $\mathrm{Max}_d(C(X))$ is boolean, but it also is an F-space.

As just mentioned (3) and (4) are equivalent by an application of [30, Theorem 2.13]. □

Remark 6.6. All of the results in this section have counterparts for any **W**-object (G, u) for which $\mathrm{cl}\,\mathrm{coz}(G) = \mathrm{cl}\,\mathrm{coz}(YG)$, i.e. $Z^\sharp(G) = Z^\sharp(YG)$. If this happens, then $\mathrm{Max}_d(G)$ is a quasi F-space as it is homeomorphic to the quasi F-cover. Therefore, if G is lamron, then $\mathrm{Min}(G)^{-1}$ and $\mathrm{Max}_d(G)$ are homeomorphic, and so any topological statement about the space $\mathrm{Max}_d(G)$ will have a corresponding statement about $\mathrm{Min}(G)^{-1}$. For example, if A is a uniformly complete vector lattice with weak unit, then $\mathrm{cl}\,\mathrm{coz}(A) = \mathrm{cl}\,\mathrm{coz}(YA)$ and so it is true that $\mathrm{Max}_d(A)$ is a quasi F-space; Theorem 3.2 of [27] can be shortened by simply pointing out that $\mathrm{Max}_d(A)$ and $QF(YA)$ are homeomorphic. We leave it to the reader to show that their proof can be modified for any **W**-object with $\mathrm{cl}\,\mathrm{coz}(G) = \mathrm{cl}\,\mathrm{coz}(YG)$.

Definition 6.7. The ℓ-group G is said to satisfy the *countable polar condition* if for any countable subset of G^+, say $S = \{g_n\}_{n\in\mathbb{N}}$, there is a $g \in G^+$ such that $S^\perp = g^\perp$. (For rings, Henriksen and Jerison [23] called this the countable annihilator condition.)

Example 6.8. Not all ℓ-groups satisfy the countable polar condition. In fact, for a compact zero-dimensional Hausdorff space X, $C(X, \mathbb{Z})$ satisfies the countable polar condition if and only if X is basically disconnected.

Lemma 6.9. *Let* (G, u) *be a* **W***-object and* $S \subseteq G^+$*. If there is some* $g \in G^+$ *such that* $S^\perp = g^\perp$*, then*

$$\mathrm{cl} \bigcup_{g_i \in S} \mathrm{coz}(g_i) = \mathrm{cl}\,\mathrm{coz}(g).$$

And conversely.

Proof. Let $p \in \mathrm{cl} \bigcup_{g_i\in S} \mathrm{coz}(g_i)$. If $p \notin \mathrm{cl}\,\mathrm{coz}(g)$, then there is some $h \in G^+$ such that $h(p) = 1$ and $h(q) = 0$ for all $q \in \mathrm{cl}\,\mathrm{coz}(g)$. Then $h \wedge g = 0$ and so for all $g_i \in S$, $h \wedge g_i = 0$. This means that for all $g_i \in S$ and for all $t \in \mathrm{coz}(g_i)$, $h(t) = 0$. Let $O = h^{-1}((\frac{1}{2}, \infty))$, an open neighborhood of p. Then $O \cap \bigcup \mathrm{coz}(g_i) \neq \emptyset$, a contradiction. Thus, $p \in \mathrm{cl}\,\mathrm{coz}(g)$.

Conversely, let $p \in \mathrm{cl}\,\mathrm{coz}(g)$. If $p \notin \mathrm{cl} \bigcup_{g_i\in S} \mathrm{coz}(g_i)$, then again there is some $h \in G^+$ such that $h(p) = 1$ and $h(q) = 0$ for all $q \in \mathrm{cl} \bigcup_{g_i\in S} \mathrm{coz}(g_i)$. It follows that $h \in S^\perp$, whence $h \in g^\perp$. So $\mathrm{coz}(g) \subseteq Z(h)$, whence $p \in Z(g)$, a contradiction. □

Proposition 6.10. *The* **W***-object* (G, u) *has the countable polar condition if and only if* $\mathrm{cl}\,\mathrm{coz}(G) = \mathrm{cl}\,\mathrm{coz}(YG)$.

