Division closed partially ordered rings

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Abstract. Fuchs [6] called a partially-ordered integral domain, say D, *division closed* if it has the property that whenever $a > 0$ and $ab > 0$, then $b > 0$. He showed that if D is a lattice-ordered division closed field, then D is totally ordered. In fact, it is known that for a lattice-ordered division ring, the following three conditions are equivalent: a) squares are positive, b) the order is total, and c) the ring is division closed. In the present article, our aim is to study ℓ-rings that possibly possess zero-divisors and focus on a natural generalization of the property of being division closed, what we call *regular division closed*. Our investigations lead us to the concept of a positive *separating* element in an ℓ-ring, which is related to the well-known concept of a positive d-element.

1. Partially Ordered Rings

Throughout, rings are assumed to be commutative. We do not assume that rings possess an identity, but when the ring R does possess an identity, we will denote it by 1_R (or 1 when it is not ambiguous).

A *partially ordered ring* (or *po*-ring for short) is a ring R equipped with a partial order, say $\leq$, such that the following two properties hold: (1) if $a \leq b$, then $a + c \leq b + c$, and (2) whenever $0 \leq a, b$ then $0 \leq ab$. An equivalent, and more useful, way of viewing a partially ordered ring is through its positive cone: $R^+ = \{a \in R : 0 \leq a\}$. As is well-known, the positive cone of a *po*-ring can be characterized algebraically.

Definition 1.1. Recall that a *positive cone* of a ring is a subset $P \subseteq R$ such that the following three conditions are satisfied:

i) $P + P \subseteq P$,
ii) $PP \subseteq P$, and
iii) $P \cap -P = \{0\}$.

Given a *po*-ring $(R, \leq)$, the set R^+ is a positive cone. Conversely, a positive cone of R generates a partial order on R making R into a *po*-ring: define $a \leq b$ precisely when $b - a \in P$ (see [5] or [13] for more information.) We shall use both notations $(R, \leq)$ and (R, R^+) to denote that R is a partially ordered ring.

A partially ordered ring whose partial order is a lattice order is called a *lattice-ordered ring* (or ℓ-ring for short). In particular, an ℓ-ring is a lattice-ordered group and so the theory of such groups is useful in studying ℓ-rings. If (R, R^+) is an ℓ-ring, we denote the least upper bound (resp., greatest lower bound) of $a, b \in R$ by $a \vee b$ (resp., $a \wedge b$). The positive part of a is $a^+ =$

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$a \vee 0$, the negative part of a is $a^- = -a \vee 0$, and the absolute value of a is $|a| = a^+ + a^- = a^+ \vee a^-$. It is known that $a^+ \wedge a^- = 0$, and whenever a pair of elements satisfies the equality $a \wedge b = 0$, we say a and b are *disjoint*. In general, for any $a \in R$, $a = a^+ - a^-$. For more information on the theory of lattice-ordered groups we recognize the following two references as excellent sources on the subject: [5] and [2].

Within the class of ℓ-rings there are many interesting subclasses. We recall the most well-known ones now.

Definition 1.2. Suppose R is an ℓ-ring.

(1) R is called an *almost f-ring* if $a \wedge b = 0$ implies $ab = 0$.
(2) R is called a *d-ring* if $a \wedge b = 0$ and $c \geq 0$, then $ac \wedge bc = 0$.
(3) R is called an *f-ring* if $a \wedge b = 0$ and $c \geq 0$, then $ac \wedge b = 0$.

It is known that every f-ring is a d-ring and every d-ring is an almost f-ring. For more information the reader is encouraged to read [1] or [3]. At the end of the section we shall recall some examples showing that the implications are strict. The class of f-rings contains some of the most well-behaved ℓ-rings and is very well understood.

As for elements in an ℓ-ring R, we say that $x \in R^+$ is a *d-element* if whenever $a, b \in R$ and $a \wedge b = 0$, then $xa \wedge xb = 0$. One can check (or see Lemma 1 of [3]) that $x \in R^+$ is a d-element precisely when for all $a, b \in R^+$, $x(a \wedge b) = xa \wedge xb$. Notice that R is a d-ring if and only if every positive element is a d-element. For an arbitrary ℓ-ring R, we let $d(R)$ denote the set of d-elements of R. Similarly, we define an *f-element* as an $x \in R^+$ so that whenever $a, b \in R$ and $a \wedge b = 0$, then $xa \wedge b = 0$. We denote the set of f-elements of R by $f(R)$.

Next, we recall some results involving the relationships between the three classes of ℓ-rings. Recall that a positive element $u \in R^+$ is called a *weak order unit* if $u \wedge x = 0$ implies $x = 0$. A ring is called *semiprime* if it has no nonzero nilpotent elements. An ℓ-group R is called *archimedean* if $a, b \in R^+$ and, for all $n \in \mathbb{N}$ $na \leq b$ implies $a = 0$. Whenever, for $0 \leq a, b$, $na \leq b$ for all $n \in \mathbb{N}$, we write $a \ll b$. Thus, archimedean can be phrased to say that if $0 \leq a, b$ and $a \ll b$, then $a = 0$.

Proposition 1.3. [3, Theorem 15] *Suppose R is an ℓ-ring with $1 > 0$. Then R is an almost f-ring if and only if 1 is a weak order unit.*

Proposition 1.4. [3, Theorem 14] *A d-ring R with the property that for all $a \in R, Ra = 0$ implies $a = 0$, is an f-ring. In particular, a d-ring with a regular element is an f-ring.*

Proposition 1.5. [1, Theorem 1.11] (1) *Any semiprime almost f-ring is an f-ring.*

(2) *Any archimedean almost f-ring with identity is an f-ring.*

Proposition 1.6. [3, p.57, Corollary 1] *Suppose R is an ℓ-ring. Then R is a d-ring if and only if for all $a, b \in R$, $|ab| = |a||b|$. In particular, $b \in R^+$ is a d-element if and only if for all $a \in R$, $|a|b = |ab|$.*

Proposition 1.7. [1, Proposition 1.3] *Let R be an ℓ-ring and $a \in R$. Then $a^+a^- = 0$ if and only if $a^2 = |a|^2$. In either case $a^2 \geq 0$. Therefore, R is an almost f-ring if and only if $a^2 = |a|^2$ for all $a \in R$. Moreover, in an almost f-ring, squares are positive.*

Example 1.8. (1) Example 16 of [3] is an example of an almost f-ring with positive identity which is not a d-ring. We recall a variant of this example in Example 4.3.

(2) We recall Example Consider $R = \mathbb{R}[x]$. Recall that for $f(x) = a_0 + a_1x + \ldots + a_nx^n \in \mathbb{R}[x]$ (with $a_n \neq 0$), we call a_0 (resp., a_n) the constant term (resp., leading coefficient) of $f(x)$. In R, we let $\mathcal{P}_1$ be the positive cone defined by letting $f(x)$ be positive if $f(x) = 0$ or both the leading coefficient and constant term are non-negative real numbers. Notice that $1 \wedge x^n = 0$ for all $n \geq 1$ and $x \ll x^2 \ll x^3 \ll \ldots$. This ℓ-ring has positive squares but is not an almost f-ring (see [15, p.366] for more information).

