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MARTÍNEZ FRAMES

PAPIYA BHATTACHARJEE, LEE KLINGLER, AND WARREN WM. MCGOVERN

Abstract. We call a frame *Martínez* if it is an algebraic frame with FIP in which every element is a d-element. The study of Martinez frames and the d-operator in this article has led to a better understanding of the q-nucleus defined in [15]. We generalize the construction of the q-nucleus to arbitrary sets of primes and investigate this operator from a topological perspective on the prime spectrum.

1. Introduction

This article is aimed at generalizing the work done by the current authors in [7] to the theory of algebraic frames. In [7], the authors constructed rings of the form $A+B$ satisfying certain properties on prime ideals. The works of Jorge Martínez and Eric Zenk were instrumental in the investigation of these properties: [16] and [17]. With the passing of J. Martínez in 2020, we are privileged to honor his memory by investigating a class of algebraic frames with FIP defined by the d-operator, calling these Martínez frames.

Here is the general arc of the paper. We begin by investigating the class of Martínez frames and the d-operator. The results will show a connection to the study of prime elements. This then will allow us to pivot and study a more general construction applied to arbitrary collections of primes; connections to the patch topology and Skula topology will be made. Finally, we will focus on one specific collection of prime elements which is the study of the q-operator as developed by [15] and later in [6].

We assume the reader is familiar with the theory of frames. Our main reference for the theory of frames is the book of Picado and Pultr [20]. We would like to remind the reader of some useful definitions and notations from the theory of algebraic frames. The readers familiar with [18, Definition & Remark 1.1] should feel free to skip Definition 1.1.

Definition 1.1. Let L be a frame.

1. Two elements, say $a, b \in L$, are said to be *disjoint* if $a \wedge b = 0$.
2. The *polar* of $a \in L$ is the largest element of L disjoint from a; we denote the polar of a by $a^{\perp}$. The polar of a is also known as the pseudo-complement of a.

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3. If $a^\perp = 0$, then a is said to be *dense*.
4. An element $x \in L$ is called *compact* if whenever $x \leq \bigvee S$, then $x \leq \bigvee S'$ for some finite subset $S' \subseteq S$. The frame is called *compact* if the element $1 \in L$ is compact. We denote the collection of compact elements by $\mathfrak{K}(L)$. Notice that $\mathfrak{K}(L)$ is a join semi-lattice, that is, the join of two compact elements is again compact.
5. A frame is called *algebraic* if every element is a join of compact elements.
6. A frame L is said to satisfy FIP (short for *the finite intersection property*) if the (nonempty) finite meet of compact elements is again compact. FIP is equivalent to the statement that the collection of compact elements $\mathfrak{K}(L)$ is a sub-lattice of L. In the general theory of lattices, these are also known as *arithmetic frames* (see [12], and other authors (including some of the current ones) have called these M-frames. A compact algebraic frame satisfying FIP is known as a *coherent* frame.
7. We shall refer to a dense compact element by the term *unit*. Frames satisfying FIP need not possess units.
8. An element $p < 1$ in L is called *prime* if for all $a, b \in L$, $a \wedge b \leq p$ implies that either $a \leq p$ or $b \leq p$. The set of prime elements of L is denoted by $\mathrm{Spec}(L)$.
9. Recall that each frame has a well-defined Heyting implication. Namely, $a \to b$ is defined to be $\bigvee\{x \in L : (x \wedge a) \leq b\}$. Obviously, $a \to 0 = a^\perp$.
10. The frame L is said to be *spatial* if it is frame isomorphic to the frame of open sets of some topological space. Every algebraic frame is spatial. There is a well-known theorem that a frame is spatial if and only if every element is a meet of prime elements.
11. An element $x \in L$ for which $x \vee x^\perp = 1$ is called *complemented*. The frame L is called *zero-dimensional* if every element is a join of complemented elements.
12. An algebraic frame L is called *projectable* if for every compact element $c \in \mathfrak{K}(L)$, $c^{\perp\perp}$ is complemented.
13. Let $h : L \to M$ be a frame homomorphism. h is *dense* if for any $a \in L$, $h(a) = 0$ implies $a = 0$. A dense onto frame homomorphism satisfies the following: for each $a \in L$, $h(a^\perp) = h(a)^\perp$.

Example 1.2. We provide the reader with an inventory of examples of algebraic frames.

1) Given a topological space (X, τ), the topology as a lattice is a frame. It is algebraic precisely when X has a base of compact open subsets, e.g. the prime spectrum of a commutative ring with identity.

2) For a commutative ring with identity, say R, its collection of radical ideals ordered by inclusion is a coherent frame. According to the celebrated theorems of Hochster [13] and Banaschewski [4], every coherent frame is the frame of radical ideals for some commutative ring with identity.

3) For any lattice-ordered group G, its collection of convex ℓ-subgroups ordered by inclusion is an algebraic frame satisfying FIP which need not possess a unit. These arithmetic frames have a property known as disjointification. Recently, F. Wehrung [24] has shown that not every algebraic frame satisfying FIP is the frame of convex ℓ-subgroups for some abelian ℓ-group.

4) If one takes a bounded distributive lattice K, then the collection of ideals of K, ordered by inclusion, is a coherent frame whose compact elements is lattice isomorphic to K.

Definition 1.3. Let L be a spatial frame. Recall that the hull-kernel topology on $\mathrm{Spec}(L)$ is the topology of sets of the form $U(x) = \{p \in \mathrm{Spec}(L) : x \nleq p\}$ for each $x \in L$. In fact, spatiality is characterized by observing that the assignment $x \mapsto U(x)$ is a frame isomorphism of L onto the frame of open subsets of $\mathrm{Spec}(L)$.

The operator $U(\cdot)$ satisfies the following properties: i) $U(a) \subseteq U(b)$ if and only if $a \le b$, ii) $U(a \wedge b) = U(a) \cap U(b)$, iii) $U(a \vee b) = U(a) \cup U(b)$.

Now, when L is algebraic we can consider the collection $\mathcal{B} = \{U(a) : a \in \mathfrak{K}(L)\}$. The collection $\mathcal{B}$ is closed under finite unions and forms a base for the hull-kernel topology on $\mathrm{Spec}(L)$. Furthermore, when L is an algebraic frame satisfying FIP the collection $\mathcal{B}$ is also closed under finite intersections. We let $V(x)$ denote the set-theoretic complement of $U(x)$ in $\mathrm{Spec}(L)$.

For an algebraic frame satisfying FIP, say L, we may define the patch topology on $\mathrm{Spec}(L)$ as the topology generated by the base of open sets $\{U(a) \cap V(b) : a, b \in \mathfrak{K}(L)\}$.

Notation 1.4. We shall have a need to distinguish between the closures of sets with regards to the hull-kernel and patch topologies. Let $S \subseteq \mathrm{Spec}(L)$. First, we shall use

$$\mathrm{cl}\, S$$

to denote the closure of S in $\mathrm{Spec}(L)$ with respect to the hull-kernel topology, and we shall use

$$\mathrm{cl}_p\, S$$

to denote the closure of S in $\mathrm{Spec}(L)$ with respect to the patch topology.

As is customary we let $\mathrm{Min}(L)$ and $\mathrm{Max}(L)$ denote the collection of minimal prime elements and maximal elements of L, respectively. We should point out that it is possible for $\mathrm{Max}(L) = \emptyset$. A nice consequence of FIP is the following Lemma on Ultrafilters (see [16, Lemma 2.2]) which characterizes the minimal prime elements of an algebraic frame satisfying FIP.

Lemma 1.5 (Lemma on Ultrafilters)**.** *Suppose L is an algebraic frame satisfying FIP and let $p \in \mathrm{Spec}(L)$. Then p is a minimal prime element if and only if*

$$F_p = \{c \in \mathfrak{K}(L) : c \nleq p\}$$

is an ultrafilter on $\mathfrak{K}(L)$. In this case,

$$p = \bigvee_{c \in F_p} c^{\perp}.$$

Corollary 1.6. *Suppose L is an algebraic frame satisfying FIP and let $p \in \mathrm{Min}(L)$. For any $c \in \mathfrak{K}(L)$, exactly one of $c \nleq p$ or $c^{\perp} \nleq p$.*

We now turn to d-elements. The abstract notion of a d-element was defined in [16], and the authors gave a thorough treatment of the topic including the origins of the notion of a d-ideal from the theory vector lattices. In particular, they showed that the d-operator is a nucleus on a frame. Recently, in [8] Dube and Sithole studied the sublocale dL and functorial properties of the operator as well as providing some novel characterizations of d-elements. It should also be noted that T. Dube has done some interesting work on d-elements in the frame of radical ideals of a commutative ring.

Definition 1.7. Suppose L is an algebraic frame. We call the element $x \in L$ a *d-element* if whenever $c \leq x$ and $c \in \mathfrak{K}(L)$, then $c^{\perp\perp} \leq x$.

It follows from Corollary 1.6 that every minimal prime element of an algebraic frame satisfying FIP is a d-element. Also, notice that for any $x \in L$, the element $x^{\perp\perp}$ is a d-element. The infimum of any collection of d-elements is again a d-element, which leads to the definition of the d-operator: $d : L \to L$ by taking $x \in L$ to $d(x)$ the least d-element above x. Note that if $c \in \mathfrak{K}(L)$, then $d(c) = c^{\perp\perp}$. Hence, a compact c is a unit precisely when $d(c) = 1$.