Proof. Recall Lemma 2.2 of [2] which states and proves that every cozero-set in YG is a countable union of G-cozero-sets. So starting with a $C \in \mathrm{coz}(YG)$, there is a countable subset of G^+, say $S = \{g_n : n \in \mathbb{N}\}$, such that

$$C = \bigcup_{n\in\mathbb{N}} \mathrm{coz}(g_n).$$

If G has the c.p.c., then there is some $g \in G^+$ such that $S^\perp = g^\perp$. It follows, by Lemma 6.9, that

$$\operatorname{cl} C = \operatorname{cl} \bigcup_{n\in\mathbb{N}} \operatorname{coz}(g_n) = \operatorname{cl}\operatorname{coz}(g).$$

Therefore, $\operatorname{cl}\operatorname{coz}(G) = \operatorname{cl}\operatorname{coz}(YG)$.

Conversely, suppose that $\operatorname{cl}\operatorname{coz}(G) = \operatorname{cl}\operatorname{coz}(YG)$ and let $S = \{g_n : n \in \mathbb{N}\} \subseteq G^+$. Set $C = \bigcup_{n\in\mathbb{N}} \operatorname{coz}(g_n)$, a cozero-set of YG. By hypothesis, there is some $g \in G^+$ such that $\operatorname{cl} C = \operatorname{cl}\operatorname{coz}(g)$. Then $S^\perp = g^\perp$ and so G has the c.p.c. □

Acknowledgements. The authors would like to thank the Harriet L. Wilkes Honors College of Florida Atlantic University for hosting the first author during her sabbatical year.

References

[1] Azarpanah, F. and A.R.Salehi. *Ideal structure of the classical ring of quotients of* $C(X)$, Top. Appl. **209** (2016) 170-180

[2] Ball, R. N. and A. W. Hager. *Archimedean kernel distinguishing extensions of archimedean* ℓ*-groups with weak order unit*, Indian J. Math. **29** (1987) 351–368.

[3] Banaschewski, B. *On Wallman's method of compactification.* Math. Nachr **27** (1963) 105-114.

[4] Bhattacharjee, P. and K. Drees. *Filters of* $\operatorname{Coz}(X)$, Categ. Gen. Algebr. Struct. Appl. (to appear).

[5] Bhattacharjee, P., K. Drees, and W. Wm. McGovern. *Extensions of commutative rings*, Top. Appl. **158** (2011) 1802–1814.

[6] Bhattacharjee, P. and W. Wm. McGovern. *When* $\operatorname{Min}(A)^{-1}$ *is Hausdorff*, Comm. Alg. **41** (2013) 1–10.

[7] Bhattacharjee, P. and W. Wm. McGovern. **W**, in preparation.

[8] Bhattacharjee, P. and W. Wm. McGovern. *Lamron* ℓ*-groups*, Quast. Math. (to appear).

[9] Bigard, A., K. Keimel, and S. Wolfenstein. Groupes et Anneaux Réticulés. Lecture Notes in Mathematics, Vol. **608**. Springer-Verlag, Berlin-New York, 1977.

[10] Conrad, P. *The essential closure of an Archimedean lattice-ordered group*, Duke Math. J. **38** (1971) 151-160.

[11] Conrad, P. and J. Martìnez. *Complemented lattice-ordered groups*, Indag. Mathem., N.S. **1(3)** (1990) 281–298.

[12] Darnel, M. R. Theory of Lattice-ordered Groups. Monographs and Textbooks in Pure and Applied Mathematics 187, Marcel Dekker, 1995.

[13] de Pagter, B. *On* z*-ideals and* d*-ideals in Riesz spaces. III*, Nederl. Akad. Wetensch. Indag. Math. **43(4)** (1981) 409422.

[14] Dow, A., M. Henriksen, R. Kopperman, and J. Vermeer. *The space of minimal prime ideals of* $C(X)$ *need not be basically disconnected*, Proc. Amer. Math. Soc. **104 (1)** (1988) 317-320.

