Appl Categor Struct (2013) 21:751–780
DOI 10.1007/s10485-012-9289-0

Saturation, Yosida Covers and Epicompleteness in Compact Normal Frames

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Received: 19 February 2012 / Accepted: 22 June 2012 / Published online: 22 July 2012

Abstract In this article the frame-theoretic account of what is archimedean for order-algebraists, and semisimple for people who study commutative rings, deepens with the introduction of $\mathcal{J}$*-frames*: compact normal frames that are join-generated by their saturated elements. Yosida frames are examples of these. In the category of $\mathcal{J}$-frames with suitable skeletal morphisms, the strongly projectable frames are epicomplete, and thereby it is proved that the monoreflection in strongly projectable frames is the largest such. That is news, because it settles a problem that had occupied the first-named author for over five years. In compact normal Yosida frames the compact elements are saturated. When the reverse is true one gets the *perfectly saturated* frames: the frames of ideals $\mathrm{Idl}(E)$, when E is a compact regular frame. The assignment $E \mapsto \mathrm{Idl}(E)$ is then a functorial equivalence from compact regular frames to perfectly saturated frames, and the inverse equivalence is the saturation quotient. Inevitable are the *Yosida covers* (of a $\mathcal{J}$-frame L): coherent, normal Yosida frames of the form $\mathrm{Idl}(F)$, with F ranging over certain bounded sublattices of the saturation SL of L. These Yosida frames cover L in the sense that each maps onto L densely and codensely. Modulo an equivalence, the Yosida covers of L form a poset with a top $\mathcal{Y}L$, the latter being characterized as the only Yosida cover which is perfectly saturated. Viewed correctly, these Yosida covers provide, in a categorical setting, another (point-free) look at earlier accounts of coherent normal Yosida frames.

Keywords Frames and frame homomorphisms · Epicompletion · Weakly closed maps · $\mathcal{J}$-frames · Yosida covers · Perfectly saturated frames

Mathematics Subject Classifications (2010) 06D22 · 13B21 · 18B15 · 18B35

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1 Introduction

The work that has become this paper was supposed to have continued in the spirit of [17], to deepen the insights of that article. Along the way Yosida frames suggested the idea of a $\mathcal{J}$-frame, and the connection with epicompletion in compact frames with skeletal maps became obvious. Many months later, the result is the present narrative.

The paper has five sections, of which the first is concerned with the basic and preliminary background materials. Section 3 works out the technical aspects of the presentation, that is to say, we settle on a category $\mathfrak{J}$: it will have $\mathcal{J}$-frames for objects, and the maps will be weakly closed frame homomorphisms that carry saturated elements to saturated elements. We learn enough about the monomorphisms and epimorphisms of $\mathfrak{J}$, to show that the strongly projectable frames in $\mathfrak{J}$ are epicomplete.

Section 4 introduces the perfectly saturated frames; they are the ideal frames of compact regular frames. Each $\mathcal{J}$-frame L has a system of Yosida covers—morphisms $e : Y \longrightarrow L$ that are dense, codense and surjective (and therefore monic!) The largest of these, $\mathcal{Y}L$, is perfectly saturated. It is shown here that a projective frame in $\mathfrak{J}$ is perfectly saturated and, further, that the projectives of $\mathfrak{J}$ are of the form $\mathrm{Idl}(E)$, where E is a projective compact regular frame.

Section 3 delivers a surprise: Proposition 3.3.6, which leads to Section 5, which solves the problem raised in [23] with the monoreflection ψ of a suitably chosen category of compact normal frames with skeletal frame maps in their full subcategory of strongly projectable frames. This monoreflection is the functorial epicompletion. The odyssey begun in [22], and long held suspended since [16], is over.

In Section 6 a number of applications are considered. It is shown that the frame of z-ideals of $C(X)$ (with X compact) is perfectly saturated if and only if X is perfectly normal.

2 The Basic Ingredients

This section reviews the principal elements that form the background for our presentation. Our main references for such information on frames are [12] and [24]. Any unexplained categorical terms may surely be found in [10].

2.1 Basic Catalog

A *frame* is a complete lattice in which the *frame law* holds:

$$a \wedge \left(\bigvee S\right) = \bigvee \{a \wedge x \,:\, x \in S\},$$

for each $S \subseteq L$.

Definition 2.1.1 Throughout, L is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in L$, denote the set of elements of L less than or equal to (resp. greater than or equal to) x by $\downarrow x$ (resp. $\uparrow x$).

- In a frame, a is *well below* b, written $a \preceq b : b \vee a^{\perp} = 1$; equivalently, if there is a c such that $b \vee c = 1$, while $c \wedge a = 0$.

- $x \in L$ is *regular*: $x = \vee\{a \in L : a \preceq x\}$. Let $\mathrm{Reg}(L)$ denote the subset of all regular elements of L. A frame L is *regular*: each element of L is regular.
- A frame L is *normal*: whenever $x \vee y = 1$, there exist disjoint $u, v \in L$, such that $u \leq x$ and $v \leq y$, and $1 = x \vee v = u \vee y$.
- A frame L is *joinfit* if for each $a \in L$ with $a \neq 0$, there is a $b < 1$ such that $a \vee b = 1$. Joinfit frames are introduced in [14].
- Let j be a closure operator j on a frame L.
  - j is *dense*: $j(0) = 0$.
  - Let $jL = \{x \in L : j(x) = x\}$. Then j is dense if and only if $0 \in jL$.
  - j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$, for all $a, b \in L$.

Finally, in this concise dictionary, there is the phrase "$\vee$-generated by a set T", or synonymously "join-generated by T", which will be used, mostly out of convenience, to indicate that each member of the structure being looked over may be written as a join of elements of T.

Regarding frame homomorphisms, the following remarks are in order as background material for this paper.

Definition & Remarks 2.1.2 A *frame homomorphism* (or, synonymously, *frame map*) is a function between frames which preserves all suprema and all finite infima (including the empty ones).

1. Throughout, $\mathfrak{Frm}$ denotes the category of all frames and all frame maps. If $h : L \longrightarrow M$ is a $\mathfrak{Frm}$-morphism, then $h_* : M \longrightarrow L$ denotes its right adjoint; that is, the map defined by

$$x \leq h_*(y) \iff h(x) \leq y, \quad \text{for all } x \in L, \; y \in M.$$

2. h is one-to-one if and only if $h_* \cdot h = 1_L$, and that h is surjective, if and only if $h \cdot h_* = 1_M$.
3. The frame map h is *codense* if $h(x) = 1$ implies that $x = 1$.
4. Next, we remind the reader of the notion of a closed frame homomorphism. First, a map of the form $x \mapsto x \vee a$ from L onto $\uparrow a$ is a *closed quotient*. In general, suppose that $h : L \longrightarrow M$ is a frame homomorphism, and let $q : M \longrightarrow F$ be a frame surjection. Factor $q \cdot h = m \cdot e$ through the image, as indicated in the commutative square below:

$$\begin{array}{ccc} L & \xrightarrow{\;h\;} & M \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ E & \xrightarrow{\;m\;} & F \end{array} \qquad (2.1.2.1)$$

The frame homomorphism h is *closed* if for each closed quotient q, e too is a closed quotient. It is shown in [24, Chapter II, §5] that h is closed if and only if

$$h(b) \leq h(a) \vee y \implies b \leq a \vee h_*(y).$$

5. Of prime importance in the sequel, however, are the frame maps which satisfy a more relaxed condition than that of closed maps. Suppose that in the characterization of closed maps given in the previous display one restricts to $b = 1$; that is, all that is assumed is

$$h(a) \vee y = 1 \implies a \vee h_*(y) = 1.$$

Frame maps with this property are said to be *weakly closed*. This concept was introduced in [16]. Here is a list of the basic features of weakly closed maps. For proofs of these, we refer the reader to [16, 3.2].
Let $h : L \longrightarrow M$ denote a frame homomorphism.

(a) The composite of two weakly closed maps is weakly closed.
(b) The frame map h is weakly closed if and only if for each $a \in L$ and $y \in M$

$$h(a) \vee y = 1 \iff a \vee h_*(y) = 1,$$

(c) The saturation s is weakly closed.
(d) If h is a dense weakly closed frame map, then it is also codense. Conversely, if h is surjective and also codense, then it is weakly closed.

2.2 Compact Normal Frames with s-Maps

We begin this section with a review of the saturation quotient. What we refer to here simply as "Banaschewski's Theorem" (Corollary 2.2.5) is the statement that for compact normal frames L, there is an isomorphism between $\mathrm{Reg}(L)$, the subframe of regular elements of L, and SL, the saturation quotient of L. One of the accomplishments of [17] was to place Banaschewski's Theorem in a suitable categorical context, in order to make the isomorphism of the theorem natural.

Definition 2.2.1 Let L be a compact frame. For each $a \in L$, consider all $b \in L$ such that $b \vee y = 1$ implies $a \vee y = 1$, Let $s(a)$ denote the join of all such b. By a standard compactness argument, $s(a) \vee y = 1$ implies $a \vee y = 1$. $s(a)$ is the *saturation* of a.

Note that s is a closure operator, and, in fact a nucleus, as s preserves finite intersection. Thus, SL, the set of *saturated* elements—those for which $a = s(a)$—is a frame, in which the infima agree with those of L. The reader should also bear in mind that although the finite join of saturated elements is generally not saturated, there will be instances when certain subsets of saturated elements are closed under finite joins, (such as we will encounter in Yosida frames, in which the compact elements are saturated.)

