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Semi-clean group rings

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ABSTRACT

Recall that a ring is said to be a *clean ring* if every element can be expressed as the sum of a unit and an idempotent. In one variant of this definition, a ring is said to be a *semi-clean ring* if every element can be expressed as the sum of a unit and a periodic element. Ye's Theorem [12] states that the group ring $\mathbb{Z}_{(p)}[C_3]$ is semi-clean, where p is a prime integer and C_3 is a cyclic group of order 3. In this article, we generalize Ye's Theorem by demonstrating that, if R is a local ring, then the group ring $R[G]$ is semi-clean if and only if G is a torsion abelian group.

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1. Introduction

Throughout the article we assume that R is a commutative ring with identity and G is an abelian group with operation written multiplicatively. We use $R[G]$ to denote the group ring

$$R[G] = \left\{ \sum_{i=1}^{n} r_i g_i : r_i \in R, g_i \in G \right\}$$

with coefficients from R. Clearly, $R[G]$ is also a commutative ring with identity. Recall that an element $a \in R$ is called *periodic* if there are $k < n \in \mathbb{N}$ such that $a^k = a^n$.

Definition 1.1. Let $a \in R$. We say a is

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(1) *clean* if a can be written as the sum of a unit and an idempotent;
(2) *feebly clean* if a can be written as the sum of a unit and the difference of two orthogonal idempotents;
(3) *semi-clean* if a can be written as the sum of a unit and a periodic element.

The ring R is called *clean* (respectively *feebly clean* or *semi-clean*) if every element is clean (respectively feebly clean or semi-clean). Clearly, every clean ring is feebly clean, and since the difference of two orthogonal idempotents is periodic, every feebly clean ring is semi-clean.

The notion of a clean ring was originally defined by Nicholson [11], while feebly clean was created by Arora and Kundu [1]. Restricting to integral domains, we note that a domain is clean if and only if it is local. The same can be said for indecomposable rings. Since units, idempotent elements, and periodic elements retain those properties under homomorphic image, the property that a ring is clean, feebly clean, or semi-clean is preserved under homomorphic image. For more information on clean rings, see [10].

In [6] and [5], the authors characterized when $\mathbb{Z}_{(p)}[C_q]$ is a clean group ring. Recently, in [7], the authors generalized the result to the case of $R[G]$ where G is a finite abelian group and R is a localization of the ring of integers $\mathcal{O}_K$ of K, where K is either a quadratic field or a cyclotomic field. In [8], a characterization was given of when the group ring $\mathbb{Z}_{(p)}[C_q]$ is feebly clean for distinct primes p and q, generalizing Ye's Theorem [12] by showing that any group ring of the form $\mathbb{Z}_{(p)}[C_3]$ is feebly-clean. The goal of this article is to characterize when the group ring $R[G]$ is semi-clean.

Our basic references for group rings are [3] and [9]. For a given prime p, by a p-group we mean a group G with the property that every element of G has order p^k for some k. A group ring $R[G]$ is said to have *trivial units* if the only units of $R[G]$ are of the form ug for some unit coefficient $u \in R$ and some group element $g \in G$. Of course, by a local ring we mean a ring with a unique maximal ideal.

Recall the following facts regarding group rings.

Theorem 1.2.

(1) *[3, Theorem 8.1] $R[G]$ is an integral domain if and only if R is an integral domain and G is torsion-free.*
(2) *[3, Theorem 19.1] $R[G]$ is a local ring if and only if R is a local ring and G is a p-group, where p is the (prime) characteristic of the residue field of R. Furthermore, if $R[G]$ is local, and M is the maximal ideal of R, then the maximal ideal of $R[G]$ is $M[G]+I$, where I is the augmentation ideal (i.e., the kernel of the ring homomorphism from $R[G]$ to R that maps each element of G to 1), and the characteristic of the residue field of $R[G]$ is also p.*
(3) *[9, Theorem 8.5.3] If F is a field and G is torsion-free, then $F[G]$ has trivial units.*

In one direction, it is easy to prove necessary conditions that $R[G]$ be semi-clean.