It is a consequence of the inequality $a^{\perp\perp} \vee b^{\perp\perp} \leq (a \vee b)^{\perp\perp}$, that

$$d(x) = \bigvee\{c^{\perp\perp} : c \in \mathfrak{K}(L) \text{ and } c \leq x\}.$$

When L is an algebraic frame satisfying FIP, then operator d is the prototypical example of a special kind of nucleus on L. In the third section we shall discuss nuclei in general. The reader is urged to see [16] for proofs as well as a more general discussion on the topic.

2. Martínez Frames

As we already mentioned, we call an algebraic frame satisfying FIP a *Martínez frame* if $dL = L$. In this section we shall assume that all frames are algebraic and satisfy FIP; it should be apparent why we do so. Our first result of the section characterizes Martínez frames through an algebraic lens. Following our first result we then proceed to show that there is a way of characterizing Martínez frames through topological means. The collection of prime d-elements of L shall be denoted by $\mathrm{Spec}_d(L)$.

Theorem 2.1. *Suppose L is an algebraic frame satisfying FIP. The following statements are equivalent.*

1. *L is a Martínez frame, that is, every element is a d-element.*
2. $\mathrm{Spec}(L) = \mathrm{Spec}_d(L)$.
3. *For all $c \in \mathfrak{K}(L)$, $c = c^{\perp\perp}$.*
4. *Every compact element is a d-element.*

Proof. 1. $\Rightarrow$ 2. If L is a Martínez frame, then $dL = L$ and so $\mathrm{Spec}(L) = \mathrm{Spec}(dL)$. Since $\mathrm{Spec}(dL) = \mathrm{Spec}_d(L)$ the result follows.

2. $\Rightarrow$ 3. Let $c \in \mathfrak{K}(L)$, then $d(c) = c^{\perp\perp}$. Since an algebraic frame satisfying FIP is spatial it follows that each compact is a meet of prime elements. By hypothesis, each prime is a d-element and since the infimum of d-elements is again a d-element, we gather that each compact element is a d-element. Therefore, $c = c^{\perp\perp}$.

3. $\Rightarrow$ 1. Obvious.

1. $\Rightarrow$ 4. Patent.

4. $\Rightarrow$ 1. If every element compact element is a d-element, then in particular for any $c \in \mathfrak{K}(L)$, $c \leq c$ implies $c^{\perp\perp} \leq c$, whence $c = c^{\perp\perp}$. Thus, if $x \in L$ and $c \leq x$ with $c \in \mathfrak{K}(L)$, then $c^{\perp\perp} = c \leq x$. It follows that every element of L is a d-element, i.e. $dL = L$.

□

Condition 3. of Theorem 2.1 can be used to construct examples of Martínez frames. It is a fact that an algebraic frame satisfying FIP is zero-dimensional if and only if every compact element is complemented. Thus, a zero-dimensional algebraic frame is projectable. Furthermore, a zero-dimensional algebraic frame with FIP is a Martínez frame. It also follows that an algebraic frame satisfying FIP is zero-dimensional if and only if it is both a Martínez frame and a projectable frame. Obviously, every boolean frame is trivially zero-dimensional.

The article [8] does a thorough job of considering categorical questions surrounding the d-operator and so we urge the interested reader to peruse this paper. In the hopes of supplying more examples of Martínez frames we point out that the direct product of Martínez frames is again a Martínez frame; proof left to the reader. Certain frame homomorphisms will preserve the Martínez condition.

Proposition 2.2. *Suppose $h : L \to M$ is a dense onto frame homomorphism and that both L and M are algebraic and satisfy FIP. If L is a Martínez frame, then so is M.*

Proof. Let $k \in \mathfrak{K}(M)$, then there exists $a \in L$ such that $h(a) = k$. Since L is algebraic,

$$k = h(a) = h(\bigvee c_\alpha) = \bigvee h(c_\alpha),$$

for $c_\alpha \in \mathfrak{K}(L)$. Using the compactness of k there exists some $c \in \mathfrak{K}(L)$ with $c \leq a$, such that $k = h(c)$; hence, $k^{\perp\perp} \leq h(c)^{\perp\perp}$. Because L is a Martínez frame $c = c^{\perp\perp}$. Finally, using that h is dense onto and so $h(a^{\perp\perp} = h(a)^{\perp\perp}$

$$k^{\perp\perp} = h(c)^{\perp\perp} = h(c^{\perp\perp}) \leq h(c) = k.$$

Applying Theorem 2.1, we conclude that M is a Martínez frame. □

Example 2.3. In [7], we supplied a way of constructing commutative rings whose frame of radical ideals are Martínez frames. We called such rings Martínez rings. We used the $A + B$ construction to construct different kinds of Martínez rings. In particular, if $A = \mathbb{Z}$ and $\mathcal{P} = \mathrm{Max}(A)$, then the frame of radical ideals of $A + B$ is a Martínez frame that is not zero-dimensional. In general, the contraction map $h : \mathrm{Rad}(A + B) \to \mathrm{Rad}(A)$ is onto but not dense. Since $\mathrm{Rad}(\mathbb{Z})$ is not a Martínez frame, it follows that the condition that h is dense in Proposition 2.2 is needed.

To understand Martínez frames through a topological lens our next result is pivotal; it generalizes a result about commutative rings and radical ideals [2, Theorem 3.3].

Theorem 2.4. *Suppose L is an algebraic frame satisfying FIP. The closure of $\mathrm{Min}(L)$ in $\mathrm{Spec}(L)$ with respect to the patch topology is $\mathrm{Spec}_d(L)$.*

Proof. Let $p \in \mathrm{cl}_p(\mathrm{Min}(L))$. Let $b \in \mathfrak{K}(L)$ such that $b \leq p$. We want to show that $b^{\perp\perp} \leq p$. If not, then there is some compact $a \leq b^{\perp\perp}$ such that $a \nleq p$. It follows that $p \in U(a) \cap V(b)$. By hypothesis, there exists some $r \in \mathrm{Min}(L)$ such that $r \in U(a) \cap V(b)$. Thus, $b \leq r$. Since r is a d-element we gather that and $b^{\perp\perp} \leq r$, whence $a \leq r$, the desired contradiction. Hence, p is a d-element, i.e. $p \in \mathrm{Spec}_d(L)$.

On the other hand, let $r \in \mathrm{Spec}_d(L)$. Let $r \in U(a) \cap V(b)$ be a basic open neighborhood in the patch topology for some $a, b \in \mathfrak{K}(L)$. So, $a \nleq r$ and $b \leq r$. Since r is a d-element, $b^{\perp\perp} \leq r$. We claim that there exists some $s \in \mathfrak{K}(L)$ such that $s \leq b^{\perp}$ and $s \wedge a \neq 0$. If that is not the case, then $b^{\perp} \leq a^{\perp}$ since L is an algebraic frame. This in turn will imply that $a \leq a^{\perp\perp} \leq b^{\perp\perp} \leq r$, a contradiction.

Since $0 < s \wedge a$ it follows that there is some minimal prime $p \in \mathrm{Min}(L)$ such that $s \wedge a \nleq p$. It follows $a \nleq p$, and so $p \in U(a)$. Since $s \wedge b = 0$ and $s \nleq p$, it follows that $b \leq p$. Consequently, $p \in U(a) \cap V(b)$, and we conclude that $r \in \mathrm{cl}_p \mathrm{Min}(L)$. □

Recall that a *π-base* for a topology is a collection of nonempty open subsets such that any arbitrary nonempty open subset contains some set in the collection. It is prudent that we point the reader in the direction of two resultsto which we owe a debt of gratitude: [25, Theorem 7] and [2, Theorem 4.5].

Theorem 2.5. *Suppose L is an algebraic frame satisfying FIP. The following statements are equivalent.*

1. *L is a Martínez frame.*
2. $\mathrm{Spec}(L) = \mathrm{Spec}_d(L)$.
3. *The set of minimal prime elements is patch-dense in* $\mathrm{Spec}(L)$.
4. *The set $\{U(a) : 0 < a\}$ is a π-base for the patch topology on* $\mathrm{Spec}(L)$.
5. *The set $\{U(a) : 0 < a \in \mathfrak{K}(L)\}$ is a π-base for the patch topology on* $\mathrm{Spec}(L)$.
6. *Distinct compact open subsets of* $\mathrm{Spec}(L)$ *have distinct closures (with respect to the hull-kernel topology).*

Proof. 1. and 2. are equivalent by 2.1.

2. ⇒ 3. By Theorem 2.4, $\mathrm{cl}_p \mathrm{Min}(L) = \mathrm{Spec}_d(L)$. By hypothesis, $\mathrm{Spec}_d(L) = \mathrm{Spec}(L)$. Consequently, the patch closure of $\mathrm{Min}(L)$ is all of $\mathrm{Spec}(L)$.

3. ⇒ 4. Suppose that $\mathrm{Min}(L)$ is patch-dense in $\mathrm{Spec}(L)$ and let $U(a) \cap V(b)$ (with $a, b \in \mathfrak{K}(L)$) be a non-empty basic open set in the patch topology. Without loss of generality we assume that $0 < b$. By hypothesis, there is some $p \in \mathrm{Min}(L)$ so that $p \in U(a) \cap V(b)$. This implies that $a \nleq p$ and $b \leq p$. Knowing that b is non-zero we also know that $0 < p$. Since p is a minimal prime we know that $b^{\perp} \nleq p$. Therefore, by primality, $a \wedge b^{\perp} \nleq p$. Clearly, $0 < a \wedge b^{\perp}$.