Proposition 2.2.2 *For any compact frame L, the saturation s_L is dense precisely when L is joinfit.*

Proof To say that $s_L(0) > 0$ is to say that there is an $x \in L$, $x > 0$, so that $x \vee y = 1$ implies that $y = 1$. Thus $s_L(0) > 0$ is tantamount to L not being joinfit. □

In order to regard the saturation quotient as a functor one must restrict the maps. Needed are the frame homomorphisms $h : L \longrightarrow M$ between compact frames for which there is a frame homomorphism $S(h) : SL \longrightarrow SM$, obviously unique, making the square below commute:

$$\begin{array}{ccc} L & \xrightarrow{\quad s_L \quad} & SL \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle S(h)} \\ M & \xrightarrow{\quad s_M \quad} & SM \end{array} \qquad (\dagger)$$

It is easy to see that $S(h)$ exists if and only if $s_L(x) = s_L(y)$ in L implies that $s_M(h(x)) = s_M(h(y))$, and then, necessarily, $S(h)(s_L(x)) = s_M(h(x))$, for each $x \in L$. A frame map with this property was called an *s-natural* map kn [20], and simply an *s-map* in [17], a label which we will keep in this presentation.

Observe that, if h is an s-map, then $S(h)_* = h_*|_{SM}$.

We quote from [17] a proposition describing the relationship between s-maps and weakly closed ones.

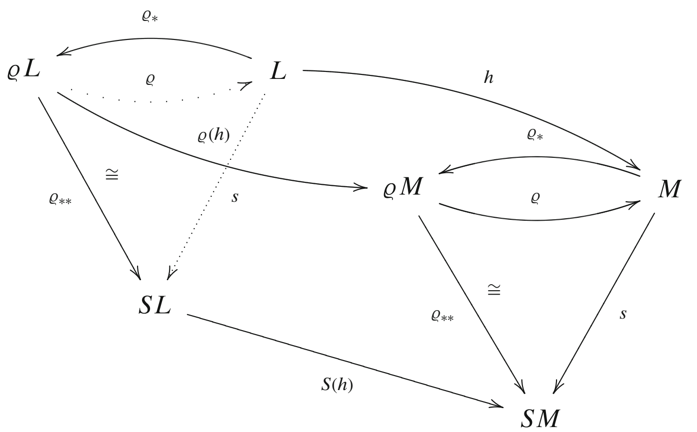
Proposition 2.2.3 *Let $h : L \longrightarrow M$ denote a frame homomorphism between compact frames. Regarding the three conditions listed below,*

- (a) *implies* (b)*;*
- *with Choice,* (b) *implies* (c)*, and if L is also normal, then the reverse is true;*
- *if both L and M are normal, then* (c) *implies* (a)*, and thus all three are equivalent.*

(a) *h is weakly closed.*
(b) *h_* maps maximal elements of M to maximal elements of L.*
(c) *h is an s-map.*

The following theorem from [17] describes the highlights of the relationship between a compact normal frame and its regular coreflection and the saturation.

Theorem 2.2.4 *For any s-map $h : L \longrightarrow M$ between compact normal frames, all faces of the prism below commute—the top face and two triangles commuting in both directions.*



(2.2.4.1)

Further,

1. *s defines an epireflection of the category $\mathfrak{KN}_s$ of compact normal frames and s-maps in the full subcategory $\mathfrak{KReg}_s$ of compact regular frames.*
2. *ϱ_* and s are natural transformations (in $\mathfrak{KN}_s$), and the functor ϱ, (regarded as a reflection) is naturally equivalent to S.*

We extract Banaschewski's Theorem out of Theorem 2.2.4 as a separate result.

Corollary 2.2.5 (Banaschewski) *For each compact normal frame L,* $\mathrm{Reg}(L)$ *and SL are isomorphic. The isomorphism is natural as indicated in diagram* (2.2.4.1).

2.3 Maps with Isomorphic Reducts

It is convenient to refer to SL as the *saturated reduct* of L. In like fashion, we will also call the induced $S(h) : SL \longrightarrow SM$ (for each weakly closed map $h : L \longrightarrow M$), the *saturated reduct* of h.

In this very brief section, we have some observations about the saturation nucleus itself, followed by a result which reveals when a reduct is an isomorphism. First, a well known characterization of the saturation; the reader is referred to [3].

Proposition 2.3.1 *For compact normal frames, the saturation quotient is, up to isomorphism, the unique codense frame homomorphism onto a compact regular frame.*

And this characterization leads to the following.

Proposition 2.3.2 *Suppose that $h : L \longrightarrow M$ is a weakly closed frame surjection of compact normal joinfit frames. Then $S(h)$ is an isomorphism if and only if h is dense.*

Proof If $S(h)$ is an isomorphism, then since the saturations on L and M are dense, we are able to conclude that (as a first factor of a dense map) h must be dense.

Conversely, if h is dense and weakly closed, then it is also codense, which implies that $s_M \cdot h$ is a codense frame surjection onto SM. Applying Proposition 2.3.1, there must be an isomorphism $u : SL \longrightarrow SM$ such that $u \cdot s_L = s_M \cdot h$. It is then clear that $u = S(h)$, and we are done. □

3 $\mathcal{J}$-Frames

We have come to the heart of the paper, that is, to settle among the objects we wish to study. They are the $\mathcal{J}$-frames, a formal definition of which will presently be given. This class of frames prominently includes all coherent Yosida frames. We proceed to detail what this means.

3.1 Yosida Frames, for Motivation

Foremost, and for the record, here is the definition of a $\mathcal{J}$-frame.

Definition & Remarks 3.1.1 A compact normal frame in which every element is a supremum of saturated elements is called a $\mathcal{J}$*-frame*. It will be seen that compact normal frames are necessarily $\mathcal{J}$-frames.

(a) In [21, Section 3] it is shown—albeit with the assumption that coherent frames are spatial—that any coherent normal Yosida frame can be described as the frame of ideals of a bounded sublattice of the frame $\mathfrak{O}(\mathrm{Max}(L))$ which is a base for the open sets. The setup we will describe in this section is an extension of that correspondence, and we begin with the "base".
Before delving into a study of $\mathcal{J}$-frames, we remind the reader of some basic notions in frames.
  1. *algebraic frame:* a frame that is join-generated by its compact elements.
  2. An algebraic frame is *coherent* If the finite meet of compact elements (including the empty meet) is always compact.

These frames bear the name "Yosida" because of the connection to Yosida spaces, so crucial in the development of archimedean ℓ-groups with a unit, a connection which is somewhat tenuous in retrospect. Even at the time of [21] it had become obvious that coherent Yosida frames were actually the natural generalization of the frame of z-ideals of a ring $C(X)$ of continuous functions on a topological space; (see Section 6.2.)

(b) The concept of a Yosida frame is actually defined in [21] for any compact frame L: L is Yosida if and only if, for each pair of compact elements $a < b$, there is an $x \in L$—which may be taken compact as well—such that $a \vee x < 1$, while $b \vee x = 1$.
However, for purposes of this paper, it is more than sufficient to study coherent Yosida frames, and then, in the context of compact normal frames, we have that each compact element is saturated. Note as well that any coherent normal Yosida frame is a $\mathcal{J}$-frame.

Motivated by Yosida frames, we make the following introductions. To emphasize: throughout, all frames are compact and normal.

Definition & Remarks 3.1.2 Begin with a $\mathcal{J}$-frame L. Suppose that F is a bounded sublattice of SL such that each $x \in L$ is a join (in L) of elements of F. We shall refer to such a bounded sublattice F of SL as a *base of saturated elements*. It is easy to see that SL itself is a base of L. We will also refer to the base as the signature of L.

Each coherent normal Yosida frame has the *standard* signature $\mathfrak{k}(L)$, the sublattice of compact elements. As we shall see, this is the smallest signature.

In fact, concerning bases, there is the following observation.

Lemma 3.1.3 *Suppose that L is a $\mathfrak{J}$-frame. Every saturated compact element is a member of any base of L. Therefore, every complemented element of L belongs to every base, and if L is a coherent Yosida frame, then $\mathfrak{k}(L)$ is the least base of saturated elements.*

Proof Only the first claim needs proof, since every complemented element is compact and saturated, and, when L is a Yosida frame, the set of compact elements is a base.

Now if $a \in L$ is saturated and compact, then for any base F, $a = \bigvee Y$, for some finite subset of the base. Applying the saturation nucleus, one sees that a is also the join in SL of the members of Y. We conclude that $a \in F$. □

3.2 Ideal Frames

Given a bounded distributive lattice F, one may form the frame of its ideals $\mathrm{Idl}(F)$, which is a coherent frame. This construct is prominently employed in [21, Section 3], albeit with the standing assumption that all the coherent frames are spatial. The central result of that section of [21] (Theorem 3.5) will be reproduced here without any such assumptions. Some proofs will be sketched, while others will be left to the reader.

The frame of ideals $\mathrm{Idl}(F)$ is regarded as a frame with the meet operation being set-theoretic intersection, while the join $\bigvee \mathcal{S}$ of a family of ideals $\mathcal{S}$ is described as follows:

$$\bigvee \mathcal{S} = \{x \in F \;:\; x \leq a_1 \vee \cdots \vee a_n,\; a_i \in J_i,\; J_i \in \mathcal{S}\}.$$

Here is a technical, yet useful observation. It is followed by a proposition listing the basic features of $\mathrm{Idl}(F)$, when F is a base of saturated elements of the $\mathcal{J}$-frame L.

Lemma 3.2.1 *If $J \preceq K$ in* $\mathrm{Idl}(F)$*, then there exists an $x \in K$ such that $a \preceq x$, for all $a \in J$.*

Proof If $J \preceq K$ then $1 = y \vee x$, with $y \in J^{\perp}$ and $x \in K$. Thus, for each $a \in J$, $a \wedge y = 0$, so that $a \preceq x$. □

The first claim in the proposition that follows is [21, Theorem 3.5], assuming the frames in question are spatial.