(3) Example 3.3 and Example 3.5 of [1] are examples of d-rings which are not f-rings. Certainly, these rings do not have any regular elements.

2. Rings of Quotients

Our interest in rings of quotients stems from the program of characterizing those lattice-ordered integral domains which embed into lattice-ordered fields. Several authors have studied this problem; we list here a few references for the interested reader: [4], [12], [14], [11], [8], [9], [6], [7]. In this article, an integral domain that is an ℓ-ring (respectively, *po*-ring) shall be called an *integral ℓ-domain* (respectively, *po-domain*). Of course, by an ℓ-field we mean a field which is also an ℓ-ring.

The most convenient way of embedding an integral ℓ-domain R into an ℓ-field would be to order the classical field of fractions of the domain, denoted $q(R)$, in such a way that the map is an embedding. Formally, given two *po*-rings R and T we say *R is po-embeddable in T* if there is an injective ring homomorphism $\phi : R \to T$ such that $T^+ \cap \phi(R) = \phi(R^+)$. Most of the time, the embedding ϕ will be understood, and so we shall abuse notation and assume that R is identified with its image inside of T. If both R and T are ℓ-rings, we shall use the term *ℓ-embeddable*. To date, there is no general method of constructing a partial order on $q(R)$ making it into an ℓ-ring and having R be ℓ-embeddable in $q(R)$.

Example 2.1. Observe that the notion of *po*-embeddable is stronger than saying that T is a *po-extension* of R, that is, $\phi(R^+) \subseteq T^+$. For example, equip $\mathbb{Z}$ with the positive cone $P = \{0\} \cup \{n \in \mathbb{N} : n \geq 2\}$. Then the identity map is a *po*-extension of $(\mathbb{Z}, P)$ into $\mathbb{Z}$ (with the usual order) which is not a *po*-embedding.

For completeness sake, we remind the reader that if R is a ring and S is a multiplicatively closed set of regular elements, then the quotient ring of R with respect to S is the ring $S^{-1}R = \{\frac{a}{b} : a \in R, b \in S\}$ with the usual addition and multiplication of fractions. We use $\mathrm{Reg}(R)$ to denote the multiplicatively closed set of all regular elements of R; $\mathrm{Reg}(R)^+$ is the set of positive regular elements. The quotient ring induced by $\mathrm{Reg}(R)$ is the *classical ring of quotients* of R (a.k.a. the total ring of quotients); we denote it by $q(R)$. Observe that as long as R has regular elements, then $q(R)$ possesses an identity.

From this point forward, we make the assumption that R **possesses a regular element** to ensure that $q(R) \neq \emptyset$.

Notice that if we simply want to partially order $q(R)$, we can take R^+ as a positive cone on $q(R)$ in which case R is *po*-embeddable in $q(R)$. However, this choice is not very enlightening; consider, for example, $R = \mathbb{Z}$ with the usual order. Thus, as in most natural cases, we would like to find an order on $q(R)$ so that if $r \in \mathrm{Reg}(R)^+$, then its inverse is also positive in $q(R)$. In this way, we are led to consider the following subset of $q(R)$:

$$\mathcal{P}_{R^+} = \{q \in q(R) |\text{ there exists } a, b \in R^+, \text{ with } b \text{ regular, such that } q = \frac{a}{b}\}.$$

(When no confusion arises we shall drop the subscript and simply call this set $\mathcal{P}$.)

We observe that $\mathcal{P}$ might be trivial, i.e. R has no positive regular elements. (For example, the ring $\mathbb{Z} \times \mathbb{Z}$ equipped with the partial order $P = \{(a, 0) : a \in \mathbb{N}\}$ has no positive regular elements.) We will make a stronger case after Definition 2.3 as to why we will assume that $\mathrm{Reg}(R)^+ \neq \emptyset$. For now, observe that in the case $\mathrm{Reg}(R)^+ \neq \emptyset$, then $R^+ \subseteq \mathcal{P} \cap R$ since for any $a \in R^+$ and any $s \in \mathrm{Reg}(R)^+$, we can write $a = \frac{as}{s} \in \mathcal{P} \cap R$.

Lemma 2.2. *Given a po-ring R, $\mathcal{P}$ is a positive cone on $q(R)$.*

Proof. Notice that $\mathcal{P}$ trivially satisfies conditions i) and ii) of the definition of a positive cone on $q(R)$ (this assertion can be checked by using addition and multiplication of fractions). We show that $\mathcal{P}$ satisfies condition iii). To that end, let $q \in \mathcal{P} \cap -\mathcal{P}$, which means that there are $a, b, c, d \in R^+$ with b, d regular such that $q = \frac{a}{b} = -\frac{c}{d}$. Hence, the following string of equalities hold: $ad = (-c)b = -(cb)$. We know that $ad, cb \in R^+$ and hence $ad = -(cb) \in R^+ \cap -R^+ = \{0\}$ so that $ad = (-c)b = 0$. Since b, d are regular it follows that $a = 0 = c$, whence $q = 0$. □

We now turn our attention to the *po*-embeddability of (R, R^+) in $(q(R), \mathcal{P})$, i.e. the embeddability of R into $q(R)$ in such a way that $R^+ = \mathcal{P} \cap R$. We are led to one of the main definitions of the article. In [6], the author called a *po*-domain *division closed* if it had the feature that for any $a, b \in R$ if $0 < ab$ and either a or b is positive, then the other is also positive.

Definition 2.3. We say that the *po*-ring (R, R^+) is *regular division closed* if R^+ satisfies the following condition: whenever $a \in R$ and $b \in \mathrm{Reg}(R)$ satisfy $ab, b \in R^+$, then $a \in R^+$.

As we mentioned before, it is possible that no regular element of R is positive, in which case, R is trivially regular division closed. **We shall henceforth make the assumption that there is some positive regular element.** This assumption is equivalent to the not unreasonable assumption that $R^+ \neq \emptyset$. (For integral domains, this assumption is equivalent to assuming that $R^+ \neq \{0\}$, which is not unreasonable.) Under this assumption, it follows that if R is regular division closed and unital, then in R, $0 < 1$. Also, if $u \in R^+$ is invertible, then $u^{-1} \in R^+$.

Our next result shows that the class of regular division closed *po*-rings is non-trivial.

Proposition 2.4. *Let R be a po-ring. Then $(q(R), \mathcal{P})$ is regular division closed.*

Proof. Suppose $q_1 \in \mathcal{P}$ is a regular element and $q_2 \in q(R)$ such that $q_1q_2 \in \mathcal{P}$. Notice that q_1 is invertible in $q(R)$. Therefore, there exist elements $c \in R^+, b \in R, a, s, u \in \mathrm{Reg}(R)^+, t \in \mathrm{Reg}(R)$ such that $q_1 = \frac{a}{s}, q_2 = \frac{b}{t}$, and $q_1q_2 = \frac{c}{u}$. Since $q_1^{-1} = \frac{s}{a} \in \mathcal{P}$, it follows that $q_2 = \frac{sc}{au} \in \mathcal{P}$. □

We now characterize when (R, R^+) is *po*-embeddable in $(q(R), \mathcal{P})$.