Next, observe that $U(a \wedge b^{\perp}) = U(a) \cap U(b^{\perp})$. Since $U(b^{\perp}) \subseteq V(b)$ it follows that $U(a \wedge b^{\perp})$ is a non-empty subset of $U(a) \cap V(b)$.

4. ⇒ 5. For any $0 < a \in L$ we can choose a compact $c \in \mathfrak{K}(L)$ such that $0 < c < a$. it follows that $U(c) \subseteq U(a)$.

5. ⇒ 2. Suppose that the collection of sets of the form $U(c)$ for c compact and nonzero is a π-base for the patch topology on $\mathrm{Spec}(L)$. We aim to show that every prime element is a d-element. To that

end, let $p \in \operatorname{Spec}(L)$ and suppose $c \in \mathfrak{K}(L)$ satisfies $c \leq p$. We would like to show that $c^{\perp\perp} \leq p$. If not, then there is some compact $d \in \mathfrak{K}(L)$ such that $d \leq c^{\perp\perp}$ and $d \nleq p$. Thus, $p \in U(d) \cap V(c)$. By hypothesis, there is some nonzero compact $a \in \mathfrak{K}(L)$ such that $U(a) \subseteq U(d) \cap V(c)$. Since a is nonzero, there is some $q \in \operatorname{Min}(L)$ such that $a \nleq q$. Consequently, $q \in U(a)$ and hence $q \in U(d) \cap V(c)$. Since q is a minimal prime element it is a d-element. Thus, $c^{\perp\perp} \leq q$. But we chose d to be below $c^{\perp\perp}$ and so $d \leq q$, a contradiction.

5. $\Rightarrow$ 6. Suppose that the collection $\{U(a) : 0 < a \in \mathfrak{K}(L)\}$ is a π-base for the patch topology on $\operatorname{Spec}(L)$ and that $U(a)$ and $U(b)$ are distinct compact open subsets (with respect to the hull-kernel topology). Without loss of generality, we shall assume that $a, b \in \mathfrak{K}(L)$ and both are nonzero. Furthermore, we assume that $U(a) \nsubseteq U(b)$. Therefore, there is some $p \in \operatorname{Spec}(L)$ such that $p \in U(a) \smallsetminus U(b)$. In other words, $p \in U(a) \cap V(b)$, a patch basic open set. By hypothesis, there is some nonzero compact $c \in \mathfrak{K}(L)$ such that $U(c) \subseteq U(a) \cap V(b)$. Observe that $U(c) \cap U(b) = \emptyset$. Also, since $0 < c$ there is some minimal prime element $q \in U(c)$, whence $q \in U(a) \cap V(b)$.

Observe that $q \in U(c)$ together with $U(c) \cap U(b)$ implies that $q \notin \operatorname{cl} U(b)$. Obviously, $q \in \operatorname{cl} U(a)$. Therefore, $U(a)$ and $U(b)$ have distinct closures.

6. $\Rightarrow$ 5. Let $U(c) \cap V(d)$ be a nonempty patch basic open subset of $\operatorname{Spec}(L)$; so $c, d \in \mathfrak{K}(L)$. Without loss of generality, we may assume that $U(d) \neq \emptyset$. Observe that $U(c)$ and $U(d)$ are distinct compact open subsets, and thus by hypothesis have distinct closures. Thus, there exists a $p \in \operatorname{Spec}(L)$ for which $p \in \operatorname{cl} U(c) \smallsetminus \operatorname{cl} U(d)$. This in turn, forces there to be some compact element $k \in \mathfrak{K}(L)$ so that $p \in U(k)$ and $U(k) \cap U(d) = \emptyset$. This implies that $U(k) \subseteq V(d)$. We also know that $U(k) \cap U(c) \neq \emptyset$ which means that $0 < k \wedge c$. Since L is satisfies FIP, $k \wedge c \in \mathfrak{K}(L)$. Finally, $U(k \wedge c) \subseteq U(c)$. Whence $\emptyset \neq U(k \wedge c) \subseteq U(c) \cap V(d)$.

□

3. Nuclei Induced by Primes

M. Hochster [13] originally defined and studied the patch topology on a spectral space, and thus the prime spectrum of a commutative ring with identity. Later, Banschewski and Brümmer [3] discussed the patch topology in a more general context. Others including Escardo [10] and Sexton and Simmons [21] were instrumental in reformulating the construction in connection to certain nuclei, and demonstrating a connection between spatial sublocales, the patch topology and the patch frame. Our hope is to reframe (pun intended) the ideas here so that our view of the patch topology described in the next section can be understood in the proper context.

The two special examples of nuclei are the d-nucleus and the q-nucleus. Recall the definition of a nucleus on a frame L as an operator $j : L \to L$ which idempotent, inflationary, and preserves the meet (see [20]). The range of the nucleus j will be denoted by jL instead of $j(L)$. Notice that this is the set of fixed points of j; denoted by $\operatorname{fix}(j)$ in [16] and [17]. It is known that if j is a nucleus on an algebraic frame L, then jL is an algebraic frame whose compact elements are precisely of the form

$j(c)$ for some $c \in \mathfrak{K}(L)$. If L has FIP, then so does jL. In general, the lattice operations on jL do not agree with the lattice operations of L.

Martínez and Zenk [16] showed that the d-operator is a nucleus on an algebraic frame satisfying FIP. Consequently, idempotency yields the following.

Corollary 3.1. *For any algebraic frame satisfying FIP, say L, the frame dL is a Martínez frame.*

The d-nucleus is an example of what is called an inductive nucleus; see [16] for more information on inductive nuclei.

Definition 3.2. The nucleus $j : L \to L$ is called *inductive* if for every $x \in L$,

$$j(x) = \bigvee \{j(c) : c \in \mathfrak{K}(L), c \leq x\}.$$

Remark 3.3. The notion of an inductive nucleus on an algebraic frame can be viewed within a more generalized concept. For any frame L, not necessarily algebraic, a nucleus $j : L \to L$ is called *Scott continuous* if for any upward directed subset D of L,

$$j(\bigvee D) = \bigvee_{d \in D} j(d).$$

Banaschewski [3] called such nuclei *finitary* and Escardo [10] has called these *perfect*. Relating to what we are doing here the following lemma was noted in [18].

Lemma 3.4. *Suppose L is an algebraic frame. A nucleus in L is inductive if and only if it is finitary.*

Our interest in this section is an attempt to study the q-nucleus and how it relates to the d-operator. We remind the reader of the discussion from [16, Section 4].

Let j be a nucleus on the algebraic frame L. We call an element $x \in L$ *j-absorbing* if for all $c \in \mathfrak{K}(L)$, if $c \leq x$, then $j(c) \leq x$. The collection of j-absorbing elements, denoted by $\mathrm{Ab}(j)$, contains jL. In fact, there is an inductive nucleus $\widehat{j} : L \to L$ such that $\widehat{j}L = \mathrm{Ab}(j)$. In particular, $\widehat{j}$ is defined as follows. For $x \in L$

$$\widehat{j}(x) = \bigvee \{j(c) : c \in \mathfrak{K}(L), c \leq x\}.$$

Clearly, if j is inductive, then $j = \widehat{j}$. Moreover, for every $x \in L$, $\widehat{j}(x) \leq j(x)$.

The set $\mathrm{Ab}(j)$ is closed under arbitrary infima, and so $\widehat{j}(x)$ is also the least j-absorbing element of L above x.

Proposition 3.5. [16, Theorem 4.12] *Suppose L is an algebraic frame and that $j : L \to L$ is a nucleus on L. Then $\widehat{j}$ is the largest inductive nucleus satisfying $\widehat{j} \leq j$.*

We are now in position to discuss the q-operator. The motivation of studying this operator comes from the theory of localizations and prime ideals in commutative rings with identity. With such a ring A and a saturated multiplicative set S, the prime ideals of $S^{-1}A$ are precisely the extension of the prime ideals of A that are disjoint from S. Of course, the most interesting example is the passage of a commutative ring with identity to its classical ring of quotients. In doing so, one highlights the prime ideals of A which do not contain any regular elements. Then this collection of prime ideals is in one-to-one correspondence with the prime ideals of the classical ring of quotients of A. Interestingly, this can be generalized to any collection of prime elements in a spatial frame.

Definition 3.6. Suppose L is a spatial frame. Throughout, $\mathcal{P}$ shall denote a non-empty collection of prime elements of L. For each $x \in L$, we denote by $V_{\mathcal{P}}(x) = \mathcal{P} \cap V(p)$, the set of prime elements in $\mathcal{P}$ which exceed x. We recall the operator defined in [17, Remark 2.2], and denote it by $q_{\mathcal{P}} : L \to L$. For each $x \in L$, let

$$q_{\mathcal{P}}(x) = \bigwedge V_{\mathcal{P}}(x).$$

Remark 3.7. In [6] and [15], the authors studied the q-operator. The nucleus q can be described as $q_{\mathcal{P}}$ where $\mathcal{P} = Q(L)$, the collection of prime elements of L that do not exceed a unit. We will have more to say about this nucleus in the last section.