Proposition 3.2.2 *For each base F of saturated elements of L, we have the following:*

1. $\mathrm{Idl}(F)$ *is a $\mathcal{J}$-frame, and, in fact, it is Yosida.*
2. *The function $\Theta : \mathrm{Idl}(F) \longrightarrow SL$ defined by $\Theta(J) = \bigvee J$ (calculated in SL) is a codense frame homomorphism onto SL.*
3. *Let $J \in \mathrm{Idl}(F)$; J is regular if and only if for each $x \in J$, there is a $y \in J$ such that $x \preceq y$.*
4. *$J \in \mathrm{Idl}(F)$ is saturated if and only if J is of the form*

$$J = (\downarrow x) \cap F, \quad x \in SL.$$

Proof It is well known that $\mathrm{Idl}(F)$ is coherent. As to the normality, suppose that $J \vee K = F$, in $\mathrm{Idl}(F)$. Then $x \vee y = 1$, for choices of $x \in J$ and $y \in K$. Use the normality of L to produce $a, b \in L$, with $a \wedge b = 0$, and $a \vee y = x \vee b = 1$. Next, apply the fact that each member of L is a join of elements in F together with the compactness of 1 to show that a and b may, in fact, be taken from F. As $a \leq x$ and $b \leq y$, we are able to conclude that the ideals J' and K' generated, respectively, by a and b witness the normality of $\mathrm{Idl}(F)$.

The assertion that $\mathrm{Idl}(F)$ is Yosida follows from 4 in the proposition, which is settled below. It is a consequence of 4, since the compact elements of $\mathrm{Idl}(F)$ are the principal ideals.

Let us now address the rest of what is asserted. To establish 2, we note the following:

- Θ *is codense.* Suppose that $\Theta(J) = 1$, for some ideal J of F. The compactness of 1 in SL implies that the join of a finite subset $F_0 \subseteq J$ is 1, if calculated in SL. But then $\bigvee F_0 = 1$ in L as well. Thus $1 \in J$ and $J = F$.
- We leave to the reader the proof that Θ preserves joins, which is straightforward.
- Now, let us show that Θ preserves finite intersections. For any two ideals J and K, it is obvious that $\Theta(J \cap K) \leq \Theta(J) \wedge \Theta(K)$. Reversing,

$$\Theta(J) \wedge \Theta(K) = \left(\bigvee J\right) \wedge \left(\bigvee K\right) = \bigvee \{a \wedge b \; : \; a \in J, \; b \in K\},$$

  by the frame law, and each such $a \wedge b \in J \cap K$, whence $\Theta(J) \wedge \Theta(K) \leq \Theta(J \cap K)$.

For 3, assuming J is regular in $\mathrm{Idl}(F)$ and $x \in J$, we have finitely many ideals $I_1, I_2, \ldots, I_n$, all $I_j \preceq J$, with $x \in I \equiv I_1 \vee \cdots \vee I_n$. But $I \preceq J$, and so, by Lemma 3.2.1, there is a $y \in J$ such that $x \preceq y$. Conversely, observe that since J is the join of the principal ideals it contains, the hypothesis insures that these principal ideals are well below J.

Regarding 4, observe that if $J \in \mathrm{Idl}(F)$ and $y = \bigvee J$ (computed in SL), then $J \subseteq (\downarrow y) \cap F$, and for each ideal K,

$$J \vee K = F \iff 1 = a \vee b, \; a \in J, \; b \in K \iff ((\downarrow y) \cap F) \vee K = F.$$

We underscore: the arrows from left to right are straightforward. In the reverse direction, the reader should realize that $(\downarrow y) \cap F) \vee K = F$ does imply that $1 = a \vee b$, for suitable $a \in J$, and $b \in K$, because 1 is compact and by the way y is defined. This proves that if J is saturated, then $J = (\downarrow y) \cap F$.

Conversely, if $J = \{x \in F : x \leq y\}$, with $y \in SL$, then if J' is an ideal that properly contains J, we have an $a \in J'$ such that $a \not\leq y$. This implies that there is some $b \in SL$ satisfying $b \preceq a$ but not well below y. In turn, we may find a $c \in SL$, such that $c \wedge b = 0$, $c \vee a = 1$, and $c \vee y < 1$. Next, we may replace c with an element of F if necessary, as follows, in two steps:

1. Using the fact that the members of F generate L, there exist $x_1, \ldots, x_m$ in F, such that $x_1 \vee \cdots \vee x_m \vee a = 1$, with each $x_i \leq c$.
2. Now, let c' be the join of the x_i in SL, and observe that $c' \leq c$, and $c' \vee a = 1$ (in both L and SL!), with $c' \in F$.

Thus, we have $((\downarrow c) \cap F) \vee J < F$, whereas $((\downarrow c) \cap F) \vee J' = F$. □

Propositions 2.3.1 and 3.2.2.2 imply the following.

Corollary 3.2.3 *Suppose that F is a base of saturated elements of the $\mathfrak{J}$-object L. Then $S(\mathrm{Idl}(F)) = SL$.*

We conclude the section with a result that shows how an algebraic property of a $\mathcal{J}$-frame is reflected in its saturation quotient. The reader is reminded that a frame is said to be *strongly projectable* if every polar is complemented. This property is also referred to in the literature as "extremally disconnected". These frames are also called "de Morgan" frames.

Proposition 3.2.4 *A compact normal joinfit frame L is strongly projectable if and only if SL has that property.*

Proof It suffices to note that $x^{\perp\perp} \vee x^{\perp} = 1$ *in* L if and only if $x^{\perp\perp} \vee x^{\perp} = 1$ *in* SL, and recall that polars are saturated [14, Corollary 4.2]. □

3.3 Monomorphisms, et al.

This section zeroes in on monomorphisms and epimorphisms of $\mathfrak{J}$. Our purpose is to understand the connection between surjective frame maps which are both dense and codense, and those which induce an isomorphism on the saturations. There will be some redundancy in the proofs, establishing the same fact in different ways. Yet here we have a circumstance in category theory, in which monomorphisms are not-necessarily one-to-one, and for this reason alone the details are interesting.

Let us begin with epimorphisms, with a little lemma that does enough.

Lemma 3.3.1 *Let $h : L \longrightarrow M$ be a frame homomorphism. Then*

1. *If h is a morphism in $\mathfrak{J}$, then it is an epimorphism if and only if $S(h)$ is epic in $\mathfrak{KReg}$.*
2. *If h is a morphism and $S(h)$ is surjective, then so is h.*

Proof

(1) Observe, to start, that since the morphisms of $\mathfrak{J}$ map saturated elements to saturated elements, $S(h) = h_{|_{SL}}$.
Now, it follows from the commutativity in (†) that if h is epic, then $S(h)$ is epic. Conversely, suppose $S(h)$ is epi, and j and k are $\mathfrak{J}$-morphisms, with $j \cdot h = k \cdot h$. Then

$$S(j) \cdot S(h) = S(j \cdot h) = S(k \cdot h) = S(k) \cdot S(h),$$

whence $S(j) = S(k)$. Because M is a $\mathcal{J}$-frame and therefore generated by its saturated elements, $j = k$.

(2) If $S(h)$ is surjective, then, since SM (resp. $E(M)$) join-generates M, we may conclude that h is also surjective. □

The following observation should stand out, as highlighted here.

Corollary 3.3.2 *Suppose that $h : L \longrightarrow M$ is a $\mathfrak{J}$-morphism. If $S(h)$ is an isomorphism, then h is a dense surjection.*

The following observation about monomorphisms seems to be indispensable.

Proposition 3.3.3 *Let $h : L \longrightarrow M$ be a frame map. If h is a $\mathfrak{J}$-morphism, we have regarding the following statements, that* 1 *and* 2 *are equivalent, and each implies* 3*:*

1. *h is dense.*
2. *The saturated reduct $S(h)$ is one-to-one.*
3. *h is monic.*

Proof

(1) $\Longrightarrow$ (2): If h is dense, then from (†), it follows that $S(h)$ is dense. Thus, in $S(h)$ we have a frame homomorphism between compact regular frames which is dense and closed (as is each frame map between compact regular frames.) Therefore $S(h)$ is one-to-one.

(2) $\Longrightarrow$ (3): If $S(h)$ is one-to-one, it is monic. Now if $h \cdot g_1 = h \cdot g_2$, we have $S(h) \cdot S(g_1) = S(h) \cdot S(g_2)$, whence $S(g_1) = S(g_2)$. In turn, this implies that g_1 and g_2 agree on the saturated elements of their common domain. Since they $\vee$-generates A, we conclude that $g_1 = g_2$, and so h is monic, thus proving (3).

(2) $\Longrightarrow$ (1): Assume $S(h)$ is one-to-one, yet $h(x) = 0$, for some nonzero x. Since ϱL is $*$-dense in L, we may suppose without loss of generality that $x \in \varrho L$; that is, $\varrho(h)$ is not one-to-one. But then, according to the natural equivalence of Theorem 2.2.4, neither is $S(h)$, which is a contradiction. □

Under much tighter circumstances, one may improve upon the conclusions in Proposition 3.3.3. The *diamond frame* is the four-element frame, denoted Δ, consisting of 0 as bottom, 1 as top, and a and b as mutually complementing nontrivial elements. Observe that the diamond frame is Yosida, and it has only one base, itself.

Proposition 3.3.4 *Every monomorphism out of a strongly projectable frame is dense (and hence one-to-one.).*

Proof Suppose $h : L \longrightarrow M$ is monic in $\mathfrak{J}$. If there is an $x \neq 0$ in L such that $h(x) = 0$, it may be assumed that x is saturated. For, if $h(s(x)) > 0$, then since h is weakly closed and the codomain of h is joinfit, there is a $u < 1$ such that $h(s(x)) \vee u = 1$, whence $s(x) \vee h_*(u) = 1$. Thus,

$$1 = x \vee h_*(u) = h(x) \vee u = u,$$

a contradiction.