Theorem 2.5. *Let R be a po-ring. The embedding of (R, R^+) into $(q(R), \mathcal{P})$ is a po-embedding if and only if R is regular division closed.*

Proof. Suppose that the embedding is a *po*-embedding and that $a \in R$ and $b \in \mathrm{Reg}(R)$ satisfy $ab, b \in R^+$. Notice that $a = \frac{ab}{b} \in \mathcal{P} \cap R = R^+$ (by assumption). Thus, R is regular division closed.

Conversely, we aim to show that $R^+ = \mathcal{P} \cap R$. As noted immediately before Lemma 2.2, $R^+ \subseteq \mathcal{P} \cap R$. For the other inclusion, if $a \in \mathcal{P} \cap R$, then we can write $a = \frac{x}{s}$ for some $x \in R^+$ and $s \in \mathrm{Reg}(R)^+$, which means that $as = x \geq 0$. Thus, by hypothesis, $a \in R^+$. □

Example 2.6. We find the following example instructive. Let R be a *po*-field in which $0 < 1$ but R is not division closed. Consider the ordering $\mathcal{P}$ on $q(R) = R$. This ordering enlarges the original ordering in such a way that every nonzero element of R^+ now has a positive inverse. For example, consider $(\mathbb{R}, \mathbb{N})$. Then $\mathcal{P} = \mathbb{Q}^+$.

Remark 2.7. Suppose (R, R^+) is a *po*-ring. For $s \in \mathrm{Reg}(R)^+$, in an abuse of notation, we will write $\frac{1}{s} \in \mathcal{P}$ instead of $\frac{s}{s^2}$ for the inverse of s in $q(R)$, even though we do not assume R is unital.

Proposition 2.8. *Suppose (R, R^+) is a po-ring such that there is a positive cone P on $q(R)$ that satisfies $R^+ \subseteq P \cap R$. Then $\mathcal{P} \subseteq P$ if and only if for every $s \in \mathrm{Reg}(R)^+$, $\frac{1}{s} \in P$.*

Proof. Let $\mathcal{P} \subseteq P$. If $s \in \mathrm{Reg}(R)^+$, then $\frac{1}{s} \in \mathcal{P} \subseteq P$.

Conversely, let $\frac{a}{s} \in q(R)$ with $a \in R^+$ and $s > 0$, a regular element. Since $R^+ \subseteq P \cap R$ it follows that $a \in P$. By hypothesis, $\frac{1}{s} \in P$. Therefore, $\frac{a}{s} = a\frac{1}{s} \in P$. □

Corollary 2.9. *Suppose R is a po-ring. Then $\mathcal{P}$ is the smallest positive cone on $q(R)$ containing R^+ with the property that the inverse of a positive regular element of R is positive in $q(R)$.*

We recall a theorem of Steinberg which is pivotal to our discussion.

Theorem 2.10. [14, Proposition 1] *Suppose R is an ℓ-ring and Σ is a multiplicative set of positive regular elements. The quotient ring $\Sigma^{-1}R$ can be made into an ℓ-ring extension of R such that the inverse of an element in Σ is positive if and only if multiplication by each element of Σ is an ℓ-homomorphism.*

According to the proof of Proposition 1 of [14], the positive cone of $\Sigma^{-1}R$ is the set $\mathcal{P}_\Sigma = \{q \in \Sigma^{-1}R : q = \frac{a}{b}$ for some $a \in R^+$ and $b \in \Sigma\}$. We would like to say that Theorem 2.5 is a generalization of Steinberg's theorem. However, it is possible that $q(R)$ is not the same as $\Sigma^{-1}R$ where $\Sigma = \mathrm{Reg}(R)^+$.

We are led to consider the question of when, for a given *po*-ring R, $q(R) = \Sigma^{-1}R$ where $\Sigma = \mathrm{Reg}(R)^+$. As a general fact, the two rings coincide precisely when the saturation of $\mathrm{Reg}(R)^+$ is $\mathrm{Reg}(R)$, that is, every regular element of R is a factor of a positive regular element. In this case, we say the *po*-ring R is *positively saturated*. One nice class of positively saturated rings is the class of *po*-rings with positive squares, that is, for all $a \in R$, $a^2 \in R^+$. In fact, we only need that the squares of regular elements are positive to be positively saturated. We have the following string of inclusions of classes of ℓ-rings:

f-rings $\subset$ d-rings $\subset$ almost f-rings $\subset$ positive squares $\subset$ positively saturated

Consider the next three examples which show that there are *po*-rings and ℓ-rings which are positively saturated but do not have positive squares.

Example 2.11. Let $R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Partially order R be setting $R^+ = \mathbb{Z}^+$. Notice that R is a *po*-domain which is not an ℓ-ring. It is not the case that R has positive squares since $(1+\sqrt{2})^2 = 3+2\sqrt{2} \notin R^+$. However, every element is a factor of a positive element. To see this let $r = a+b\sqrt{2} \in R$. Consider $x = a^2 - 2b^2 \in \mathbb{Z}$. If $x > 0$, then set $z = a - b\sqrt{2}$. Otherwise, set $z = -a + b\sqrt{2}$. Then $rz \in \mathbb{Z}^+$.

Notice that R is division closed since if $r \in \mathbb{Z}^+, q \in R$ and $rq \in \mathbb{Z}^+$, then $q \in \mathbb{Z}^+$. Thus, R is *po*-embeddable in $q(R) = \mathbb{Q}(\sqrt{2})$ where $\mathcal{P} = \mathbb{Q}^+$. Furthermore, $\Sigma^{-1}R = q(R)$ where $\Sigma = R^+$. Thus, $(\mathbb{Z}[\sqrt{2}], \mathbb{Z}^+)$ is positively saturated and does not have positive squares.

Example 2.12. Let $R = \mathbb{Z}[\sqrt{2}]$ and take $R^+ = \mathbb{Z}^+[\sqrt{2}] = \{a + b\sqrt{2} \in R : a, b \geq 0\}$. Then R is an integral ℓ-domain and $q(R) = \mathbb{Q}(\sqrt{2})$. Consider the positive cone $P = \mathbb{Q}^+(\sqrt{2})$. Since $P \cap R = R^+$ we can say that (R, R^+) is ℓ-embeddable in $(q(R), P)$. By a similar proof as in the previous example, R is positively saturated and does not have positive squares. However, R is not division closed and multiplication by a positive element need not be an ℓ-homomorphism. For example, one the one hand $1 \wedge \sqrt{2} = 0$, while on the other hand $(1 + \sqrt{2}) \wedge (2 + \sqrt{2}) > 0$ (multiplication by $1 + \sqrt{2}$).