In [16] it is stated without proof that $q_{\mathcal{P}}$ is a nucleus. We provide a proof for completeness sake. The authors were not so much interested in the nucleus $q_{\mathcal{P}}$ as they were in $\widehat{q_{\mathcal{P}}}$: the inductive nucleus induced by $q_{\mathcal{P}}$.

Proposition 3.8. *The operator $q_{\mathcal{P}}$ is a nucleus on L.*

Proof. i) Suppose $x, y \in L$ and $x \le y$. Then $V_{\mathcal{P}}(y) \subseteq V_{\mathcal{P}}(x)$, whence $q_{\mathcal{P}}(x) = \bigwedge V_{\mathcal{P}}(x) \le \bigwedge V_{\mathcal{P}}(y) = q_{\mathcal{P}}(y)$.

ii) Clearly, $x \le \bigwedge V_{\mathcal{P}}(x) = q_{\mathcal{P}}(x)$.

iii) We aim to show that $V_{\mathcal{P}}(x) = V_{\mathcal{P}}(q_{\mathcal{P}}(x))$, from which it follows that $q_{\mathcal{P}}(x) = q_{\mathcal{P}}(q_{\mathcal{P}}(x))$. If $p \in V_{\mathcal{P}}(x)$, then $q_{\mathcal{P}}(x) = \bigwedge V_{\mathcal{P}}(x) \le p$. Thus, $V_{\mathcal{P}}(x) \subseteq V_{\mathcal{P}}(q_{\mathcal{P}}(x))$. The reverse containment is clear since $x \le q_{\mathcal{P}}(x) \le p$.

iv) Notice that $V_{\mathcal{P}}(a \wedge b) = V_{\mathcal{P}}(a) \cup V_{\mathcal{P}}(b)$. Since $q_{\mathcal{P}}$ preserves order, $q_{\mathcal{P}}(a \wedge b) \le q_{\mathcal{P}}(a) \wedge q_{\mathcal{P}}(b)$. Set $x = q_{\mathcal{P}}(a) \wedge q_{\mathcal{P}}(b)$, and observe that $x \le q_{\mathcal{P}}(a)$ and $x \le q_{\mathcal{P}}(b)$. Let $p \in V_{\mathcal{P}}(a \wedge b)$. Then either $p \in V_{\mathcal{P}}(a)$ or $p \in V_{\mathcal{P}}(b)$. But then either

$$x \le q_{\mathcal{P}}(a) \le p \text{ or } x \le q_{\mathcal{P}}(b) \le p.$$

Thus, $x \le p$ for every $p \in V_{\mathcal{P}}(a \wedge b)$. It follows that $x \le q_{\mathcal{P}}(a \wedge b)$. □

We note that the next lemma is part of [16, Lemma 4.2 (b)]. It appears as if the authors are assuming that j is inductive however, the proof only uses the fact that j is a nucleus. This is an important fact as it demonstrates that using nuclei to create new frames does not create new prime elements. We include a proof for completeness sake.

Lemma 3.9. *Suppose $j : L \to L$ is a nucleus on the spatial frame L. Then* $\operatorname{Spec}(jL) = \operatorname{Spec}(L) \cap jL$.

Proof. Clearly, $\operatorname{Spec}(L) \cap jL \subseteq \operatorname{Spec}(jL)$. Suppose that $p \in \operatorname{Spec}(jL)$. We aim to show that p is a prime of L. To that end suppose that $a \wedge b \le p$. Then $j(a) \wedge j(b) = j(a \wedge b) \le j(p) = p$. Therefore, either $a \le j(a) \le p$ or $b \le j(b) \le p$. □

We now argue that in order to study inductive nuclei it is paramount to understand the $q_{\mathcal{P}}$-operator. This is due to the next result, every inductive nucleus is induced by an appropriate $q_{\mathcal{P}}$-nucleus. Later we shall have even more to say.

Theorem 3.10. *Let L be an algebraic frame and $j : L \to L$ an inductive nucleus. Set $\mathcal{P} = \mathrm{Spec}(jL)$, then*

$$j = \widehat{q_{\mathcal{P}}}.$$

Proof. We remind the reader that in a spatial frame, and so in particular an algebraic frame, every element is the meet of all of the primes above it.

Next, we claim that any prime in $\mathcal{P}$ above x is also above $j(x)$. To see this, let $p \in \mathcal{P}$ be above x. Then $j(x) \leq j(p) = p$.

Now, $j(x)$ is the meet (in jL) of the primes in jL above $j(x)$ (and hence above x). By the previous lemma, each prime in jL is a prime in L. So $j(x)$ is a lower bound in L for the collection of primes in $\mathcal{P}$ above x. Therefore, $j(x)$ is below the infimum of the primes in $\mathcal{P}$ above x, that is

$$j(x) \leq \bigwedge_L V_{\mathcal{P}}(x) = q_{\mathcal{P}}(x)$$

Since j is assumed to be inductive it follows by Proposition 3.5, that

$$j \leq \widehat{q_{\mathcal{P}}}.$$

Therefore, $q_{\mathcal{P}}L \subseteq jL$. Now, for each $p \in V_{\mathcal{P}}(x)$, $\widehat{q_{\mathcal{P}}}(x) \leq q_{\mathcal{P}}(x) \leq p$. This is also taking place in jL, whence $\widehat{q_{\mathcal{P}}}(x)$ is below the meet of $V_{\mathcal{P}}(x)$ in jL which as we pointed out is $j(x)$. Therefore,

$$\widehat{q_{\mathcal{P}}} \leq j.$$

□

Example 3.11. We start with $\mathcal{P} = \mathrm{Min}(L)$ and consider $q_{\mathcal{P}}$. The collection of fixed elements of $q_{\mathcal{P}}$ is not all of the $q_{\mathcal{P}}$-absorbing elements, since the latter collection is precisely the d-elements. In fact, the set of primes that are fixed by $q_{\mathcal{P}}$ is exactly $\mathrm{Min}(L)$. However, for each $c \in \mathfrak{K}(L)$, $q_{\mathcal{P}}(c) = c^{\perp\perp}$. Therefore, $\widehat{q_{\mathcal{P}}} = d$.

We investigate $q_{\mathcal{P}}$ further.

Proposition 3.12. *Let L be a spatial frame and suppose that $\mathcal{P}$ is a non-empty collection of prime elements of L. For $p \in \mathrm{Spec}(L)$, the following are equivalent.*

1. $p \in \mathrm{Spec}(q_{\mathcal{P}}L)$.
2. $p = \bigwedge V_{\mathcal{P}}(p)$.
3. $p \in \mathrm{cl}\, V_{\mathcal{P}}(p)$.

Proof. 1. ⇔ 2. This is clear.

2. ⇒ 3. Let $p \in \mathrm{Spec}(L)$. Consider $V_{\mathcal{P}}(p) = \{t \in \mathcal{P} : p \leq t\}$ and suppose $p = \bigwedge V_{\mathcal{P}}(p)$. We show that p is in the closure of $V_{\mathcal{P}}(p)$ with respect to the hull kernel topology on $\mathrm{Spec}(L)$. To that end, let $x \in L$ and $p \in U(x)$. Then $x \nleq p$ and so there is some $t \in V_{\mathcal{P}}(p)$ such that $t \in U(x)$. (Otherwise, $x \leq t$ for all such t and thus, x would be smaller than the infimum of these.) Therefore, $p \in \mathrm{cl}\, V_{\mathcal{P}}(p)$.

3. ⇒ 2. Now, suppose that $p \in \mathrm{cl}\, V_{\mathcal{P}}(p)$. We aim to show that $q_{\mathcal{P}}(p) = p$. If not, then $p \in U(q_{\mathcal{P}}(p))$. By hypothesis, there is some $t \in U(q_{\mathcal{P}}(p)) \cap V_{\mathcal{P}}(p)$. But clearly, $q_{\mathcal{P}}(p) \leq t$, contradicting that $t \in U(q_{\mathcal{P}}(p))$.

□

Remark 3.13. It is worth noting that we are not claiming that a prime element is a meet of elements of $\mathcal{P}$ if and only if it belongs to the hull-kernel closure of $\mathcal{P}$. This is categorically false as we know that the closure of the collection of minimal prime elements is all of $\mathrm{Spec}(L)$: if $q \in \mathrm{Spec}(L)$ and $q \in U(c)$, then any minimal prime element below q will also live in $U(c)$.

What we have just proved is the following. Start with a collection $\mathcal{P}$ of primes and take a prime $q \in \mathrm{Spec}(L)$. Next, take the set of primes in $\mathcal{P}$ that exceed q. Then q is in the hull-kernel closure of this set if and only if q is the meet of elements in $\mathcal{P}$.

These ideas lead us to construct a topological closure operator on a given topological space. We are now aware that what we had constructed is known as the Skula topology on a space X; the topology generated by the collection of open sets and closed sets. This topology is also known as the front topology; we thank the referee for suggesting this article by Simmons [22]). In a more recent article [14] the authors call this the strong topology. We include our construction for completeness sake.