Next, SL is strongly projectable and regular, so that $x = \bigvee Y$, for some subset Y of nonzero polars, which are complemented and, therefore, saturated. Note the following, keeping in mind that the maps here are $\mathfrak{J}$-morphisms:

1. $S(h)(x) = S(h)(s(x)) = s(h(x)) = s(0) = 0$, and so
2. for each $a \in Y$, $S(h)(a) = s(h(a)) = h(a)$, since (as argued above) $h(a)$ is saturated in M.
3. Conclusion:

$$\bigvee_{a \in Y} h(a) = S(h)(x) = 0,$$

   and therefore, without loss of generality, one may suppose that x is complemented.

We define two frame maps, f_1 and f_2, from Δ into L. Let

$$f_1(a) = x,\ f_1(b) = x^{\perp}, \text{ and } f_2(a) = 0,\ f_2(b) = 1.$$

This defines two frame homomorphisms, and the reader can plainly see that that, while $f_1 \neq f_2$, $h \cdot f_1 = h \cdot f_2$, contradicting that h is monic. □

Thanks to Lemma 3.3.1, we have the following corollary, which also invokes the two propositions concerning monomorphisms.

Corollary 3.3.5 *Let $h : L \longrightarrow M$ be a $\mathfrak{J}$-morphism, with a strongly projectable domain. Then $S(h)$ is an isomorphism if and only if h is a monic surjection.*

A straightforward application of Corollary 3.3.5 will lead the way, and also lead us in a direction and to a result we had not foreseen.

Proposition 3.3.6 *If L is a strongly projectable $\mathfrak{J}$-object, then it is epicomplete.*

Proof Suppose that the $\mathcal{J}$-frame L is strongly projectable and $m : L \longrightarrow M$ is monic and epic in $\mathfrak{J}$. Applying Proposition 3.3.4, together with Lemma 3.3.1.1, we have that the reduct $S(m)$ is an epic embedding in $\mathfrak{KReg}$. The proof of [22, Lemma 5.4], may be repeated, almost verbatim, to prove that $S(m)$ is an isomorphism, whence m is an isomorphism. □

We return to condition (2) in Proposition 3.2.2, in order to motivate, along with the preceding results of this section, the following definition. Let F be a base of saturated elements for the $\mathcal{J}$-frame L. Define $\epsilon : \mathrm{Idl}(F) \longrightarrow L$ by

$$\epsilon(J) = \bigvee J,$$

the above supremum calculated in L, for each ideal J in $\mathrm{Idl}(F)$.

It is a straightforward matter to check that ϵ is a frame map, and since L is a $\mathcal{J}$-frame, ϵ is surjective. Further, since ϵ is dense as well, and every dense codense surjective frame map is weakly closed, we have the following.

Proposition 3.3.7 *Let L be a $\mathcal{J}$-frame, and suppose that F is a base of saturated elements of L. Then $\epsilon : \mathrm{Idl}(F) \longrightarrow L$ is a dense surjective $\mathfrak{J}$-morphism. In addition, the right adjoint ϵ_* of ϵ is the function defined by*

$$\epsilon_*(x) = (\downarrow x) \cap F.$$

Proof Computing the right adjoint easily shows that ϵ_* is as described. □

3.4 Yosida Covers

Borrowing from the intuition in the theory of covers in topology, when $g : Y \longrightarrow L$ is a dense and codense frame surjection, we say that Y is a *cover* of L; if Y is also Yosida, then it is a *Yosida cover*. If, by circumstance, one should need to keep track of the maps that witnesses the covering, then the usage that calls the pair (Y, g) a *cover* of L should be clear.

The reader might guess by now that all Yosida covers are of the form $\epsilon : \mathrm{Idl}(F) \longrightarrow L$, for a suitable base F of the frame L. This is correct. This depends on the fact that if h is a surjective $\mathfrak{J}$-morphism, and F is a base for the domain, then $h(F)$ is a base for the range. The proof of that observation is straightforward; it is left to the reader. The claim made in this paragraph is best proved with the help of additional information (Proposition 3.4.4).

We have succeeded in associating, with each $\mathcal{J}$-frame L and each base F, a Yosida cover $f : Y(F, L) \longrightarrow L$. The map f is an isomorphism, if and only if L is Yosida and F is the standard signature. In Theorem 3.4.5 we give a more formal account, and round out the view of these Yosida covers. Leading up to that goal, the next few observations supply the missing elements.

Remark 3.4.1 One matter of some importance is whether one can get an upper bound on the size of the Yosida covers of a given $\mathcal{J}$-frame L. Now, according to Proposition 2.3.2, the dense, codense frame-surjections upon L are precisely the maps that induce an isomorphism between the respective saturations. As we will see in Theorem 3.4.5, a bound for $|L|$ is $|\mathrm{Idl}(SL)|$.

Proposition 3.4.2 *Suppose that ν is a dense, codense nucleus on the $\mathcal{J}$-frame L. Then*

1. *νL is a $\mathcal{J}$-frame, and*
2. *As a morphism in $\mathfrak{J}$, ν is monic.*

Proof Once it has been verified that the quotient νL is a $\mathcal{J}$-frame, the second claim then follows immediately from Proposition 3.3.3. As to the first claim, it suffices to show that if $x \in L$ is saturated, then so is $\nu(x)$. This is straightforward, and it is left to the reader. □

The above result, together with the propositions (in Section 2.3) on the subject of when a dense codense surjective frame map has a reduct which is an isomorphism, now implies the following.

Proposition 3.4.3 *Suppose that $h : L \longrightarrow M$ is a dense and codense frame surjection in $\mathfrak{J}$. Then h is monic.*

Proof We have from the proof of Proposition 2.3.2 that $s_M \cdot h = S(h) \cdot s_L$, and that $S(h)$ is an isomorphism. Thus, since s_L and s_M are monic according to Proposition 3.4.2, h too must be monic. □

Some formalism then, to properly close this section of the paper, again reminiscent of the structuring of covers of compact Hausdorff spaces.

Suppose that L is a $\mathfrak{J}$-object and (Y, f) and (Z, g) are Yosida covers of L. Say that the two covers are *equivalent* if there is an isomorphism $u : Y \longrightarrow Z$ such that $g \cdot u = f$. It is straightforward to show that this does define an equivalence relation. Next, for the above covers, we define $(Y, f) \geq (Z, g)$ if there is a frame map $v : Y \longrightarrow Z$ such that $g \cdot v = f$. Observe that v is necessarily monic. The relation is obviously reflexive and transitive, and if $(Y, f) \leq (Z, g)$ and $(Z, g) \leq (Y, f)$, then the two covers are equivalent.

We can now easily establish that all Yosida covers arise from the choice of a base of saturated elements.

Proposition 3.4.4 *Suppose $g : Y \longrightarrow L$ is a Yosida cover of L. Let $F = g(\mathfrak{k}(L))$. Then F is a base of saturated elements, and there is an isomorphism $g' : Y \longrightarrow$ Idl(F) such that $\epsilon \cdot g' = g$, and Y is equivalent to* Idl(F).

Proof That F is a bounded sublattice of saturated elements is clear because Y is Yosida and the map g preserves saturated elements. Since the frame map g is surjective, we obtain that F is a base of saturated elements.

Now define $g' : Y \longrightarrow \mathrm{Idl}(F)$ by

$$g'(y) = \langle g(a) \, : \, a \in \mathfrak{k}(Y) \, a \leq y \rangle,$$

the ideal generated in F by the images under g of all compact elements below y. In the reverse direction, assign $h : \mathrm{Idl}(F) \longrightarrow Y$, as

$$h(J) = \bigvee \{ c \in \mathfrak{k}(L) \, : \, g(c) \in J \} \, .$$

One easily demonstrates that h and g' are frame homomorphisms, and that $\epsilon \cdot g' = g$, while $g \cdot h = \epsilon$. Note also that g and ϵ are monic, which implies that g' and h are mutual inverses of one another. □

We therefore have a poset of equivalence classes of covers. Now, here is a formal account of what has been accomplished. This summary is entirely in $\mathfrak{J}$.

Theorem 3.4.5 *Suppose that L is an $\mathfrak{J}$-object. There is a poset of equivalence classes of Yosida covers. Moreover,*

1. *each is of the form $Y = \mathrm{Idl}(F)$.*
2. *For bases F and G, if G is a sublattice of F, then there is a dense and codense frame map $h : \mathrm{Idl}(F) \longrightarrow \mathrm{Idl}(G)$ as follows:*

$$h(J) = J \cap G, \; J \in \mathrm{Idl}(F).$$

3. *The largest Yosida cover is (Y, e), where $Y = \mathrm{Idl}(SL)$ and $e(J) = \bigvee J$, for each ideal J of SL.*
4. *There is a least Yosida cover precisely when L is already Yosida.*

Proof Remark 3.4.1 shows why this collection of equivalences is a set; it is clearly a poset. The preceding proposition takes care of the first claim.

Regarding the map h, if J is a nonzero ideal of F, then as F is a base, any nonzero element of J exceeds a nonzero member of G, which is necessarily in J. This shows that h is dense. The codensity is obvious.

The rest is left for the reader. □

The next section of the paper is about the maximum member of the poset of equivalence classes of Yosida covers.

4 Perfect Saturation

The maximum member among the Yosida covers of a $\mathcal{J}$-frame L can be characterized in several ways. Recall that each Yosida cover is $\mathrm{Idl}(F)$ for a suitable base of saturated elements F. For the largest one, let $F = SL$.