Notice that $\mathcal{P} \nsubseteq P$ so that (in contrast to Proposition 2.8) even though (R, R^+) is ℓ-embeddable in $(q(R), P)$, positive elements of R need not have positive inverses in $(q(R), P)$.

Example 2.13. Let $R = \mathbb{Z} \times \mathbb{Z}$ equipped with component-wise operations. Set $P = \{(0,0)\} \cup \{(a,b) : 0 < a, b\}$. It is straightforward to show that R is a *po*-ring with the property that the squares of regular elements are positive, and R does not have positive squares.

Proposition 2.14. *Suppose R is a po-ring that is positively saturated. If there is a positive cone on $q(R)$, say P, which makes (R, R^+) po-embeddable in $(q(R), P)$, then $P \subseteq \mathcal{P}$.*

Proof. Let $x \in P$. Then $x = \frac{a}{b}$ for appropriate $a, b \in R$ with b regular. Choose $c \in R$ such that $cb \in \mathrm{Reg}(R)^+$, and observe that $x = \frac{ca}{cb}$ with $cb \in \mathrm{Reg}(R)^+$. Therefore,

$$ca = (cb)x \in P \cap R = R^+.$$

Consequently, $x \in \mathcal{P}$. □

The combination of Propositions 2.14 and 2.8 yields the following result.

Corollary 2.15. *Suppose R is a po-ring which is positively saturated. If there is a positive cone on $q(R)$, say P, which makes R po-embeddable in $(q(R), P)$ such that for every $s \in \mathrm{Reg}(R)^+$, $\frac{1}{s} \in P$, then $P = \mathcal{P}$.*

We finish this section by discussing what is known about *po*-rings and their embeddings into their classical ring of quotients. In [10], the authors investigated a nice class of integral domains. They called a *po*-domain R a *complete integral domain* if for all $a, b \in R$ such that $ab < 0$ and $b > 0$ we have $a < 0$. Notice that a *po*-domain is a complete domain if and only if it is division closed. They then showed in Theorem 7 that if $(R, \leq)$ is a complete *po*-domain satisfying the condition

$$(*) \text{ for each } 0 \neq a \in R \text{ there is a } b \in R \text{ such that } ab \geq 0,$$

then the relation on $q(R)$ defined as follows makes $q(R)$ into a partially ordered ring: for $\frac{a}{b}, \frac{c}{d} \in q(R)$ define $\frac{a}{b} \preceq \frac{c}{d}$ when there exists $x, y \in R$ such that $xb = dy > 0$ and $ax \leq cy$. Furthermore, the authors prove that the order $\preceq$ extends the order on R. We denote $q(R)$ equipped with this order by $(q(R), \preceq)$.

Observe that what the authors of [10] have called complete is a reformulation of Fuch's division closed. Also, the condition $(*)$ is what we have called here positively saturated.

Theorem 2.16. [10, Theorem 7] *Suppose the po-domain $(R, \leq)$ is a complete integral domain satisfying $(*)$. Then $(R, \leq)$ is po-embeddable in $(q(R), \preceq)$.*

We would like to share some comments about the relation $\preceq$. Note that although the authors of [10] start with a different definition of the order, they show that it is equivalent to the one given above. Unfortunately, we are unable to prove that their original definition is well-defined. Thus, we modify their definition slightly and produce the positive cone as

$$\begin{aligned} q(R)^+ &= \{\frac{a}{b} : 0 \preceq \frac{a}{b}\} \\ &= \{0\} \cup \{q \in q(R) : q = \frac{c}{d} \text{ and } \exists y > 0 \text{ such that } cy > 0, dy > 0\}. \end{aligned}$$

Proposition 2.17. *Suppose R is a po-domain. Then $q(R)^+ = \mathcal{P}$.*

Proof. Suppose $0 \preceq \frac{c}{d}$. Without loss of generality, assume that $0 \neq \frac{c}{d}$. Thus, there is a $0 < y \in R$ such that $0 < cy, dy$. Then $\frac{c}{d} = \frac{cy}{dy} \in \mathcal{P}$. Conversely, if $q \in \mathcal{P}$, then there are $c, d \in R^+$ such that $q = \frac{c}{d}$. We can assume that $c > 0$, and $cd > 0$ and $d^2 > 0$. Thus, $q \in q(R)^+$. □

Remark 2.18. Proposition 2.17 shows that it was not necessary that the authors in [10] restrict themselves to positively saturated *po*-domains. It also means that one direction of Theorem 2.5 is due to [10].

3. Integral Domains and ℓ-fields

In this section we define a notion which is stronger than that of positively saturated; we call the notion strongly positively saturated. We then characterize when an integral ℓ-domain is strongly positively saturated. We begin by recalling a characterization of totally ordered ℓ-fields which shall play a pivotal role in what follows.

Theorem 3.1. [3, p. 59, Corollary 2] *An ℓ-field R has positive squares if and only if it is totally ordered.*

We are interested in determining when $(q(R), \mathcal{P})$ is totally ordered. We will show that this condition is true when R has the following property.

Definition 3.2. We call a *po*-ring (R, R^+) *strongly positively saturated* if for each $0 \neq r \in R$ there is an $s \in \mathrm{Reg}(R)^+$ such that $sr > 0$ or there is a $t \in \mathrm{Reg}(R)^+$ such that $tr < 0$.

A *po*-ring that is strongly positively saturated is certainly positive saturated. Below we shall supply an example of a positively saturated *po*-ring which is not strongly positively saturated. First, we show the importance of this new condition.

Theorem 3.3. *Suppose R is a non-trivial po-ring. Then $(q(R), \mathcal{P})$ is totally ordered if and only if R is strongly positively saturated.*

Proof. Suppose that $(q(R), \mathcal{P})$ is totally ordered and let $0 \neq a \in R$. By hypothesis, either $a \in \mathcal{P}$ or $-a \in \mathcal{P}$. If $a \in \mathcal{P}$, then this means $a = \frac{x}{s}$ for $0 < x \in R$ and $s \in \mathrm{Reg}(R)^+$. Then $sa = x > 0$. If $-a \in \mathcal{P}$, then $-a = \frac{y}{t}$ for $0 < y \in R$ and $t \in \mathrm{Reg}(R)^+$. Thus, $ta = -y < 0$. It follows that R is strongly positively saturated.

Conversely, suppose R is strongly positively saturated, and let $0 \neq \frac{a}{q} \in q(R)$. So there are $q_1, q_2 \in \text{Reg}(R)^+$ such that $q_1 a > 0$ or $q_2 a < 0$. Similarly, there are $p_1, p_2 \in \text{Reg}(R)^+$ such that $p_1 q > 0$ or $p_2 q < 0$. If $q_1 a > 0$ and $p_1 q > 0$, then $\frac{a}{q} = \frac{q_1 p_1 a}{q_1 p_1 q} \in \mathcal{P}$. If $q_1 a > 0$ and $p_2 q < 0$, then $\frac{a}{q} = \frac{q_1 p_2 a}{q_1 p_2 q} \in -\mathcal{P}$. If $q_2 a < 0$ and $p_1 q > 0$, then $\frac{a}{q} = \frac{q_2 p_1 a}{q_2 p_1 q} \in -\mathcal{P}$. If $q_2 a < 0$ and $p_2 q < 0$, then $\frac{a}{q} = \frac{q_2 p_2 a}{q_2 p_2 q} \in \mathcal{P}$. Thus, $(q(R), \mathcal{P})$ is totally ordered. □

We can say more when R is an integral ℓ-domain.