Construction 3.14. Let $(X, O(X))$ be a topological space. For $S \subseteq X$, we define

$$\overline{S} = \{x \in X : x \in \mathrm{cl}(S \cap \mathrm{cl}\{x\})\}.$$

It should be obvious that i) $\overline{\emptyset} = \emptyset$, ii) $S \subseteq \overline{S}$. We show that iii) the operator is idempotent and iv) $\overline{S \cup T} = \overline{S} \cup \overline{T}$ for all subsets S and T of X.

iii) Let $y \in \overline{\overline{S}}$. In order to show that $y \in \overline{S}$ we take an open set $O \in O(X)$ containing y. Now, we know that there is some $t \in O \cap (\overline{S} \cap \mathrm{cl}\{y\})$. Since $t \in \overline{S}$ and O is an open set containing t it follows that there is some $s \in O \cap (S \cap \mathrm{cl}\{t\})$. Since $t \in \mathrm{cl}\{y\}$ it follows that $s \in \mathrm{cl}\{t\} \subseteq \mathrm{cl}\{y\}$. Therefore, $s \in O \cap (S \cap \mathrm{cl}\{y\})$.

iv) Finally, observe that

$$\begin{aligned}
\overline{S \cup T} &= \{x \in X : x \in \mathrm{cl}(\mathrm{cl}\{x\} \cap (S \cup T))\} \\
&= \{x \in X : x \in \mathrm{cl}((\mathrm{cl}\{x\} \cap S) \cup (\mathrm{cl}\{x\} \cap T))\} \\
&= \{x \in X : x \in \mathrm{cl}(\mathrm{cl}\{x\} \cap S) \cup \mathrm{cl}(\mathrm{cl}\{x\} \cap T)\} \\
&= \{x \in X : x \in \mathrm{cl}(\mathrm{cl}\{x\} \cap S)\} \cup \{x \in X : x \in \mathrm{cl}(\mathrm{cl}\{x\} \cap T)\} \\
&= \overline{S} \cup \overline{T}
\end{aligned}$$

Theorem 3.15. *Let $(X, O(X))$ be a topological space. The topology generated by the just defined topological closure operator is the Skula topology.*

Proof. We begin by showing that any subset of X, say S, which is either open or closed satisfies $\overline{S} = S$. Clearly, if S is closed and $x \in \overline{S}$, then since $S \cap \mathrm{cl}\{x\}$ is closed it follows that $x \in \mathrm{cl}(S \cap \mathrm{cl}\{x\}) = S \cap \mathrm{cl}\{x\}$, whence $x \in S$.

Next, suppose S is an open subset of X, and $x \in \overline{S}$. If $x \notin S$, then since $X \smallsetminus S$ is closed it follows that $\mathrm{cl}\{x\} \subseteq X \smallsetminus S$. Therefore, $S \cap \mathrm{cl}\{x\} = \emptyset$.

We have so far shown that our topological closure operator generates a topology that is finer than the Skula topology.

Next, let $A \subseteq X$ satisfy $A = \overline{A}$, and take $x \notin A$. This means that $x \notin \mathrm{cl}(A \cap \mathrm{cl}\{x\})$. Thus, there is some open subset $T \in O(X)$ such that $x \in T$ and $T \cap (A \cap \mathrm{cl}\{x\}) = \emptyset$. But this means that x belongs to the Skula open subset $T \cap \mathrm{cl}\{x\}$, and that $A \cap (T \cap \mathrm{cl}\{x\}) = \emptyset$. Consequently, A is Skula closed.

□

We make a few observations. We should mention that (5) below is an equivalent form of the well-known T_D-separation property.

Proposition 3.16. *Suppose X is a topological space. The following hold.*

1. *The Skula topology on X is finer that the original topology.*
2. *For any set $S \subseteq X$, $\overline{S} \subseteq \mathrm{cl}\, S$.*
3. *For any $x \in X$, $\overline{\{x\}} = \{y \in X : \mathrm{cl}\{y\} = \mathrm{cl}\{x\}\}$.*
4. *The topology $O(X)$ is T_0 if and only if its Skula topology is T_1.*
5. *The Skula topology on X is discrete if and only if for all $x \in X$, $x \notin \mathrm{cl}(\mathrm{cl}\{x\} \smallsetminus \{x\})$.*

For more information and proofs we suggest the interested reader check [20].

Remark 3.17. Recall that every T_0-space can be made into a partially ordered set. Given a T_0-space X, one can define a partial order $x \leq y$ if $y \in \mathrm{cl}\{x\}$. Conversely, a partially ordered set can be topologized with the upper topology making it into a T_0-space.

It is also well-known that to characterize those spaces which arise as the prime spectrum of a frame are precisely the sober T_0-spaces. We have chosen to focus on these spaces simply as spectra of frames.

Here we consider the hull-kernel topology on $\mathrm{Spec}(L)$.

Lemma 3.18. *Suppose L is a spatial frame and consider $\mathrm{Spec}(L)$ equipped with the hull-kernel topology. Let $\mathcal{P}$ be a nonempty subset of $\mathrm{Spec}(L)$. Then p belongs to the Skula closure of $\mathcal{P}$ if and only if $p \in \mathrm{cl}\, V_{\mathcal{P}}(p)$ if and only if $p = \bigwedge V_{\mathcal{P}}(p)$.*

Proof. Let $p \in \mathrm{Spec}(L)$. The proof hinges on the following simple fact together with Proposition 3.12.

$$\begin{aligned} \mathrm{cl}(\mathcal{P} \cap \mathrm{cl}\{p\}) &= \mathrm{cl}(\mathcal{P} \cap \{q \in \mathrm{Spec}(L) : p \leq q\}) \\ &= \mathrm{cl}\, V_{\mathcal{P}}(p) \end{aligned}$$

□

Corollary 3.19. *Let L be a spatial frame and $\mathcal{P} \subseteq \mathrm{Spec}(L)$. The following statements hold*

1. $q_{\mathcal{P}} = q_{\overline{\mathcal{P}}}$.
2. $\mathrm{Spec}(q_{\mathcal{P}} L) = \overline{\mathcal{P}}$.
3. *The set of minimal primes is Skula closed.*
4. *The Skula topology on $\mathrm{Spec}(L)$ is discrete if and only if no prime is a meet of a collection of primes properly exceeding the said prime.*

Remark 3.20. Our viewpoint came from looking at commutative rings with identity, so in particular coherent frames. We attempted to classify when a prime is the meet of collection of primes that properly contain it. This led to our definition and construction defined above. One interesting class of rings are the Hilbert rings. A ring is said to be a *Hilbert* ring (aka Jacobson ring) if every prime ideal is the meet of maximal ideals. It follows that $\mathrm{Max}(A)$ is Skula-dense in $\mathrm{Spec}(A)$ if and only if A is a Hilbert ring. In some sense, then, if $\mathrm{Max}(A)$ is Skula-closed, then one could call such a ring *anti-Hilbert*. Interestingly, in this case, a ring which is both Hilbert and anti-Hilbert is a zero-dimensional ring, and conversely.

Proposition 3.21. *Suppose L is an algebraic frame. The Skula topology on the hull-kernel topology on $\mathrm{Spec}(L)$ is finer than the patch topology on $\mathrm{Spec}(L)$. Moreover, the Skula topology applied to the patch topology on $\mathrm{Spec}(L)$ equals the Skula topology applied to the hull-kernel topology.*

Proof. This is immediate from Theorem 3.15. □

Corollary 3.22. *Suppose L is an algebraic frame. The Skula topology on $\mathrm{Spec}(L)$ is Hausdorff.*

We now investigate the equation $q_{\mathcal{P}}L = L$.

Proposition 3.23. *Let L be a spatial frame and $\mathcal{P} \subseteq \mathrm{Spec}(L)$. The following statements are equivalent.*

1. $q_{\mathcal{P}}L = L$.
2. *Every prime of L is a meet of elements in $\mathcal{P}$.*
3. *The collection $\mathcal{P}$ is Skula-dense in $\mathrm{Spec}(L)$.*
4. $\mathrm{Spec}(q_{\mathcal{P}}L) = \mathrm{Spec}(L)$.

Proof. 1. ⇒ 2. Suppose that $q_{\mathcal{P}}L = L$. This implies that every element of L is a meet of primes in $\mathcal{P}$. Thus, every prime of L is a meet of elements in $\mathcal{P}$.

2. ⇒ 3. Suppose every prime of L is a meet of elements in $\mathcal{P}$. Then every $p \in \mathrm{Spec}(L)$ satisfies $p \in \mathrm{cl}\, V_{\mathcal{P}}(p)$. It follows that $\mathcal{P}$ is Skula-dense.

3. ⇒ 4. Clearly, $\mathrm{Spec}(q_{\mathcal{P}}L) \subseteq \mathrm{Spec}(L)$. Let $p \in \mathrm{Spec}(L)$. Then p is in the Skula closure of $\mathcal{P}$. By Lemma 3.18, $p = \bigwedge V_{\mathcal{P}}(p) = q_{\mathcal{P}}(p) \in q_{\mathcal{P}}L$.