4.1 Yosida Frames with a Unique Base

We begin with a proposition which gives a list of equivalent ways to define the notion of a Yosida frame with a unique base, such a frame is called a *perfectly saturated* frame. The proposition is quite straightforward, given the information in Proposition 3.2.2. For each $\mathfrak{J}$-object L one has the perfectly saturated Yosida cover, $\mathrm{Idl}(SL)$, which is abbreviated $\mathcal{Y}L$.

Proposition 4.1.1 *Suppose that L is a $\mathcal{J}$-frame. Then the following are equivalent:*

1. *L is Yosida and has a unique base.*
2. $SL = \mathfrak{k}(L)$.
3. *L is a coherent frame in which the subset of compact elements is a compact regular frame and, indeed, a complete meet subsemilattice of L.*

Proof By Lemma 3.1.3, (1) implies (2), and conversely. It is obvious that (2) implies (3), and from (3) and the fact that every coherent frame is isomorphic to the ideal frame of its sublattice of compact elements, (3) follows. □

A little further reflection upon the meaning of Proposition 4.1.1 will reveal the following. The reader should reflect as well that the map $x \mapsto (\downarrow x)$ that realizes the equivalence in one of the directions preserves the *saturated suprema*.

Proposition 4.1.2 *The category $\mathfrak{KReg}$ of compact regular frames and all frame homomorphisms is equivalent to the category $\mathfrak{PS}$ of all perfectly saturated frames and all the coherent frame maps. More precisely, the functors* Idl *and S carry out this equivalence as the maps*

$$\mathrm{Idl}(SL) \longrightarrow L \ \ by \ \ J \mapsto \bigvee J, \ \ J \in \mathrm{Idl}(SL)$$

and

$$E \longrightarrow S(\mathrm{Idl}(E)) \ \ by \ \ x \mapsto (\downarrow x), \ \ x \in E$$

witness this equivalence.

Proof It is well known that, for each bounded distributive lattice F, $\mathrm{Idl}(F)$ is coherent frame. In fact, the passage $F \longrightarrow \mathrm{Idl}(F)$ witnesses the equivalence of the category of all bounded distributive lattices (and all lattice homomorphisms) and the category of all coherent frames (and all coherent frame maps). The inverse equivalence is realized by the assignment $L \mapsto \mathfrak{k}(L)$.

Now according to Proposition 4.1.1, if L is perfectly saturated, then $\mathfrak{k}(L) = SL$, and so

$$\mathrm{Idl}(SL) = \mathrm{Idl}(\mathfrak{k}(L)) \cong L,$$

the isomorphism being precisely the one given in the statement above. Conversely, if E is a compact regular frame, then $\mathfrak{k}(\mathrm{Idl}(E))$ is the lattice of principal ideals, which (arguing as in the proof of Proposition 3.2.2(4),) is, in fact, $S(\mathrm{Idl}(E))$. Finally, we conclude that $E \cong S(\mathrm{Idl}(E))$ (as bounded distributive lattices, initially) using the map we have noted in the proposition. However, this isomorphism is also one of complete meet semilattices, and therefore $x \mapsto (\downarrow x)$ (with $x \in E$) also carries the given supremum of E to that of the frame of principal ideals.

Thus it is seen that the general equivalence between bounded distributive lattices and coherent frames, restricts to the equivalence of categories as specified in the proposition. □

4.2 Projectivity

The goal in this section is to characterize the projective frames of $\mathfrak{J}$. Theorem 4.2.1 simplifies the arguments leading up to that characterization, although in the end the final result—Proposition 4.2.4—is not really satisfying.

Let us first highlight the coreflective feature of the perfectly saturated frames.

Theorem 4.2.1 *$\mathcal{Y}$ defines a monocoreflection of $\mathfrak{J}$ in the full subcategory $\mathfrak{PS}$ of all perfectly saturated frames.*

Proof For each weakly closed frame homomorphism $h : L \longrightarrow M$, consider the map $\mathcal{Y}(h) : \mathcal{Y}L \longrightarrow \mathcal{Y}M$ defined by

$$\mathcal{Y}(h)(J) = \{\, y \in M \,:\, y \leq h(x), \text{for some } x \in J \,\}.$$

The reader will quickly realize that $\mathcal{Y}(h)(J)$ is none but the ideal of SM generated by the image $h(J)$. It is actually well known that $\mathcal{Y}(h)$ is a coherent frame map between Yosida frames, and is therefore a member of $\mathfrak{J}$.

Next, the diagram below commutes:

$$\begin{array}{ccc} \mathcal{Y}L & \xrightarrow{\mathcal{Y}(h)} & \mathcal{Y}M \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ L & \xrightarrow[h]{} & M \end{array} \tag{4.2.1.1}$$

To see this calculate for each ideal J of SL:

$$e(\mathcal{Y}(h)(J)) = \bigvee h(J) = h\Big(\bigvee J\Big) = h(e(J)).$$

Since the vertical maps are monic, it is clear that $\mathcal{Y}$ is a covariant functor. Moreover, for the same reason, $\mathcal{Y}(h)$ is the only map across the top of the diagram that makes it commute. Then take L to be perfectly saturated, so that $\mathcal{Y}$ is easily seen to be a coreflection. □

The term "projective" here is the usual one: a frame P which has the property that for any surjective frame map $g : M \longrightarrow N$ and each $f : P \longrightarrow N$ (both maps in the category $\mathfrak{J}$) there is an $\mathfrak{J}$-morphism $f' : P \longrightarrow M$ such that $g \cdot f' = f$. By elementary principles, it follows that for any projective frame P in $\mathfrak{J}$ and any map $e : L \longrightarrow P$ of the category onto P, e is a retraction; that is, there is an $\mathfrak{J}$-morphism $u : P \longrightarrow L$ such that $e \cdot u = 1$.

It is easy to give a tidy necessary condition for projectivity in $\mathfrak{J}$.

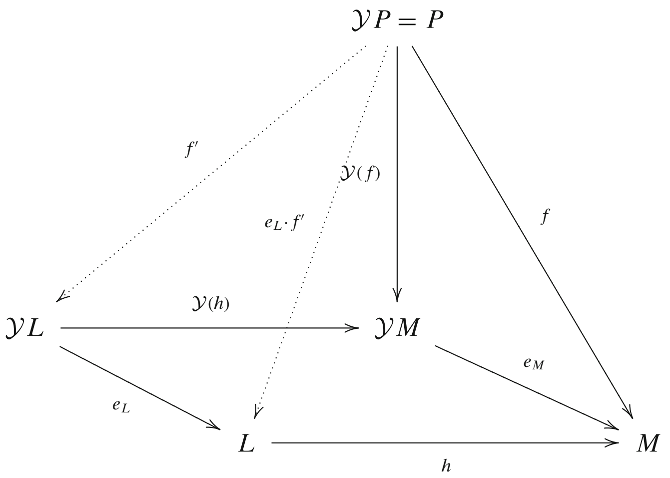
Lemma 4.2.2 *If P is projective then* $P = \mathrm{Idl}(SP)$.

Proof As has been shown, P is a dense codense quotient of $\mathrm{Idl}(SP)$, and the witnessing map q is monic. Since P is assumed to be projective, q is a retraction. Every monic retraction is an isomorphism. □

Next, we reduce the problem of identifying the projectives of $\mathfrak{J}$ entirely to the subcategory of perfectly saturated frames.

Lemma 4.2.3 *If P is projective in the subcategory $\mathfrak{PS}$, then it is projective in $\mathfrak{J}$.*

Proof Suppose that $h : L \longrightarrow M$ is a codense surjection, mapping saturated elements to saturated elements, and $f : P \longrightarrow M$ be a $\mathfrak{J}$-morphism. Lift these maps by applying $\mathcal{Y}$. In the following commutative diagram we display the situation.



(4.2.3.1)

As P is projective among perfectly saturated frames, there is a coherent frame map $f' : P \longrightarrow \mathcal{Y}L$ such that $\mathcal{Y}(h) \cdot f' = \mathcal{Y}(f)$. Then it is easy to see that $e_L \cdot f'$ is the map we need. □

And now, applying Proposition 4.1.2, here is a characterization of the projectives in $\mathfrak{J}$. The interested reader should look, for example, at [4] where the frames are not necessarily compact, to see that the results are different.

Proposition 4.2.4 *A $\mathfrak{J}$-object P is projective if and only if P is perfectly saturated and its saturation SP is projective in $\mathfrak{KReg}$.*

Remark 4.2.5 Proposition 4.2.4 has a tidy feel to it. However, the knowledgeable reader may perhaps already be aware that the story of projective compact regular frames—which in the presence of Zorn's Lemma *is* the dual of the category of compact Hausdorff spaces—is a complicated one, because the corresponding account of injective spaces is complicated. Already for zero-dimensional compact spaces and injectives, vs. projective boolean algebras, the matter is not easily summarized. The reader who is interested in pursuing this matter further is referred to S. Koppelberg's chapter in the *Handbook of Boolean Algebra, Volume 3* [13].

On the other hand, it is not hard to argue that in the category of compact Hausdorff spaces, the injectives are described as the *coretractions* of products of copies of the closed unit interval: that is to say, *the space X is injective $\Longleftrightarrow$ there is an embedding m of X into $J = [0, 1]^\kappa$, for suitable κ, such that for some continuous surjection $r : J \longrightarrow X$, $r \cdot m = 1$.*

Does this mean that the projective compact regular frames are retractions of coproducts of copies of the localic interval?

4.3 Forth and Back

The tight interplay between a $\mathcal{J}$-frame and its saturation is surely evident to the reader. We now briefly consider this interplay for its own sake. Here is a preface, recalling some facts from [18] about properties that are closely connected to regularity. From Proposition 3.2.2 we recall that $\mathrm{Idl}(F)$ always is normal.