Theorem 3.4. *Suppose R is an integral ℓ-domain. The following statements are equivalent:*

(a) *$(q(R), \mathcal{P})$ is an ℓ-ring.*
(b) *$(q(R), \mathcal{P})$ is a d-ring.*
(c) *$(q(R), \mathcal{P})$ is an ℓ-ring that has positive squares.*
(d) *$(q(R), \mathcal{P})$ is a totally ordered field.*
(e) *R is strongly positively saturated.*

Proof. It suffices to show that if $(q(R), \mathcal{P})$ is an ℓ-ring, then it is a d-ring since the other conditions are known to be equivalent, and the others surely imply that $(q(R), \mathcal{P})$ is an ℓ-ring.

So assume that $(q(R), \mathcal{P})$ is an ℓ-ring. For each $s \in \mathcal{P}$, $\frac{1}{s} \in \mathcal{P}$ by Proposition 2.8, and so $s \in d(q(R))$ by Theorem 2.10). It follows that $(q(R), \mathcal{P})$ is a d-ring. □

Remark 3.5. Since it is known that there is no total ordering on $\mathbb{C}$ making it into an ℓ-field, it follows that any *po*-ordering on $\mathbb{C}$ making it into a *po*-ring is not strongly positively saturated.

Next, we supply a partial generalization of Theorem 3.4.

Proposition 3.6. *Suppose R is a d-ring. Then R is strongly positively saturated if and only if R is totally ordered.*

Proof. Clearly, if R is totally ordered, then it is strongly positively saturated. Conversely, suppose that the d-ring is strongly positively saturated and let $0 \neq r \in R$. First, consider the case that there is an $s \in \text{Reg}(R)^+$ such that $rs > 0$. By hypothesis and Proposition 1.6,

$$sr = |sr| = |s||r| = s|r|.$$

Since s is regular, it follows that $r = |r| \geq 0$. Similarly, if there is a $t \in \text{Reg}(R)^+$ such that $tr < 0$, then $r < 0$. Consequently, R is totally ordered. □

Example 3.7. (1) The ℓ-ring $\mathbb{Q}(\sqrt{2})$, ordered component-wise, is a strongly positively saturated field which does not have positive squares. The ordering $\mathcal{P}$ is the usual total ordering. It follows (Theorem 3.3) that $\mathbb{Z}[\sqrt{2}]$, ordered component-wise, is also strongly positively saturated.

(2) Consider the group ring $\mathbb{R}[G]$ with $G = \langle a \rangle$ being a cyclic group of order 2, ordered component-wise. An element $x + ya$ with $x, y \in \mathbb{R}$ is regular if

and only if $y \neq \pm x$. So if $x + ya$ is regular, then $x - ya$ is also regular and $(x + ya)(x - ya) = x^2 - y^2 > 0$ or < 0. Thus, $\mathbb{R}[G]$ is a positively saturated ℓ-ring. However, for $1 - a$ and any regular element $x + ya$, $(1 - a)(x + ya) = (x - y) + (y - x)a$ is not comparable to 0. Consequently, $\mathbb{R}[G]$ is not strongly positively saturated.

(3) Example 1.8 (2) is an example of an ℓ-ring that has positive squares, and that is also strongly positively saturated since for any polynomial $f(x) \neq 0$, $xf(x)$ is comparable to 0.

(4) Example 2.13 is an example of a *po*-ring for which the square of each regular element is positive. However, it is not strongly positively saturated. Of course, any such example cannot be an integral ℓ-domain.

4. Separating elements in ℓ-rings

We turn our attention to ℓ-rings. We first show that the class of regular division closed ℓ-rings is significant in connection to d-rings. We then isolate and explore a property shared by elements in such rings; we call this property "separating". We use the separating property to classify regular division closed ℓ-rings. Next, we shall consider some related properties and discuss how they relate to each other, as well as to other well-known properties.

Proposition 4.1. *Let R be an ℓ-ring. If $\mathrm{Reg}(R)^+ \subseteq d(R)$, then R is regular division closed. In particular, a d-ring is regular division closed.*

Proof. Suppose $ab > 0$ and $b > 0$ is regular. Then, by Proposition 1.6, $ab = |ab| = |a|b$ and hence $(a - |a|)b = 0$. Since b is regular it follows that $a - |a| = 0$, and so, $a = |a| > 0$. □

Proposition 4.2. *Let R be an ℓ-ring. Then $\mathrm{Reg}(R)^+ \subseteq d(R)$ if and only if the set $\mathcal{P}$ is a lattice with the lattice operations defined by: for all $\frac{a}{q}, \frac{b}{p} \in \mathcal{P}$, $\frac{a}{q} \wedge \frac{b}{p} = \frac{ap \wedge bq}{qp}$ and $\frac{a}{q} \vee \frac{b}{p} = \frac{ap \vee bq}{qp}$, and R^+ is a sublattice of $\mathcal{P}$.*

Proof. Suppose that every positive regular element is a d-element. We show that for all $\frac{a}{q}, \frac{b}{p} \in \mathcal{P}$, $\frac{a}{q} \wedge \frac{b}{p} = \frac{ap \wedge bq}{qp}$. That $\frac{a}{q} \vee \frac{b}{p} = \frac{ap \vee bq}{qp}$ can be proved similarly. Since we are concerned with $\mathcal{P}$, we assume that our elements are positive and that our denominators are regular. Clearly, $\frac{ap \wedge bq}{qp} \leq \frac{a}{q}, \frac{b}{p}$. Suppose that $\frac{c}{h} \in \mathcal{P}$ such that $\frac{c}{h} \leq \frac{a}{q}, \frac{b}{p}$. Then $\frac{cqp}{hqp} \leq \frac{ahp}{hqp}, \frac{bhq}{hqp}$, so $cqp \leq ahp, bhq$. Thus,

$$cqp \leq ahp \wedge bhp = h(ap \wedge bq)$$

since h is a d-element, and hence $\frac{c}{h} \leq \frac{ap \wedge bq}{qp}$. Hence, $\frac{a}{q} \wedge \frac{b}{p} = \frac{ap \wedge bq}{qp}$. That R^+ is a sublattice of $\mathcal{P}$ is clear.