[15] Dow, A., M. Henriksen, R. Kopperman, R.G. Woods. *Topologies and cotopologies generated by sets of functions*, Houston J. Math. **19 (4)** (1993) 551-586.

[16] Engelking, R. General Topology, Sigma Series in Pure Mathematics, Vol. **6**, Heldermann Verlag, Berlin, 1989.

[17] Gillman, L. and M. Jerison. Rings of Continuous Functions, Graduate Texts in Mathametics, Vol. **43**, Springer Verlag, Berlin-Heidelberg-New York, 1976.

[18] Hager, A. W., C. M. Kimber, and W. Wm. McGovern, *Least integer closed groups*, Ordered Algebraic Structures, Dev. Math., 7, Kluwer Acad. Publ., Dordrecht, (2002) 245–260.

[19] Hager, A. W., C. M. Kimber, and W. Wm. McGovern. *Weakly least integer closed groups*, Rend. Circ. Mat. Palermo **52 (2)** (2003) 453-480.

[20] Hager, A.W. and J. Martinez. *Fraction-dense algebras and spaces*, Canad. J. Math. **45 (5)** (1993) 977-996.

[21] Hager, A. W. and W. Wm. McGovern. *The projectable hull of an archimedean* ℓ*-group with weak unit*, Quast. Math. (to appear).

[22] Hager, A. W. and L. C. Robertson. *Representing and ringifying a Riesz space*, Symposia Mathematica, Vol. XXI, (1977) 411-431.
[23] Henriksen, M. and M. Jerison. *The space of minimal prime ideals of a commutative ring*, Trans. A.M.S. **115** 110–130.
[24] Henriksen, M., S. Larsen, J. Martìnez, and R.G. Woods. *Lattice-ordered algebras that are subdirect products of valuation domains*, Trans. Amer. Math. Soc. **345 (1)** (1994) 195-221.
[25] Henriksen, M., J. Vermeer, and R. G. Woods. *Quasi F-covers of Tychonoff spaces*, Trans. Amer. Math. Soc. **303 (2)** (1987) 779-803.
[26] Henriksen, M. and R. G. Woods. *Cozero complemented spaces; when the space of minimal prime ideals of a $C(X)$ is compact*, Topology Appl. **141** (2004) 147-170.
[27] Huisjmans, C.B. and B. de Pagter. *Maximal d-ideals in a Riesz space*, Canad. J. Math. **35 (6)** (1983) 1010-1029.
[28] Huisjmans, C.B. and B. de Pagter. *On z-ideals and d-ideals in Riesz spaces. II*, Nederl. Akad. Wetensch. Indag. Math. **42 (4)** (1980) 391-408.
[29] Knox, M. L., R. Levy, W. Wm. McGovern, J. Shapiro. *Generalizations of complemented rings with applications to rings of functions*, J. Algebra Appl. **8** (2009) 17-40.
[30] Knox, M. L. and W. Wm. McGovern. *Feebly projectable ℓ-groups*, Alg. Univ., **62 (1)** (2009) 91–112.
[31] Levy, R. and J. Shapiro. *Rings of quotients of rings of functions*, **146/147** (2005) 253-265.
[32] Luxemburg, W. A. J., and A. C. Zaanen. Riesz Spaces. Vol. I. North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971.
[33] Martinez, J. and E.R. Zenk *When an algebraic frame is regular*, Algebra Universalis **50** (2003) 231-257.
[34] McGovern, W. Wm. *Neat rings*, Journal of Pure and Applied Algebra **205** (2006) 243–265.
[35] Porter, J. R. and R.G. Woods, R. Extensions and Absolutes of Hausdorff Spaces. Springer-Verlag, New York, 1988.
[36] Vermeer, J. *The smallest basically disconnected preimage of a space*, Topology Appl. **17** (1984) 217-232.

[1]Penn State Behrend, School of Science, Erie, PA 16563
E-mail address: pxb39@psu.edu

[2]H. L. Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458
E-mail address: warren.mcgovern@fau.edu