Remark 4.3.1 We assume that L is an algebraic frame. Theorem 2.4 of [18] compares

Reg(1) L is regular.
Reg(2) Each d-element—meaning, any updirected join of polars—is regular.
Reg(3) $a^{\perp\perp}$ is regular, for each compact a.
Reg(4) $a^{\perp}$ is regular, for each compact a.

Each of the Reg(i) listed (with $i = 2, 3, 4$) is implied by its predecessor.

In the aforementioned theorem it is shown that Reg(2) and Reg(3) are equivalent, and in turn equivalent to L being projectable; that is, $a^{\perp\perp}$ is complemented, for each compact a. As to Reg(4), it is shown to be equivalent to: for each pair a and b of disjoint compact elements of L, $a^{\perp} \vee b^{\perp} = 1$.

A simple yet important observation is put into a lemma, which is proved easily, and we do leave it for the reader.

Lemma 4.3.2 *For each $Y \in \mathrm{Idl}(F)$, one has*

$$Y^{\perp} = \left(\downarrow \bigvee Y \right)^{\perp} = \downarrow \left(\bigvee Y \right)^{\perp}.$$

In particular, every polar of $\mathrm{Idl}(F)$ *is principal, and, in fact, generated by an element of the saturation.*

For the ideal frames under consideration here we have the following.

Proposition 4.3.3 *Any perfectly saturated regular frame is finite.*

Proof Since in a regular frame every element is saturated, it is clear that in L every element is compact. On the other hand, L is an algebraic regular frame, and every compact element in a regular frame is complemented. We conclude that every element is complemented, that is to say that L is a boolean frame. Finally, L must be finite, else one may also (inductively) construct an infinite disjoint set, the supremum of which cannot be compact. □

The next theorem seems to say that the connection between a compact regular frame and its frame of ideals is quite strong.

Theorem 4.3.4 *Suppose that F is a compact regular frame. Then the following are equivalent.*

1. $\mathrm{Idl}(F)$ *has* Reg(4).
2. *F is strongly projectable.*

3. Idl(F) *is strongly projectable.*
4. Idl(F) *has* Reg(2).

Proof Assume Idl(F) has Reg(4). By our earlier observations we have that for saturated ideals I and J, with $I \wedge J = 0$, $I^\perp \vee J^\perp = F$. Applying Lemma 4.3.2, one concludes that if $x \wedge y = 0$ in F, then $x^\perp \vee y^\perp = 1$. It is well known that this property guarantees that F is strongly projectable.

With F strongly projectable and using Lemma 4.3.2 once more—or else Proposition 3.2.4, it follows that Idl(F) is strongly projectable, and so (2) implies (3). Since (3) $\implies$ (4) $\implies$ (1), are obvious, we are done. □

5 Epicompleteness with Skeletal Frames

After having spent some energy dealing with monomorphisms are not one-to-one, we are about to switch to a category by restricting the morphisms. In any discussion involving epicompletions and epicomplete objects, the epicompleteness is the usual categorical notion which stipulates that there are no monic epimorphisms coming out of the object in question. As in the category $\mathfrak{J}$, the maps out of strongly projectable objects that are monic and epic are isomorphisms: compare Proposition 3.3.6 with Theorem 5.3.6. We now proceed to capitalize on [15, 23] and [16], with Proposition 3.3.6, to arrive at the conclusion that demanding the morphisms to now also be skeletal, makes no difference. And so, unexpectedly in the end, it emerges that the intuition which early on had predicted that a monoreflection (such as ψ), in strongly projectable frames was expected to be the maximum, was correct anyway. It has remained somewhat puzzling that the proof of the epicompleteness of this class had eluded us, until now.

Throughout this section, the ambient category of $\mathcal{J}$-frames with all the morphisms of $\mathfrak{J}$ which are also skeletal. This is the category $\mathfrak{Jsk}$. Next, we briefly review the basic information on skeletal frame maps, needed in the development of this section of the paper.

5.1 Skeletal Maps

The reader will find this material introduced in [5].

Remark 5.1.1 The frame homomorphism $h : L \longrightarrow M$ is *skeletal* if $x^{\perp\perp} = 1$ implies that $h(x)^{\perp\perp} = 1$. It is easy to verify that h is skeletal if and only if

$$x_1^\perp = x_2^\perp \implies h(x_1)^\perp = h(x_2)^\perp.$$

For convenience we shall say that x is *dense* in a frame, if $x^{\perp\perp} = 1$.

It is also easy to see that h is skeletal precisely when there is a (unique) frame homomorphism $\mathcal{P}(h) : \mathcal{P}A \longrightarrow \mathcal{P}B$ making the diagram below commute:

$$\begin{CD} A @>{h}>> B \\ @V{p_A}VV @VV{p_B}V \\ \mathcal{P}A @>{\mathcal{P}(h)}>> \mathcal{P}B \end{CD} \tag{5.1.1.1}$$

In figure (5.1.1.1), p_A denotes the nucleus defined by $p_A(x) = x^{\perp\perp}$. (We do not decorate the $\perp$s to indicate which frame the complements are taken in.)

On the category $\mathfrak{Frm}_{sk}$ of frames with skeletal maps $\mathcal{P}$ is a reflection in the subcategory of boolean frames.

5.2 ψ and the Trouble it's Caused

Definition & Remarks 5.2.1 For any compact regular frame F, there is the *absolute* εF, which is a compact regular strongly projectable frame, and a skeletal embedding $\varepsilon_F : F \longrightarrow \varepsilon F$ which is the reflection map of a monoreflection ε. It is shown in [22] that this is the functorial epicompletion in the category of compact regular frames with skeletal frame maps.

Suppose that L is a compact normal joinfit frame. To define ψ, form the pushout in $\mathfrak{Frm}_{sk}$, of the maps $\varrho : \varrho L \longrightarrow L$ and $\varepsilon : \varrho L \longrightarrow \varepsilon(\varrho L)$. In the following theorem we collect the various pieces of information about ψ that are scattered over [15, 23] and [16].

Theorem 5.2.2 *Suppose that L is a compact normal joinfit frame. Consider the diagram*

$$\begin{CD} \varrho L @>{\varepsilon_{\varrho L}}>> \varepsilon(\varrho L) \\ @V{\varrho_L}VV @VV{g_L}V \\ L @>{\psi_L}>> \psi L \end{CD} \tag{5.2.2.1}$$

and assume that it forms a pushout in $\mathfrak{Frm}_{sk}$. Then

1. *Martínez and Zenk* [23] *ψ defines an epireflection of the category of compact normal joinfit frames with skeletal maps, in the full subcategory of its strongly projectable frames.*
2. *Martínez* [15] *Restricting to closed skeletal frame maps, then ψ is a monoreflection, and, in fact, each ψ_L is a closed embedding.*

By way of support of the above theorem, the reader might find the following useful. Much of this—and, in particular, items 1 and 2 below—may be retrieved from [16, Lemma 3.2.7].

Remark 5.2.3 Unless otherwise noted, all frames in this remark are compact, normal and joinfit. Suppose that $h : L \longrightarrow M$ is a frame homomorphism. Then,

1. If h is skeletal then so is $S(h)$.
2. If h is weakly closed then so is $\psi(h)$.

On the category $\mathfrak{KNJ}_{sk}^{wc}$ of compact normal joinfit frames with weakly closed, skeletal maps, the reflections s and ψ commute. The composite $\psi \cdot s$ epireflects—but does not monoreflect—in the full subcategory of compact regular strongly projectable frames.

Next, in order to study the effect of ψ on saturated elements, as the reader perhaps already expects, one should be clear about the interaction of saturation with the coproduct. In the section that follows, we develop those elements of the discussion. For background regarding the coproduct of frames the reader is referred to [24, Chapter 2], for the construction of the coproduct; to [11] and [2], for arguments leading up to the Tychonoff Theorem for frames, and to [23, Section 4], where properties of the coproduct which are specific to the current context are more amply discussed.

5.3 Coproduct Delights

In the discussion ahead, some properties of coproducts, indeed, of a particular coproduct, will play a role below. It is the coproduct $A \coprod E$, in which A is a $\mathcal{J}$-frame and E is compact regular and strongly projectable. The next two lemmas are about such coproducts.

Throughout these preliminaries, the reader would do well to remember that every $\mathcal{J}$-frame is joinfit. Useful as well is the following basic lemma about weakly closed maps.

Lemma 5.3.1 *Suppose that $h : L \longrightarrow M$ is a weakly closed frame map. Then its right adjoint h_* maps saturated elements to saturated ones.*

Proof Suppose that $y \in M$ is saturated. Let $a \in L$, with $h_*(y) < c$. Then $h(c) \not\leq y$, and there must be a $z \in M$, $y \leq z < 1$, such that $h(c) \vee z = 1$. Thus, as h is weakly closed, $c \vee h_*(z) = 1$, which is what was claimed. □

Lemma 5.3.2 *An $a \in A$ is saturated if and only if $a \otimes 1$ is saturated. In particular, the coprojection $a \mapsto a \otimes 1$ preserves saturated elements.*

Proof In $A \coprod E$, we may suppose that the elements are written as $\vee_i x_i \otimes y_i$, the y_i being complemented elements. This is a consequence of the fact that E is compact and zero-dimensional, and so each member is a join of complemented elements.