Conversely, suppose that $a \wedge b = 0$ in R. Then, as R^+ is a sublattice of $\mathcal{P}$, $\frac{a}{1} \wedge \frac{b}{1} = 0$. Let $p \in \mathrm{Reg}(R)^+$. Then

$$0 = \frac{a}{1} \wedge \frac{b}{1} = \frac{ap}{p} \wedge \frac{b}{1} = \frac{ap \wedge bp}{p}.$$

It follows that $ap \wedge bp = 0$, whence $p \in d(R)$. □

Example 4.3. Since every d-ring is regular division closed it is a natural question whether every almost f-ring is regular division closed. We recall Example 16 of [3]. Let $P = x\mathbb{R}[x]$ equipped with the lex-ordering such that $f \in P$ is positive precisely when its leading coefficient is positive. According to the terminology in [13], P is a supertesimal ℓ-ring (i.e. for all $f \in P^+$, $f \ll f^2$). Set $A = P \times \mathbb{R} \times P \times P$ with coordinate-wise addition. The ordering on A is given by the following positive cone:

$$A^+ = \{(f, m, f_1, f_2) \in A : f > 0\} \cup \{(0, m, f_1, f_2) \in A : m > 0\} \cup \{(0, 0, f_1, f_2) : f_1, f_2 \geq 0\}.$$

The multiplication on A is given by

$$\begin{aligned}(f, m, f_1, f_2)(g, n, g_1, g_2) = \\ (nf + mg + fg, mn, f(g_1 + g_2) + g(f_1 + f_2) + nf_1 + mg_1, \\ f(g_1 + g_2) + g(f_1 + f_2) + nf_2 + mg_2).\end{aligned}$$

(The above product is slightly different than the one used in [3] in that the first coordinate of their product is $nf+mg+2fg$.) Notice that $1_A = (0, 1, 0, 0)$ is a weak-order unit. Therefore, A is an almost f-ring.

One can check, by computation, that the set of d-elements of A is

$$d(A) = \{(0, m, f_1, f_2) \in A^+ : m \geq 0\}.$$

The regular elements of A are given by

$$\mathrm{Reg}(A) = \{(f, m, f_1, f_2) \in A : m \neq 0\}.$$

Moreover, we know that A is not semiprime (see Proposition 1.5); we observe that the nilradical of A is

$$n(A) = \{(0, 0, f_1, f_2) \in A : f_1, f_2 \in P\}.$$

Notice that

$$(x^2, 1, 0, 0)(0, 0, x^3, -x^2) = (0, 0, x^5 - x^4 + x^3, x^5 - x^4 - x^2) > 0,$$

while $(0, 0, x^3, -x^2) \not\geq 0$, and hence A is not regular division closed. Consequently, Proposition 4.1 cannot be generalized from d-rings to almost f-rings.

The desire to gain a deeper understanding of why the ℓ-ring A of the previous example is neither a d-ring nor regular division closed led us to consider the following types of elements. This new type of positive element will be useful in characterizing regular division closed ℓ-rings.

Definition 4.4. Let R be a *po*-ring and $0 < e \in R$.

(1) We call e a *separating element* (*s-element*) if for all $0 < a, b \in R$, if a and b are incomparable, then one of the following three statements is true: i) $ea = 0$, ii) $eb = 0$, or iii) ea and eb are incomparable. We call the ℓ-ring R an *s-ring* if each $0 < a \in R$ is an s-element.

(2) We define $0 < e \in R$ to be a *super separating element* (*ss-element*) if whenever $a \in R$ is incomparable to 0, then ea is incomparable to 0. We call the ℓ-ring R an *ss-ring* if each $0 < a \in R$ is an ss-element.

Lemma 4.5. *Let R be an ℓ-ring and $0 < e \in R$. Then e is an ss-element if and only if whenever $0 < a, b \in R$ and $a \wedge b = 0$, then ea and eb are incomparable.*

Proof. Suppose that e is an ss-element and let $0 < a, b \in R$ be disjoint. Set $x = a-b$ which is incomparable to 0. Therefore, $ea-eb = ex$ is incomparable to 0. It follows that neither $ea \leq eb$ nor $eb \leq ea$, i.e. ea and eb are incomparable.

Conversely, suppose that e satisfies the property that whenever $0 < a, b \in R$ and $a \wedge b = 0$, then ea and eb are incomparable. Let $x \in R$ be incomparable to 0. Then $0 < x^+, x^-$ so that since $0 = x^+ \wedge x^-$, we can apply the hypothesis to conclude that ex^+ and ex^- are incomparable. Then if $ex^+ - ex^- = ex \geq 0$, then $ex^+ \geq ex^-$, a contradiction. Also, if $ex \leq 0$, then $ex^+ \leq ex^-$, a contradiction. Consequently, ex and 0 are incomparable. □

The following is immediate.

Proposition 4.6. *Let R be an ℓ-ring. Each ss-element of R is an s-element.*

When looking at Lemma 4.5 and phrasing the analogous result in the context of s-elements, something interesting occurs: we shall see shortly that such a lemma is not true. Thus, we are lead to a third definition.

Definition 4.7. Let R be a *po*-ring and $0 < e \in R$. (3) We call e an *$s^\flat$-element* if whenever $0 < a, b \in R$ and $a \wedge b = 0$, then one of the following three statements is true: i) $ea = 0$, ii) $eb = 0$, or iii) ea and eb are incomparable. We call the ℓ-ring R an *$s^\flat$-ring* if each $0 < a \in R$ is an $s^\flat$-element.

Clearly, an s-element is an $s^\flat$-element. For regular elements we can say more.

Corollary 4.8. *Let R be an ℓ-ring and $0 < e \in Reg(R)$. Then e is an $s^\flat$-element if and only if e is an ss-element.*

Proof. This is a corollary to Lemma 4.5 since notice that regularity of e rules out the possibilities of $ea = 0$ or $eb = 0$. □

We are now in position to characterize regular division closed ℓ-rings.

Theorem 4.9. *Suppose R is an ℓ-ring. The following statements are equivalent.*

(a) *R is regular division closed.*
(b) *Every positive regular element of R is an ss-element.*
(c) *Every positive regular element of R is an s-element.*
(d) *Every positive regular element of R is an $s^\flat$-element.*

Proof. It is obvious (Corollary 4.8) that (b), (c), and (d) are equivalent.

Suppose that R is regular division closed and $e \in R^+$ is a regular element of R. Let $0 < a, b \in R$ be incomparable. If $ea \geq eb$, then $e(a-b) \geq 0$, so $a \geq b$, a contradiction of our choice of a and b. Similarly, the inequality $ea \leq eb$ is not possible. Thus, ea and eb are incomparable, whence e is an ss-element.

Now suppose that each positive regular element is an ss-element and let $a \in R$ and $b \in \mathrm{Reg}(R)$, satisfy $ab, b \in R^+$. If a and 0 are incomparable, then since b is an ss-element we gather that ab and 0 are incomparable, a contradiction. Therefore, $a \in R^+$, whence R is regular division closed. □

We next consider the relationship between our classes of elements.