4. ⇒ 1. Suppose 4. holds. Now every element of L is a meet of primes. Now each of those primes belongs to $q_{\mathcal{P}}L$. Since $q_{\mathcal{P}}L$ is closed under arbitrary meets it follows that each $x \in q_{\mathcal{P}}L$. □

Corollary 3.24. *Suppose that $q_{\mathcal{P}}L = L$. Then $\mathcal{P}$ is a co-final Skula-dense subset of $\mathrm{Spec}(L)$ and $\bigwedge \mathcal{P} = 0$.*

Remark 3.25. By design, for any collection of prime elements, say $\mathcal{P}$, of a given spatial frame L the nucleus $q_{\mathcal{P}}$ is producing a spatial frame $q_{\mathcal{P}}L$. Sets of the form jL for some nucleus $j : L \to L$ are interesting sets. Precisely, a *sublocale* of L is a set S which is closed under meets in L and satisfies for every $x \in L$ and $s \in S$, $x \to s \in S$. A subset of a frame is a sublocale if and only if it is the image of a nucleus (see [20]).

In 1987, Niefield and Rosenthal [19] characterized when every sublocale of a frame is spatial. Recently, Suarez [23] revisited this theorem shedding some new light on the topic. A spatial sublocale is also known as a sober sublocale.

We end this section by characterizing the spatial sublocales. Recall that for a frame L the collection of all nuclei on L forms a frame called the *assembly of* L and denoted by NL. The assembly is ordered pointwise: for nuclei $j, k \in NL$, $j \leq k$ means that $j(x) \leq k(x)$ for all x. The assembly itself is a frame. The collection of all finitary nuclei forms a subframe of the assembly; this subframe is called the *patch frame of* L. Interestingly, the name comes from the fact that this is generalizing the patch topology on a spectral space.

Proposition 3.26. *Let L be a spatial frame. There is a one-to-one correspondence between the Skula closed subsets of* $\mathrm{Spec}(L)$ *and the spatial sublocales of L. Moreover, the collection of nuclei of the form $q_{\mathcal{P}}$ ranging over Skula closed subsets of* $\mathrm{Spec}(L)$ *is a subframe of the assembly of L.*

PROOF. *Sketch.* We apply the results from this section. For a given Skula closed subset, say $\mathcal{P}$, the nucleus $q_{\mathcal{P}}$ induces a spatial sublocale of L, namely its image. Conversely, for a spatial sublocale S of L, the set of primes of S forms a Skula closed subset of $\mathrm{Spec}(L)$

□

4. $\mathcal{P}$-ELEMENTS

One of the goals of this section is to characterize the primes that belong to $\widehat{q_{\mathcal{P}}}L$. In a recent paper [7], we studied a construction in the theory of commutative rings with identity known as the $A+B$ construction. This topic was spurred on by A. Epstein's dissertation [9]. (Here, A denotes a given commutative ring with identity, and $A+B$ is an extension ring that is constructed using a fixed collections of prime ideals of A.) In her dissertation [9], Epstein defined the concept of a $\mathcal{P}$-ideal. In [7], we classified the prime $\mathcal{P}$-ideals as those primes of A which extend to prime sd-ideals of $A+B$. Post-acceptance of the article [7], we found a ring theoretic result in [2, Theorem 3.3] that has been instrumental to enhancing our understanding of algebraic frame satisfying FIP and d-elements.

Definition 4.1. Suppose L is an algebraic frame and $\mathcal{P} \subseteq \mathrm{Spec}(L)$. Let $x \in L$. We say that x is a $\mathcal{P}$*-element* if for all $c \in \mathfrak{K}(L)$ satisfying $c \leq x$ and for all $a \in \mathfrak{K}(L)$, if $V_{\mathcal{P}}(c) \subseteq V_{\mathcal{P}}(a)$, then $a \leq x$. We let $\mathrm{Spec}_{\mathcal{P}}(L)$ denote the collection of primes of L which are simultaneously $\mathcal{P}$-elements.

We present our generalization.

Theorem 4.2. *Let L be an algebraic frame and let $\mathcal{P}$ be a collection of prime elements of L. For each $p \in \mathrm{Spec}(L)$, the following statements are equivalent.*

1. *p is a $q_{\mathcal{P}}$-absorbing element of L.*
2. $p \in \mathrm{Spec}(\widehat{q_{\mathcal{P}}}L)$
3. $p \in \mathrm{cl}_p\, \mathcal{P}$.
4. *p is a $\mathcal{P}$-element of L.*

Proof. 1. $\Leftrightarrow$ 2. Clearly, the work of Martínez and Zenk shows that these two are equivalent.

1. $\Rightarrow$ 3. Suppose that p is a $q_{\mathcal{P}}$-absorbing element of L and let $a, c \in \mathfrak{K}(L)$ for which $p \in U(a) \cap V(c)$. It follows that $c \leq p$. This implies that $q_{\mathcal{P}}(c) \leq p$. Notice that since $a \nleq p$ it must be the case that $V_{\mathcal{P}}(c) \nsubseteq V_{\mathcal{P}}(a)$ since p is $q_{\mathcal{P}}$-absorbing. Thus, there is some $t \in V_{\mathcal{P}}(c) \smallsetminus V_{\mathcal{P}}(a)$; $t \in \mathcal{P}$. It follows that $c \leq t$, i.e. $t \in V(c)$, and also that $a \nleq t$, i.e. $a \in U(c)$. Consequently, $t \in V(c) \cap U(a)$. Therefore, $p \in \operatorname{cl}_p \mathcal{P}$.

3. $\Rightarrow$ 4. Suppose that $p \in \operatorname{cl}_p \mathcal{P}$. Let $c \in \mathfrak{K}(L)$ satisfy $c \leq p$ and take $a \in \mathfrak{K}(L)$ for which $V_{\mathcal{P}}(c) \subseteq V_{\mathcal{P}}(a)$. We aim to show that $a \leq p$. If not, then $p \in U(a) \cap V(c)$, a patch open subset of $\operatorname{Spec}(L)$. By hypothesis, there is some $t \in \mathcal{P} \cap U(a) \cap V(c)$. It follows that $t \in V_{\mathcal{P}}(c)$, and so $t \in V_{\mathcal{P}}(a)$. But this means $a \leq t$ even though $t \in U(a)$. Therefore, $a \leq p$, whence p is a $\mathcal{P}$-element.

4. $\Rightarrow$ 1. Suppose that p is a $\mathcal{P}$-element, and let $c \in \mathfrak{K}(L)$ satisfy $c \leq p$. We aim to show that $q_{\mathcal{P}}(c) = \bigwedge V_{\mathcal{P}}(c) \leq p$. It suffices to show that for every $a \in \mathfrak{K}(L)$ satisfying $a \leq \bigwedge V_{\mathcal{P}}(c)$ must also satisfy $a \leq p$. Now, if a is compact and satisfies $a \leq V_{\mathcal{P}}(c)$, then $V_{\mathcal{P}}(c) \subseteq V_{\mathcal{P}}(a)$. By hypothesis, $a \leq p$. ☐

Corollary 4.3. *Suppose L is an algebraic frame. For a collection of primes $\mathcal{P}$,*

$$\operatorname{Spec}(\widehat{q_{\mathcal{P}}}L) = \operatorname{cl}_p \mathcal{P}.$$

Furthermore, if $\mathcal{P}_1 = \operatorname{cl}_p \mathcal{P}$, then $q_{\mathcal{P}_1} = \widehat{q_{\mathcal{P}}}$.

In general, the nucleus $q_{\mathcal{P}}$ need not be inductive. In our next two results we complete a characterization of when it is.

Proposition 4.4. *Suppose L is an algebraic frame and $\mathcal{P} \subseteq \operatorname{Spec}(L)$. If $\mathcal{P}$ is patch closed, then*

$$\widehat{q_{\mathcal{P}}} = q_{\mathcal{P}}.$$

Proof. Recall that $\widehat{q_{\mathcal{P}}} \leq q_{\mathcal{P}}$ and that $q_{\mathcal{P}}L \subseteq \widehat{q_{\mathcal{P}}}L$. Since $\mathcal{P}$ is patch closed we know that $\operatorname{Spec}(\widehat{q_{\mathcal{P}}}L) = \operatorname{cl}_p \mathcal{P} = \mathcal{P}$. Moreover, since a patch closed set if Skula closed,

$$\operatorname{Spec}(\widehat{q_{\mathcal{P}}}L) = \operatorname{Spec}(q_{\mathcal{P}}L).$$

Let $x \in \widehat{q_{\mathcal{P}}}L$. By [16, Lemma 4.2], $\widehat{q_{\mathcal{P}}}L$ is algebraic and hence $x \in \widehat{q_{\mathcal{P}}}L$ is a meet in $\widehat{q_{\mathcal{P}}}L$ of primes in $\widehat{q_{\mathcal{P}}}L$. The primes witnessing this meet belong to $\mathcal{P}$. Now, the meet of these primes in $q_{\mathcal{P}}L$ belongs to $q_{\mathcal{P}}L$ and thus also in $\widehat{q_{\mathcal{P}}}L$. Therefore, $x \in q_{\mathcal{P}}L$, showing the reverse containment.

☐

Theorem 4.5. *Let L be an algebraic frame and $\mathcal{P} \subseteq \operatorname{Spec}(L)$. The following statements are equivalent.*

1. *The nucleus $q_{\mathcal{P}}$ is inductive.*
2. $\widehat{q_{\mathcal{P}}} = q_{\mathcal{P}}$.
3. *The patch closure of $\mathcal{P}$ equals the Skula closure of $\mathcal{P}$.*

Proof. 1. $\Leftrightarrow$ 2. Patent

2. $\Rightarrow$ 3. By Corollary 4.3 the set of prime elements of $\widehat{q_{\mathcal{P}}}L$ is $\mathrm{cl}_p\,\mathcal{P}$, whereas the set of prime elements of $q_{\mathcal{P}}L$ is the Skula closure of $\mathcal{P}$, by Corollary 3.19. 2. guarantees that these are equal.