Now suppose that $a \otimes 1 < \vee_i x_i \otimes y_i$. Then for some index j, $x_j \otimes y_j \not\leq a \otimes 1$, which in turn implies that $x_j \not\leq a$. If a is saturated, there is a $z \in A$ such that $z \vee x_j = 1$, while $z \vee a < 1$. To continue, put $s = (z \otimes y_j) \vee (1 \otimes y_j^{\perp})$. Thus, in $A \coprod E$ we have,

$$
\begin{aligned}
(a \otimes 1) \vee s &= (a \otimes 1) \vee \left(z \otimes y_j\right) \vee \left(1 \otimes y_j^{\perp}\right) \\
&\leq ((a \vee x) \otimes 1) \vee \left(1 \otimes y_j^{\perp}\right) < 1.
\end{aligned}
$$

By contrast,

$$
\begin{aligned}
(x_j \otimes y_j) \vee s &= \left(x_j \otimes y_j\right) \vee \left(z \otimes y_j\right) \vee \left(1 \otimes y_j^{\perp}\right) \\
&= \left(1 \otimes y_j\right) \vee \left(1 \otimes y_j^{\perp}\right) \ = \ 1.
\end{aligned}
$$

Conversely, suppose that $a \otimes 1$ is saturated. By [16, Lemma 4.3], the coprojection $u : A \longrightarrow A \coprod E$ is closed, and so u_* maps saturated elements to saturated elements (Lemma 5.3.1). To conclude, notice that $u_*(a \otimes 1) = a$. □

What is really needed, in preparation for the main theorem, is a way to decide when a member of the coproduct is saturated. The following lemma is a big step forward.

If (for $i = 1, 2$) $h_i : A_i \longrightarrow B_i$ are frame homomorphisms, then $h_1 \coprod h_2$ will stand for the unique homomorphism on $A_1 \coprod A_2$ for which

$$
\left(h_1 \coprod h_2\right)(a_1 \otimes a_2) = h_1(a_1) \otimes h_2(a_2).
$$

Lemma 5.3.3 *Suppose that A and B are compact normal and joinfit frames. Then:*

1. *The map $s_A \coprod s_B : A \coprod B \longrightarrow SA \coprod SB$ is a weakly closed frame map onto a compact regular frame.*
2. *If $h : A \coprod B \longrightarrow C$ is any weakly closed frame map into a compact regular frame C, then there is a unique weakly closed frame homomorphism $h^o : SA \coprod SB \longrightarrow C$ such that $h^o \cdot (s_A \coprod s_B) = h$.*

It follows that $S(A \coprod B) \cong SA \coprod SB$, and that $s_A \coprod s_B$ is equivalent to the saturation quotient map. Finally, $x \in A \coprod B$ is saturated if and only if it can be expressed as $x = \vee a_i \otimes b_i$, with the a_i saturated in A and the b_i saturated in B.

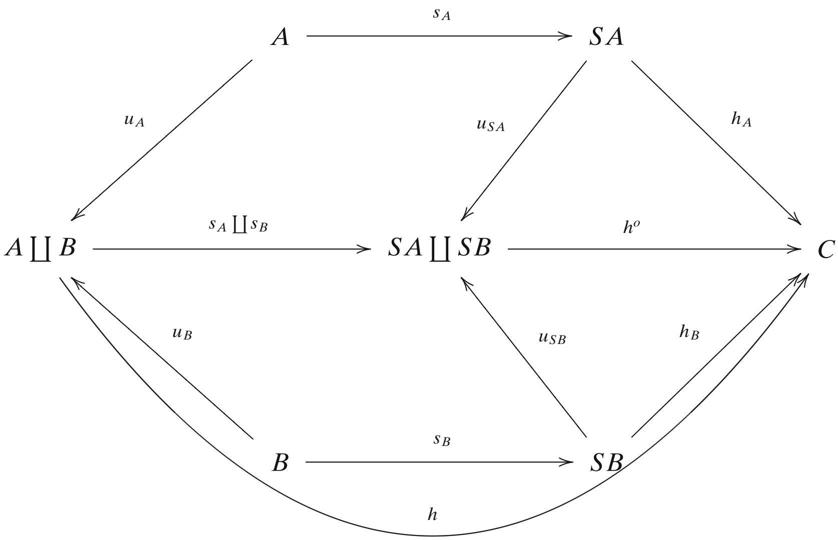
Proof With (1) and (2) in hand the rest follows easily.

It is well known that, because SA and SB are compact regular, $SA \coprod SB$ is compact regular. Likewise, by [23, Proposition 4.4], $A \coprod B$ is compact normal and joinfit because the factors are assumed to have those properties. Note that the map $s_A \coprod s_B$ is induced by coproduct properties applied to the pair $w_A : A \longrightarrow SA \coprod SB$ and $w_B : B \longrightarrow SA \coprod SB$, which in turn are

$$w_A = u_{SA} \cdot s_A, \quad \text{where } A \xrightarrow{s_A} SA \xrightarrow{u_{SA}} SA \coprod SB,$$

and similarly for B, and one should observe that u_{SA} denotes a coprojection. The point is that w_A and w_B are compositions of a closed map following a weakly closed map, which guarantees that they are weakly closed, and also $s_A \coprod s_B$.

Now we turn to diagram chasing! The maps u decorated with a subscript are coprojections (out of the subscripted object.) The horizontal maps that are edges of the two parallelograms are the two saturations and their coproduct. Our objective is to show that the long arched map h ultimately gives us the horizontal h^o such that the middle composite agrees with h.



Observe the following: first, the two parallelograms commute, by definition of the maps. Next, h_A and h_B exist using the reflective properties of the two saturations, so as to give $h \cdot u_A = h_A \cdot s_A$ (resp. $h \cdot u_B = h_B \cdot s_B$). The map h^o is the one induced by h_A and h_B on the coproduct $SA \coprod SB$. It is then a straightforward calculation to show that

$$h \cdot u_A = h^o \cdot \left(s_A \coprod s_B\right) \cdot u_A,$$

as well as the identity that replaces u_A above with u_B. We may now conclude that $h = h^o \cdot (s_A \coprod s_B)$.

The reader ought to reflect—and perhaps check in [16]—to be certain at every stage of the development in the preceding paragraph that the maps are all weakly closed. The uniqueness is, likewise, left for the reader to verify. □

Corollary 5.3.4 *Suppose that $h : A \longrightarrow B$ is a weakly closed frame map which carries saturated elements to saturated ones. Then for any compact normal joinfit frame E, the map $h \coprod 1_E$ maps saturated elements to saturated elements.*

Remark 5.3.5 In [23] ψA is defined as the pushout in $\mathfrak{Frm}_{sk}$ of the maps $\varrho_A : \varrho A \longrightarrow A$ and $\varepsilon_{\varrho A} : \varrho A \longrightarrow \varepsilon(\varrho A)$, the first being the coreflection map of the regular coreflection, and the second the reflection map of ϱA in its absolute $\varepsilon(\varrho A)$. This effectively makes ψA a closed skeletal quotient of $\varrho A \coprod \varepsilon(\varrho A)$. We denote here, following the usage in [23], the image of a generator $a \otimes b$ under said quotient, by $a \bigcirc\!\!\!\!\wedge\, b$. Some observations are probably in order:

(a) The reflection map of A may be described as $\psi(a) = a \bigcirc\!\!\!\!\wedge\, 1$, for each $a \in A$.
(b) In ψA every element, may be expressed in the form

$$\bigvee_i a_i \bigcirc\!\!\!\!\wedge\, e_i, \quad a_i \in A, e_i \in \varepsilon(\varrho A)),$$

and since $\varepsilon(\varrho A)$ is compact regular and strongly projectable, one may assume each e_i in the above expression to be complemented.

At last, here is the kind of epicompleteness result we had been aiming for throughout the progression of articles from [22] through [16]. It is a result in $\mathfrak{Jsk}$, the category of $\mathcal{J}$-frames with skeletal weakly closed frame homomorphisms that map saturated elements to saturated elements.

Theorem 5.3.6 *The reflection ψ is a monoreflection of $\mathfrak{Jsk}$ in the subcategory $\mathfrak{SP}$ whose objects are the strongly projectable $\mathcal{J}$-frames. Furthermore, each strongly projectable $\mathfrak{Jsk}$-frame is epicomplete.*

Proof Based on what we already know about ψ, from [22] through [16], the following three conditions spell out important features of ψ.

(a) In the presence of compactness and normality, prove that if L is join-generated by its saturated elements, then ψL is too. This shows that ψ is well-defined on objects.
(b) Prove that the reflection map ψ_L preserves saturated elements (assuming that the objects are compact and normal.) This condition insures that the reflection map is a $\mathfrak{Jsk}$-morphism.
(c) If $h : L \longrightarrow M$ preserves saturated elements, then so does $\psi(h)$. And so, ψ is well-defined on the morphisms of $\mathfrak{Jsk}$.

Once these conditions have been checked, we will have that ψ is indeed a monoreflection of $\mathfrak{Jsk}$ in its full subcategory of strongly projectable frames.

(a) As we have already pointed out, ψ is the composite of the coprojection $a \mapsto a \otimes 1$, followed by a coequalizer map which is a closed frame homomorphism; Lemmas 5.3.2 and 5.3.4 then establish (a).

(b) Note that

$$(a \owedge 1) \wedge (1 \owedge e) = a \owedge e,$$

and if e is complemented in $\varepsilon(\varrho A)$, then $1 \owedge e$ is also complemented in ψA. Thus, since ψA is joinfit, each such $1 \owedge e$ is saturated. Since the saturation of a compact frame is closed under infima, it follows from (a), that if A is join-generated by its saturated elements, each element of ψA is a join of elements of the form $a \owedge e$, and saturated. This shows that (a) implies (b).