Proposition 4.10. *Let R be an ℓ-ring and $0 < d \in R$. If d is a d-element, then d is an $s^\flat$-element.*

Proof. To show that the d-element $0 < d \in R$ is an $s^\flat$-element, let $0 \le a, b$ satisfy $a \wedge b = 0$. Then $da \wedge db = 0$. If both $da \neq 0$ and $db \neq 0$, then da, db are incomparable. □

Recall that for an ℓ-ring R and $a \in R^+$, the annihilator of a is $\mathrm{Ann}(a) = \{x \in R : xa = 0\}$, while the ℓ-annihilator is the set $r_\ell(a) = \{x \in R : |x|a = 0\}$. Since in any ℓ-ring, $|xy| \le |x||y|$, it is straightforward to check that for $a \in R^+$, $r_\ell(a) \subseteq \mathrm{Ann}(a)$. We note that $r_\ell(a)$ is an ℓ-ideal of R while $\mathrm{Ann}(a)$ need not be. It is known that if a is a d-element, then $r_\ell(a) = \mathrm{Ann}(a)$ ([13, #17, p.138]). We can generalize this to $s^\flat$-elements.

Proposition 4.11. *Suppose R is an ℓ-ring and $0 < a \in R$ is an $s^\flat$-element. Then $r_\ell(a) = \mathrm{Ann}(a)$.*

Proof. We need only prove that if $xa = 0$, then $|x|a = 0$. To that end, suppose that $xa = 0$. Then $ax^+ = ax^-$. Since a is an $s^\flat$-element and $x^+ \wedge x^- = 0$, either $ax^+ = 0$, $ax^- = 0$, or ax^+, ax^- are incomparable. It follows that $ax^+ = 0$ or $ax^- = 0$ in which case both equations hold and so $a|x| = a(x^+ + x^-) = 0$. □

Corollary 4.12. *Suppose R is an ℓ-ring and $0 < a \in R$ is an $s^\flat$-element. If there exists a $b \in \mathrm{Reg}(R)$ such that $0 < b \le a$, then $a \in \mathrm{Reg}(R)$.*

Proof. Let $ax = 0$. Then by Proposition 4.11, $a|x| = 0$. But since $0 < b \le a$, then $0 \le b|x| \le a|x| = 0$, and so $b|x| = 0$. Using that b is regular it follows that $|x| = 0$, whence $x = 0$. Therefore, a is regular. □

Example 4.13. (1) Consider the trivial extension of $\mathbb{Z}$ by $\mathbb{Z}$, i.e. $R = \mathbb{Z} \propto \mathbb{Z}$ with multiplication $(a, b)(c, d) = (ac, ad + bc)$. Order R component-wise. The positive element $x = (0, 1)$ is a d-element and hence an $s^\flat$-element. We claim that x is not an s-element. Observe that the elements $a = (1, 2)$ and $b = (2, 1)$ are incomparable. However, $xa = (0, 1)$ and $xb = (0, 2)$ are comparable elements.

(2) Recall Example 1.8 (2). We claim that $x + 1$ is an ss-element. To this end, we just need to show that if a polynomial $f(x)$ is incomparable to 0, then $(x + 1)f(x)$ is incomparable to 0. Suppose that $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$. We may assume that $a_n > 0$ and $a_0 < 0$. Then $(x + 1)f(x) = a_n x^{n+1} + \cdots + (a_1 + a_0)x + a_0$ is incomparable to 0. Thus, $x+1$ is an ss-element. However, $x+1$ is not a d-element since $x \wedge 1 = 0$, but $(x^2 + x) \wedge (x+1) = x > 0$.

Theorem 4.14. *Suppose R is an $s^\flat$-ring with identity $1_R > 0$. Then R is an almost f-ring.*

Proof. Let R be an $s^\flat$-ring with $1 > 0$ We will show that 1 is a weak-order unit (Proposition 1.3). Let $a \in R$ satisfy $1 \wedge a = 0$. We show that $a = 0$. Since every regular element is an $s^\flat$-element it follows that R is regular division closed by Theorem 4.9. Notice that $1 \leq 1 + a$ and hence by Corollary 4.12, $1 + a$ is a regular element. Consider

$$(1 + a)(1 - a + a^2) = 1 + a^3 > 0.$$

We conclude that $0 < 1 - a + a^2$, i.e. $a < 1 + a^2$. Thus, by [13, Theorem 2.1.4(l)]

$$a = a \wedge (1 + a^2) \leq (a \wedge 1) + (a \wedge a^2) = a \wedge a^2$$

and hence $a \leq a^2$. Since a is an s-element and $1 \wedge a = 0$ it follows that either $a = 0$, $a^2 = 0$, or a and a^2 are incomparable. In the second case, $0 \leq a \leq a^2 = 0$, and we can rule out the third case. So $a = 0$, and it follows that 1 is a weak order unit, and so R is an almost f-ring. □

Proposition 4.15. *Suppose R is an ℓ-ring with no nonzero positive nilpotent. If $0 < x$ is an ss-element, then $x \in \mathrm{Reg}(R)^+$. In particular, a semiprime almost f-ring is an ss-ring if and only if it is a totally ordered integral domain.*

Proof. We suppose that $0 < x$ is an ss-element and that $xa = 0$. By Proposition 4.11, $\mathrm{Ann}(x) = r_\ell(x)$ and so $|a|x = 0$. Thus $(x \wedge |a|)^2 \leq x|a| = 0$, and hence $x \wedge |a| = 0$. If $|a| > 0$, then x^2 and $x|a| = 0$ are incomparable, a contradiction. Therefore, $a = 0$, whence $x \in \mathrm{Reg}(R)^+$.

Next, let R be a semiprime almost f-ring that is an ss-ring. Then, by what has just been proved, every positive element (and hence every negative element) is regular. If $a \in R$, then $a^- \wedge a^- = 0$, and hence because R is an almost f-ring, $a^+a^- = 0$. So either $a^+ = 0$ or $a^- = 0$. Therefore, every element is either positive or negative, whence R is a totally ordered integral domain. The reverse implication is clear. □

We have shown that every d-element and all s-elements are $s^\flat$-elements. There are examples of d-elements which are not s-elements and examples of s-elements which are not d-elements. Unfortunately, we have been unable to show that there is a difference between the notions of an s-element and ss-element. If such a difference does exist, it must occur in an ℓ-ring with no positive regular element as our next result illustrates.

Theorem 4.16. *Let R be an ℓ-ring with $r \in Reg(R)^+$. The element $0 < e \in R$ is an s-element if and only if $0 < e \in R$ is an ss-element.*

Proof. By Proposition 4.6, every ss-element is an s-element.

Suppose that e is an s-element, and by way of contradiction, suppose further that e is not an ss-element. By Lemma 4.5, since e is not an ss-element, there are $0 < a, b$ such that $a \wedge b = 0$ and either $ea \leq eb$ or $eb \leq ea$. Since e is an

s-element, it follows that $ea = 0$ or $eb = 0$. Without loss of generality, suppose that $ea = 0$, and so $ea \le eb$. Set $x = r + a$ and $y = r + b$ and observe that $0 < x, y$. Notice that the incomparability of a and b forces the incomparability of x and y: if $r + a \le r + b$, then $a \le b$. Next, note that

$$ex = e(r + a) = er > 0$$

where the inequality stems from the regularity of r. Moreover,

$$ey = e(r + b) = er + eb \ge er = ex > 0.$$

Since none of the following three statements are true: i) $ex = 0$, ii) $ey = 0$, or iii) ex and ey are incomparable, we conclude that e is not an s-element, the desired contradiction. ☐

Questions: (1) Is there an example of an $s^\flat$-ring that is not a d-ring?
(2) Is a regular division closed ℓ-ring an almost f-ring?