3. $\Rightarrow$ 2. We have the following

$$\begin{aligned} \widehat{q_{\mathrm{cl}_p\,\mathcal{P}}} &= q_{\mathrm{cl}_p\,\mathcal{P}} \\ &= q_{\overline{\mathcal{P}}} \\ &= q_{\mathcal{P}} \end{aligned}$$

where the first equality stems from Proposition 4.4, while the second equality holds by hypothesis of 3. The final equality is Corollary 3.19. □

Corollary 4.6. *Suppose L is an algebraic frame and $j : L \to L$ is an inductive nucleus and set $\mathcal{P} = \mathrm{Spec}(jL)$. Then $\mathcal{P}$ is a patch closed subset of $\mathrm{Spec}(L)$, and $j = q_{\mathcal{P}}$.*

Proof. Observe that every element of jL is a meet of primes elements of jL. Of course, *a priori*, this meet takes place in jL. Since jL is a sublocale it follows that the meet is taking place in L as well. Therefore, we know that $j = q_{\mathcal{P}}$ so that is we have left to demonstrate is that $\mathcal{P}$ is patch closed. Now, $\mathrm{Spec}(q_{\mathcal{P}}L)$ is the Skula closure of $\mathcal{P}$, whence we know that $\mathcal{P}$ is Skula closed. Finally, an application of Theorem 4.5 yields that $\mathcal{P}$ is patch closed. □

Part of what makes the above tick is that the infimum of the minimal prime elements of L is 0. We expand on this.

Proposition 4.7. *Let L be an algebraic frame satisfying FIP and let $\mathcal{P} \subseteq \mathrm{Spec}(L)$. The following statements are equivalent.*

1. $\mathrm{Min}(L) \subseteq \mathrm{cl}_p\,\mathcal{P}$.
2. $\mathrm{Min}(L) \subseteq \mathrm{cl}\,\mathcal{P}$.
3. $\bigwedge \mathcal{P} = 0$.

Proof. 1. $\Rightarrow$ 2. Since the patch topology is finer than the hull-kernel topology on $\mathrm{Spec}(L)$, 1. implies 2.

2. $\Rightarrow$ 3. Suppose $\mathrm{Min}(L) \subseteq \mathrm{cl}\,\mathcal{P}$. Let $c \in \mathfrak{K}(L)$ be a non-zero element of L. There exists some $q \in \mathrm{Min}(L)$ such that $c \nleq q$. Since $q \in U(c)$, an open set in the hull-kernel topology, it follows that there is some $p \in U(c) \cap \mathcal{P}$. Hence, there exists some $p \in \mathcal{P}$ such that $c \nleq p$. Therefore, the infimum of the collection $\mathcal{P}$ must be 0.

3. $\Rightarrow$ 1. Suppose $p \in \mathrm{Min}(L)$ and $a, b \in \mathfrak{K}(L)$ with $p \in U(a) \cap V(b)$, a basic open set in the patch topology. So, $a \nleq p$ and $b \leq p$. Suppose by way of contradiction that $U(a) \cap V(b) \cap \mathcal{P} = \emptyset$. This means, for each $t \in \mathcal{P}$, either $a \leq t$ or $b \nleq t$. In either case, it follows that $a \wedge b^{\perp} \leq t$ for all $t \in \mathcal{P}$. By hypothesis, $a \wedge b^{\perp} = 0$. Consequently, $a \leq b^{\perp\perp} \leq p$ since minimal prime elements are d-elements. This is the desired contradiction.

□

We end with a discussion of the q-operator in the context of algebraic frames with FIP. Recall from Remark 3.7 that $Q(L)$ is the collection of prime elements that do not exceed a unit and that

$q_{Q(L)}$ is simply referred to as the q-operator. Now, both sets $\mathrm{Spec}_d(L)$ and $Q(L)$ are patch closed subsets of $\mathrm{Spec}(L)$, $d = q_{\mathrm{Spec}_d(L)}$, and $\mathrm{Spec}_d(L) \subseteq Q(L)$. Therefore, $q \leq d$. We begin by studying arbitrary inductive nuclei below d.

Suppose j is a nucleus on L such that $j \leq d$. There is an intriguing subtlety here that we will flesh out. We can consider the d-operator on jL which, a priori, could be different than first applying j to L and then applying d, viewing jL as a subset of L. We let d' be the d-operator on jL.

Our aim is to show that the condition $j \leq d$ implies that $\mathrm{Spec}_d(jL) = \mathrm{Spec}_d(L)$. In other words, when $j \leq d$, the d-elements of the frame jL are the same as the d-elements of L. Now, if $p \in \mathrm{Spec}_d(L)$, then $p \leq j(p) \leq d(p) = p$, from which we gather that $p \in \mathrm{Spec}(jL)$. Moreover,

Proposition 4.8. *Suppose that L is an algebraic frame satisfying FIP and let $j : L \to L$ be an inductive nucleus satisfying $j \leq d$. Then*

$$\mathrm{Spec}_d(jL) = \mathrm{Spec}_d(L).$$

Proof. The proof hinges on the fact that $\mathrm{Min}(jL) = \mathrm{Min}(L)$.

Let $p \in \mathrm{Spec}_d(L)$, then $p \leq j(p) \leq d(p) = p$, from which we gather that $p \in jL$. Clearly, p is a prime element of jL: note that j is a nucleus and so the infimum in jL is the same as in L. Thus, $\mathrm{Spec}_d(L) \subseteq \mathrm{Spec}(jL)$. We can use this fact to show that $\mathrm{Min}(jL) = \mathrm{Min}(L)$.

Now, any $q \in \mathrm{Min}(L)$ is a d-element of L and hence by what we just showed belongs to jL. Since $\mathrm{Spec}(jL) = \mathrm{Spec}(L) \cap jL$ it follows that q must be minimal in jL, that is, $\mathrm{Min}(L) \subseteq \mathrm{Min}(jL)$. As to the reverse containment, if $q \in \mathrm{Min}(jL)$, then $q \in \mathrm{Spec}(L)$ and so there is a minimal prime of L beneath q. But again such a minimal prime is in jL and hence must equal q. We have thus demonstrated that $\mathrm{Min}(jL) = \mathrm{Min}(L)$.

We next show that each $p \in \mathrm{Spec}_d(L)$ also belongs to $\mathrm{Spec}_d(jL)$. Let $e \in \mathfrak{K}(jL)$ satisfy $e \leq p$. We aim to show that $d'(e) \leq p$. Since j is inductive we know that, by [16, Lemma 4.2], $e = j(c)$ for some compact element $c \in \mathfrak{K}(L)$. We claim that $d'(e) = d(c)$, the right side of course equalling $c^{\perp\perp}$ in L. It will then follow that $d'(e) \leq p$ since p is a d-element of L.

Recall that since $e \in \mathfrak{K}(jL)$, $d'(e)$ is the infimum of those minimal primes of jL which exceed e. Any minimal prime of L that exceeds e also exceeds c. Conversely, if q is a minimal prime of L that exceeds c, then $e = j(c) \leq j(q) \leq d(q) = q$. It follows that e and c share the set of minimal primes of L containing them. Since jL is closed under meets in L,

$$\begin{aligned} d'(e) &= \bigwedge_{jL}\{q \in \mathrm{Min}(jL) : e \leq q\} \\ &= \bigwedge_{L}\{q \in \mathrm{Min}(jL) : e \leq q\} \\ &= \bigwedge_{L}\{q \in \mathrm{Min}(L) : c \leq q\} \\ &= d(c) \end{aligned}$$

We have shown that $\mathrm{Spec}_d(L) \subseteq \mathrm{Spec}_d(jL)$. To see the reverse containment, let $q \in \mathrm{Spec}_d(jL)$. Then clearly, $q \in \mathrm{Spec}(L)$. If $c \in \mathfrak{K}(L)$ and $c \leq q$, then $j(c) \leq q$ and $j(c) \in \mathfrak{K}(jL)$. It follows that $d'(j(c)) \leq q$. As we showed above $d'(j(c)) = d(c)$ from which we conclude that $q \in \mathrm{Spec}_d(L)$.

□

Trivially, if L has no proper units, then $\mathrm{Spec}(L) = Q(L)$, in which case $L = qL$. The converse is true since any time L has a proper unit, say $c \in \mathfrak{K}(L)$, then $V_{Q(L)}(c) = \emptyset$, and so $q(c) = 1$. Thus, in this case $qL \neq L$. We state this formally.

Proposition 4.9. *Suppose L is an algebraic frame satisfying FIP. Then $L = qL$ if and only if L has no proper units.*

It should be apparent that a Martínez frame has no proper units. Therefore,

Corollary 4.10. *Suppose L is a Martínez frame. Then $L = qL$.*

We now can characterize when qL is a Martínez frame. What our theorem shows is that if qL is a Martínez frame, then we obtain the equality $q = d$, and conversely. Here we shall again assume that L is an algebraic frame satisfying FIP ,and so qL is also an algebraic frame satisfying FIP.