(c) Suppose that $h : L \longrightarrow M$, between $\mathcal{J}$-frames, preserves saturated elements. First, let us note that the extension $\varepsilon(\varrho(h))$ of its restriction to the regular coreflection automatically preserves saturated elements. Using Lemma 5.3.3, it is easily seen that $h \coprod \varepsilon(\varrho(h))$ preserves saturated elements. To reach the desired conclusion for $\psi(h)$, consider the commutative square below, which essentially manifests the definition of $\psi(h)$:

$$\begin{CD} L \coprod \varepsilon(\varrho L) @>{h \coprod \varepsilon(\varrho(h))}>> M \coprod \varepsilon(\varrho M) \\ @V{q}VV @VV{q}V \\ \psi L @>{\psi(h)}>> \psi M \end{CD} \tag{5.3.6.1}$$

We point out that the undecorated vertical maps are the coequalizer maps that defined ψ in the first place. The composite $q \cdot (h \coprod \varepsilon(\varrho(h)) \cdot q_*$ *is* $\psi(h)$, because diagram (5.3.6.1) commutes and q is an epimorphism. Finally, Lemma 5.3.1 tells us that the q_* in this composite takes saturated elements to saturated elements, and the other two maps also preserve saturated elements. This suffices.

What is left is merely to adjust the proof of Proposition 3.3.6 to account for the maps now being skeletal. As in that proof, suppose that the $\mathcal{J}$-frame L is strongly projectable and $m : L \longrightarrow M$ is monic and epic, but this time in $\mathfrak{Jsk}$. The reader should now verify that the reduct $S(m)$ is also skeletal.

Proposition 3.3.4 and Lemma 3.3.1.1 here imply that $S(m)$ is an epic embedding in $\mathfrak{KReg}_{sk}$, the category of compact regular frames with skeletal maps. Then [22, Lemma 5.4] itself implies that $S(m)$ is an isomorphism, whence m is an isomorphism. □

6 Applications

There are several cottage industries that one might develop over the perfectly saturated frames. In this paper we have, for the most section, kept to the frames *per sé*. On the other hand, our motivation came from algebraic settings, such as that of lattice-ordered groups viewed through the frame of convex ℓ-subgroups, or of commutative rings with identity, by way of the frame of its radical ideals.

It seems reasonable to give the curious and faithful reader a look at these motivating connections, without developing said cottage industries at the close of this narrative.

6.1 Lattice-Ordered Groups

We begin with lattice-ordered groups—as usual abbreviating them: "ℓ-groups". The understanding with the reader should be transparent. These applications are intended for the reader who is interested in digging further into these questions, which also assumes that such a reader is willing to go to references such as [6] and [7] for the necessary background material.

To be precise, we consider what the very recent literature calls *unital* ℓ-groups; that is to say ℓ-groups with a designated strong order unit. The maps between the ℓ-groups will preserve the designated unit. *All ℓ-groups in this article are abelian.*

Let us begin by interpreting the information we have about perfectly saturated frames, under the hypothesis that the frames in question are spatial, and in terms of the hull-kernel topology on the space of maximal elements. For these purposes, let L be a coherent normal frame. As is well known, $\mathrm{Max}(L)$ has the topology given by the open sets

$$u(x) = \{m \in \mathrm{Max}(L) : x \not\leq m\}, \quad x \in L.$$

The complementary closed set is denoted $v(x)$.

The following basic characterization of perfect saturation turns out to be useful. It assumes Choice, which is certainly enough to make the frame spatial.

Proposition 6.1.1 *The coherent normal frame L is perfectly saturated if and only if every closed subset of* $\mathrm{Max}(L)$ *is of the form $v(c)$, with $c \in \mathfrak{k}(L)$, and the function $c \mapsto v(c)$ is a one-to-one correspondence between compact elements and closed sets.*

Proof (Necessity) Since L is Yosida, it follows that $\bigwedge v(c) = c$, whenever c is compact, and that, clearly, implies that $c \mapsto v(c)$ is one-to-one on compact elements. On the other hand, if K is a closed set of $\mathrm{Max}(L)$, then $\bigwedge K = a$, and using the assumption that saturated elements must be compact, we have that a is compact. Thus, the map $c \mapsto v(c)$ is surjective.

(Sufficiency) Assume the map $c \mapsto v(c)$ is a one-to-one correspondence, as stated in the proposition. If $x \in L$ is saturated, then $x = \bigwedge v(x)$. Since $v(x)$ is closed in $\mathrm{Max}(L)$, we have that $v(x) = v(a)$, for some $a \in \mathfrak{k}(L)$, and it is also clear that $x \geq a$. By way of contradiction, suppose $x > a$; then there is a compact element $b \leq x$, with $b \not\leq a$. But this means that, for some maximal $q \geq a$ and hence $q \geq x$, while q does not exceed b. This is absurd, and so we have that $x = a$, proving that each saturated element is compact.

The reverse is easy, and checking that is left to the reader. □

Now let us apply the aforementioned proposition to (unital) ℓ-groups. In [21], Proposition 5.2 gives necessary and sufficient conditions for the frame $\mathcal{C}(G)$ of all (order) convex ℓ-subgroups of the ℓ-group G to be Yosida. It is well known that $\mathcal{C}(G)$ is a coherent normal frame (with even stronger properties.)

We wish to characterize the unital ℓ-groups G for which $\mathcal{C}(G)$ is perfectly saturated. To that end, we first address the matter in an archimedean ℓ-group G with strong unit. Here we use the *Yosida Representation Theorem* for archimedean unital ℓ-groups [6, Corollary 13.2.6]: this result embeds G as a unital ℓ-subgroup of $C(\mathrm{Max}(\mathcal{C}(G)))$ which separates the points of the compact Hausdorff space. (The reader is also urged to look at [9], for the most general account of this representation.)

Now we have the following corollaries of Proposition 6.1.1. The converse of the first corollary is false, as the reader will have no difficulty checking.

Corollary 6.1.2 *For any archimedean unital ℓ-group G, if $\mathcal{C}(G)$ is perfectly saturated, then* $\mathrm{Max}(\mathcal{C}(G))$ *is perfectly normal.*

Proof We use the Yosida Representation Theorem to realize G as a unital ℓ-subgroup of $C(X)$, where $X = \mathrm{Max}(\mathcal{C}(G))$. Proposition 6.1.1 then tells us that every open set of X is of the form

$$\mathrm{coz}(g) = \{ x \in X : g(x) \neq 0 \},$$

which, according to [1, Lemma 2.2] is a cozeroset. □

Corollary 6.1.3 *If G is a unital ℓ-group, and $\mathcal{C}(G)$ is perfectly saturated, then* $\mathrm{Max}(\mathcal{C}(G))$ *is perfectly normal.*

Proof The crucial observation is that if J stands for the convex ℓ-subgroup of all the infinitesimal elements of G, then G/J is archimedean with strong unit. Moreover, it is easy to see that $\mathcal{C}(G/J) \cong \uparrow J$, and also that their spaces of maximal convex ℓ-subgroups are homeomorphic.

A few small issues remain to be settled. We leave those to to the reader. □

6.2 Rings of Continuous Functions

We look at some frames associated with a ring of continuous functions. We assume that X is a compact Hausdorff space beginning with the frame $\mathcal{C}_z(X)$ of z-deals of $C(X)$.

Example 6.2.1 $\mathcal{C}_z(X)$is a Yosida frame. Now, $\mathcal{C}_z(X)$ is also isomorphic to the frame of ideals of the bounded distributive lattice $\mathrm{Coz}(X)$ of all cozerosets of X [19, Lemma 4.2].

Corollary 6.1.2 guarantees that if $\mathcal{C}_z(X)$ is perfectly saturated, then X is perfectly normal. Conversely, if X is perfectly normal then it is easy to see that the lattice of cozerosets is complete, and hence a frame, an, indeed, a compact regular frame. Since $L = \mathcal{C}_z(X)$ is isomorphic to the frame of ideals of $\mathrm{Coz}(X)$, we have that $L = \mathrm{Idl}(SL)$. (This example is what prompted us to call the frames for which $L = \mathrm{Idl}(SL)$ perfectly saturated.)

A brief comment concerning epicomplete objects in $\mathfrak{J}$ is in order. In view of all that has gone before, it would be nice to know when $\mathcal{C}_z(X)$ is strongly projectable. Since

$\mathcal{C}_z(X) \cong \mathrm{Idl}(Coz(X))$ (Theorem 4.3.4), that happens precisely when X is extremally disconnected and perfectly normal.

A little more work and one has the following. The authors thank Tony Hager, for tidying up the proof.

Proposition 6.2.2 *Suppose X is a compact Hausdorff space. If $\mathcal{C}_z(X)$ is strongly projectable and perfectly saturated, then X is finite.*

Proof By way of contradiction, suppose that X is infinite. Since X must be extremally disconnected and perfectly normal, on the one hand, there is a C^*-embedded copy of the discrete natural numbers, and, hence, a copy of $\beta\mathbb{N}$, the Stone-Čech compactification. On the other hand, that copy of $\beta\mathbb{N}$ must also be perfectly normal, a contradiction, since no $p \in \beta\mathbb{N} \setminus \mathbb{N}$ is a G_δ-point ([8, Corollary 9.6]). □

Here is an interesting example, especially in the details. It is to be noted that the outcome is the same as in the preceding result.

Example 6.2.3 $\mathrm{Rad}(C(X))$, with X compact Hausdorff, is a Yosida frame precisely when X is finite.

The important things to note are these:

- To be Yosida, any two functions which have the same zerosets generate the same radical ideals, and so some power of each one is a multiple of the other.
- If X is infinite, there is a continuous function f on X with values in the extended reals $\mathbb{R} \cup \{\infty\}$, the image of which contains a strictly increasing, unbounded sequence $p_1, p_2, \ldots$. Now form $g = \exp(-f)$, and let $h = \frac{1}{f^2+1}$. Observe that $g, h \in C(X)$ and that they have the same cozeroset, which means that, there is a positive integer k such that for some $u \in C(X)$, $h^k = ug$.
  However, if $x_n \in X$ such that $f(x_n) = p_n$, then we have
  $$u(x_n) = \frac{\exp(p_n)}{(p_n^2+1)^k},$$
  which is unbounded, and we have a contradiction.

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