5. Division Closed ℓ-rings

In this section we investigate division closed ℓ-rings. We do not assume that the ℓ-rings are integral domains. Therefore, to be clear, we restate the definition of a division closed ℓ-ring: for all $a, b \in R$, if $0 < a$ and $0 < ab$, then $b > 0$.

Theorem 5.1. *Let R be an ℓ-ring. The following statements are equivalent.*

(a) *The ℓ-ring R is division closed.*
(b) *For any $0 < x \in R^+$, and $a, b \in R^+$, if $xa > xb$, then $a > b$.*
(c) *For any $0 < x, b \in R^+$ and $0 < a \in R \setminus \mathrm{Ann}(x)$, $a \wedge b = 0$ implies that xa and xb are incomparable.*

Proof. Suppose (a) is true and let $x > 0$, $a, b \in R^+$ satisfy $xa > xb$. Then $x(a - b) > 0$ and so, by the division closed property, $a - b > 0$, i.e. $a > b$. Similarly, $xa < xb$ implies $a < b$. Therefore, (b) is true.

Suppose (b) is true and let $x > 0$ and $xa > 0$. Then $0 < x(a^+ - a^-) = xa^+ - xa^-$ and so, $xa^- < xa^+$, which by hypothesis implies that $a^- < a^+$. But this yields that $a = a^+ - a^- > 0$. Therefore, (a) is true.

Suppose (a) is true. Let $0 < x, b \in R^+$ and let $0 < a \in R \setminus \mathrm{Ann}(x)$ satisfy $a \wedge b = 0$. First, we consider what happens if $xa = xb$. In this case, $x(2a - b) = 2xa - xb = xa > 0$. Thus, by (a), $0 < 2a - b$ and so $b < 2a$. But since $a \wedge b = 0$, then also $0 = 2a \wedge b$, a contradiction. Therefore, $xa \neq xb$. If xa, xb are comparable, then either $xa > xb$ or $xa < xb$. In the first case, (b) yields that $a > b$ so that $b = 0$, a contradiction. In the second case, (b) yields that $b > a$ and so $a = 0$, a contradiction. Therefore, xa and xb are incomparable, whence (c) is true.

Suppose (c) is true. Let $0 < x \in R^+$ and $xy > 0$. Thus, $xy^+ > xy^- \ge 0$. Assume that $y^- > 0$. Note that since $y^+ \in R \setminus \mathrm{Ann}(x)$ then, by (c) (applied to $a = y^+$ and $b = y^-$) implies that xy^+ and xy^- are incomparable, a contradiction. Consequently, $y^- = 0$ and so $y = y^+ > 0$. Therefore, (a) is true. □

Proposition 5.2. *Let R be an ℓ-ring with identity 1. If R is division closed, then $0 < 1$ and 1 is a weak order unit. Therefore, R is an almost f-ring.*

Proof. Since R is an ℓ-ring it contains a positive element, say $0 < x$. Then $1 \cdot x = x > 0$ so the division closed property forces that $0 < 1$. Next, suppose that $1 \wedge a = 0$ and $a > 0$. Then

$$(1 + a)(1 - a + a^2) = 1 + a^3 > 0.$$

Again applying the division closed property yields that $1 - a + a^2 > 0$. This inequality, in turn, yields the inequality $a < 1 + a^2$. But, since 1 and a are disjoint, we can use the Riesz Decomposition Theorem [13, Theorem 2.1.4(k)]to show that that $a \leq a^2$. Then $a < 2a^2$ and, hence $a(2a - 1) > 0$. The division closed property once again yields that $2a > 1$, a contradiction of our assumption that $1 \wedge a = 0$. □

Proposition 5.3. *Let R be a po-ring that does not contain any nonzero positive nilpotent element. If R is division closed, then for any $a, b \in R^+$, $ab = 0$ implies $a = 0$ or $b = 0$.*

Proof. Let R be a *po*-ring with no nonzero positive nilpotent element and suppose that R is division closed. Let $a, b \in R^+$ such that $ab = 0$ and $a > 0$. We show that $b = 0$. Now, $a(a-b) = a^2 - ab = a^2 > 0$, so $a - b > 0$ and hence $a > b$. Multiplying both sides by b yields that $0 = ab \geq b^2$ and so $b^2 = 0$. Thus, $b = 0$. □

Corollary 5.4. (1) *Let R be a unital ℓ-ring containing no nonzero positive nilpotent element. If R is division closed, then R is a totally ordered integral domain.*

(2) *Let R be a reduced d-ring (R need not have an identity). If R is division closed, then R is a totally ordered integral domain.*

Proof. (1) By Proposition 5.2, R is an almost f-ring. So for any $x \in R$, since $x^+ \wedge x^- = 0$, then $x^+x^- = 0$, and hence $x^+ = 0$ or $x^- = 0$ by Proposition 5.3. Thus, R is totally ordered, and hence R is an integral domain.

(2) By Proposition 5.3, if $ab = 0$, then $|a||b| = |ab| = 0$ by Proposition 1.6, and thus $|a| = 0$ or $|b| = 0$ by Proposition 5.3. Thus, $a = 0$ or $b = 0$. As in the proof of (1), R is a totally ordered integral domain. □

We provide some examples related to division closed ℓ-rings.

Example 5.5. (1) Example 1.8 (2) is not division closed and not an almost f-ring.

(2) Let R be the lexicographic extension of $\mathbb{R}$ over $\mathbb{R} \times \mathbb{R}$: additively $R = \mathbb{R} \times (\mathbb{R} \times \mathbb{R})$ and the positive cone of R is the union of $\{(a,(b,c)) \in R : a > 0\}$ and $\{(0,(b,c)) \in R : b, c \geq 0\}$. Then R is a non-archimedean ℓ-group. We

equip R with the trivial extension multiplication, that is, $(a,(x,y))(b,(u,c)) = (ab,(au+bx, ac+by))$. We leave it to the interested reader to show that R is an ℓ-ring with identity: $1_R = (1,(0,0))$, a strong order unit. In fact, R is division closed but not totally-ordered. It follows that R contains nonzero positive nilpotent elements, for example, $(0,(1,0))^2 = (0,(0,0))$.

(3) A division closed *po*-ring need not be an ℓ-ring. Consider the field of complex numbers $\mathbb{C}$ equipped with $\mathbb{R}^+$ as its positive cone. It is obvious that $\mathbb{C}^+$ is division closed. Since this order is not a total order, it follows (Corollary 5.4 (1)) that this order is not a lattice-order.

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