Proposition 4.11. *Let L be an algebraic frame satisfying FIP. Then the following are equivalent.*

1. *qL is a Martínez frame.*
2. *$Q(L) = \mathrm{Spec}_d(L)$.*
3. *$qL = dL$.*
4. *$Q(L) = \mathrm{cl}_p \mathrm{Min}(L)$.*

Proof. The proof here relies heavily on Theorem 2.4.

1. ⇒ 2. In any algebraic frame satisfying FIP, $\mathrm{Spec}_d(L) \subseteq Q(L)$. For the reverse containment, let $p \in Q(L)$. Then $q(p) = p$. So by hypothesis, $d(q(p)) = q(p) = p$. Therefore, $p \in \mathrm{Spec}_d(L)$.

2. ⇒ 3. This is clear knowing that $d = q_{\mathrm{Spec}_d(L)}$.

3. ⇒ 4. We know that $\mathrm{Spec}_d(L) = \mathrm{cl}_p \mathrm{Min}(L)$ and so we need to show that every $p \in Q(L)$ is a d-element. But this follows easily from 3. since $p = q(p) = d(p)$.

4. ⇒ 1. Assuming 4. is equivalent to assuming that $Q(L) = \mathrm{Spec}_d(L)$. It follows that $qL = dL$ is a Martínez frame.

□

Example 4.12. Suppose A is a commutative ring with identity. Then the finite ring of quotients of A is a Martínez ring if and only if every prime ideal that does not contain a dense finitely generated ideal is a d-ideal. If the ring satisfies Property A then we can replace the finite ring of quotients with the classical ring of quotients of A since Property A ensures that a compact element of $\mathrm{Rad}(A)$ is a unit if and only if it contains a regular element.

We close with an interesting view of the behavior of the q-nucleus on compact elements. We shall restrict our attentions to algebraic frames with FIP that contain units. Otherwise, $Q(L) = \mathrm{Spec}(L)$ nd so q is the identity on L.

Theorem 4.13. *Let L be an algebraic frame satisfying FIP with a unit. For each $c \in \mathfrak{K}(L)$*

$$q(c) = \bigvee \{d \in \mathfrak{K}(L) : \text{ there is some unit } y \text{ such that } d \wedge y \leq c\}.$$

Proof. Let c be compact and consider $q(c)$, that is, the infimum of all primes above c which do not contain a unit. First, if c is a unit, then vacuously $q(c) = 1$. Furthermore, every compact in L belongs to the set on the right side, whence the join of these is 1. So, we next assume that c is a non-unit, and let e equal the element defined on the right side of the display.

First of all, we have pointed out that in an algebraic frame satisfying FIP every compact non-unit is contained in at least one prime which contains no units; such an element is contained in a minimal prime element. If p is a prime above c which does not contain a unit, then for any $d \in \mathfrak{K}(L)$ for which there is a unit y satisfying $d \wedge y \leq p$, it follows by primality that $d \leq p$ or $y \leq p$. So it must be the former. This shows that $e \leq q(c)$.

Next, suppose $z \in \mathfrak{K}(L)$ and $z \nleq e$. Consider the collection $\mathcal{S}$ consisting of all elements $t \in L$ such that $c \leq t$ and for all units y, $z \wedge y \nleq t$. $\mathcal{S} \neq \emptyset$ since $c \in \mathcal{S}$.

Take a chain $\mathcal{C}$ in $\mathcal{S}$ and let $T = \bigvee \mathcal{C}$. If $T \notin S$, then there is some unit $y \in L$ such that $z \wedge y \leq T$. Since L is an algebraic frame satisfying FIP, the element $z \wedge y \in \mathfrak{K}(L)$ and so there is some finite collection in $\mathcal{C}$, say $t_1, ..., t_n$, such that $z \wedge y \leq t_1 \vee ... \vee t_n = t_i$ for some i. But this contradicts that $t_i \in \mathcal{S}$.

By Zorn's Lemma $\mathcal{S}$ has a maximal element, say $m \in \mathcal{S}$. We claim m is prime. To that end, let $x, y \in L$ satisfy $x \wedge y \leq m$. If $x \nleq m$, then by maximality, $x \vee m \notin \mathcal{S}$ and so there is some unit c_1 such that $z \wedge c_1 \leq x \vee m$. Similarly, if $y \nleq m$ there is some unit c_1 such that $z \wedge c_2 \leq y \vee m$.

It follows that

$$\begin{aligned} z \wedge (c_1 \wedge c_2) &= (z \wedge c_1) \wedge (z \wedge c_2) \\ &\leq (x \vee m) \wedge (y \vee m) \\ &= (x \wedge y) \vee m \\ &= m \end{aligned}$$

Since a meet of two units is again a unit, specifically $c_1 \wedge c_2$, we obtain the desired contradiction. Therefore, we conclude that m is prime. Observe that m cannot contain any unit. And since $z \nleq m$, it follows that $q(c) \leq e$. □

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References

[1] Aliabad, A. R. and Mohamadian, R. *On sz°-ideals in polynomial rings.* Communications in Algebra, **39** (2011) 701-717,.

[2] Artico, G., Marconi, U., and Moresco, R. *A subspace of Spec(A) and its connexions with the maximal ring of quotients.* Rend. Sem. Mat. Univ. Padova **64** (1981) 93–107.

[3] Banaschewski, B. and Brümmer, G. *Stably continuous frames* . Math. Proc. Camb. Phil. Soc. (1988) **104**, 7.

[4] Banaschewski, B. *Radical ideals and coherent frames.* Comment. Math. Univ. Carolin. **37** no. 2 (1996) 349–370.

[5] Bhattacharjee, P. *Maximal d-elements of an algebraic frame.* Order **36** (2019) 377–390.

[6] Bhattacharjee, P. and Dube, T. *On fraction-dense algebraic frames.* Algebra Universalis **83** no. 1 (2022) Paper No. 6, 18 pp.

[7] Bhattacharjee, P., Klingler, L., and McGovern, W. Wm. *Yosida, Martıınez, and $A + B$ rings.* Proceedings of the Graz 2021 conference, to appear.

[8] Dube, T. and L. Sithole *On the sublocale of an algebraic frame induced by the d-nucleus.* Top. and its App. **263** (2019) 90–106.

[9] Epstein, A. *Annihilators and $A + B$ rings.* Thesis (Ph.D.)–Florida Atlantic University. (2020) 114 pp.

[10] Escardó, M. H. *Properly injective spaces and function spaces.* Topology Appl. **89** (1998) 75–120.

[11] Ghashghaei, E. and McGovern, W. Wm. *Fusible rings*, Comm. Alg. **45** no. 3 (2017) 1151–1165.

[12] Gierz, G., K. H. Hofmann , K. Keimel , J. D. Lawson , M. W. Mislove , D. S. Scott, A Compendium of Continuous Lattices, Springer 1980.

[13] Hochster, M. *Prime ideal structure in commutative rings.* Trans. Amer. Math. Soc. **142** (1969) 749–758.

[14] Keimel, K. and J. D. Lawson, *D-completions and the d-topology.* Ann. Pure Appl. Logic **159** (2009), no. 3, 292–306.

[15] Martínez, J. *Unit and kernel systems in algebraic frames.* Algebra Universalis **55** no. 1 (2006) 13–43.

[16] Martínez, J. and Zenk, E. R. *When an algebraic frame is regular.* Algebra Universalis **50** no. 2 (2003) 231–257.

[17] Martínez, J. and Zenk, E. R. *Yosida frames.* J. Pure Appl. Algebra **204** no. 3 (2006) 473–492.

[18] Martínez, J. and Zenk, E. R. *Regularity in algebraic frames.* J. Pure Appl. Algebra **211** (2007), 566-580.

[19] Niefield, S. B. and Rosenthal, K. I. *Spatial sublocales and essential primes.* Topology Appl. **26** no. 3 (1987) 263–269.

[20] Picado, J. and Pultr, A. *Frames and Locales: Topology without Points.* Frontiers in Mathematics, vol. **28**. Springer, Basel (2012)

[21] Sexton, R. A. and Simmons, H. *Point-sensitive and point-free patch constructions.* J. Pure Appl. Algebra **207** no. 2 (2006) 433–468.

[22] Simmons, H. *Spaces with Boolean assemblies.* Colloq. Math. **43** (1980), no. 1, 23–39 (1981).

[23] Suarez, A. L. *On the relation between subspaces and sublocales.* J. Pure Appl. Algebra **226** no. 2 (2022) Paper No. 106851, 24 pp.

[24] Wehrung, F. *Spectral spaces of countable Abelian lattice-ordered groups.* Trans. Amer. Math. Soc. **371** no. 3 (2019) 2133–2158.

[25] Wiegand, R. *Modules over universal regular rings.* Pacific J. Math. **39** (1971) 807–819.

Department of Mathematical Sciences, Charles E. Schmidt College of Science, Florida Atlantic University, Boca Raton, FL 33431, USA

Department of Mathematics, Oregon State University, Corvallis, OR 97331-4605, USA

Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458, USA
Email address: `pbhattacharjee@fau.edu (P. Bhattacharjee)`
Email address: `klinglel@oregonstate.edu (L. Klingler)`
Email address: `warren.mcgovern@fau.edu (W. Wm. McGovern)`