Theorem 1.3. *If $R[G]$ is a semi-clean ring, then R is a semi-clean ring and G is a torsion abelian group.*

Proof. First, we show that for a field F and a torsion-free abelian group H, if $F[H]$ is semi-clean, then H is the trivial group. To see this, suppose by way of contrapositive that H is non-trivial and let $h \in H$ be a non-identity element, so 1, h, and h^2 are distinct. The ring $F[H]$ is an integral domain by Theorem 1.2 (1), so its nonzero periodic elements are a sub-collection of its units. Since the element $1 + h + h^2 \in F[H]$ is a non-unit whose unique representation as an F-linear sum has three distinct elements of H, it cannot be written as an F-linear sum of two elements of H. Since $F[H]$ has trivial units by Theorem 1.2 (3), it follows that $1 + h + h^2$ is not the sum of a unit and a periodic element, so we conclude that $F[H]$ is not semi-clean.

Now to prove the theorem, assume that $R[G]$ is semi-clean. To see that R is semi-clean, note that R is a homomorphic image of $R[G]$. To see that G is a torsion abelian group, let M be a maximal ideal of R

and let $\tau(G)$ denote the torsion subgroup of G. Since the ring $(R/M)[G/\tau(G)]$ is a homomorphic image of $R[G]$, it is semi-clean. By the previous assertion, since R/M is a field and $G/\tau(G)$ is a torsion-free abelian group, it follows that $G/\tau(G)$ is the trivial group, i.e. that $G = \tau(G)$ is a torsion group. □

In the next section we prove sufficient conditions that $R[G]$ be semi-clean, giving a partial converse to Theorem 1.3.

2. Local ring R

Throughout this section, let R be a local ring with residue field of characteristic p, where we allow the possibility that $p = 0$, and let G be a torsion abelian group. The goal of this section is to prove that the group ring $R[G]$ is a semi-clean ring (Theorem 2.7). Through much of this section, we concentrate on the case in which G is finite.

Proposition 2.1. *Suppose that $p \neq 0$ and G is an internal direct sum of subgroups G_1 and G_2, where G_1 is a p-group. Then $R[G] = R[G_1][G_2]$, where $R[G_1]$ is a local ring with residue field of characteristic p.*

Proof. Clearly $R[G] = R[G_1][G_2]$; that $R[G_1]$ is a local ring with residue field of characteristic p follows from Theorem 1.2 (2). □

Suppose G is finite. If $p \neq 0$, then we can write $G = G_1G_2$, where G_1 is a p-group and p does not divide $|G_2|$. Thus, from Proposition 2.1, for finite G we can replace R by $R[G_1]$ and G by G_2, and assume that p does not divide $|G|$. In case $p = 0$, then obviously p does not divide $|G|$.

Proposition 2.2. *The ring R is a $\mathbb{Z}_{(p)}$-algebra, and $R[G] \cong R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[G]$.*

Proof. Since R is a local ring with residue field of characteristic p, it follows that R is a $\mathbb{Z}_{(p)}$-algebra. The isomorphism $R[G] \cong R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[G]$ then follows easily from the fact that $R[G]$ is a free R-module with basis G, and similarly for $\mathbb{Z}_{(p)}[G]$. (Note that, if $p = 0$, then $\mathbb{Z}_{(p)} = \mathbb{Q}$, the field of rational numbers.) □

Proposition 2.3. *If G is a finite abelian group and $p \nmid |G|$, then $\mathbb{Z}_{(p)}[G] \cong \bigoplus_m \mathbb{Z}_{(p)}[\omega_m]$, where the direct sum ranges over some collection of divisors m of $|G|$ (possibly with repetitions), and each ω_m is a primitive m-th root of unity in $\mathbb{C}$, the field of complex numbers.*

Proof. First note that $\mathbb{Q}[G] \cong \bigoplus F$, a finite direct sum of finite field extensions of $\mathbb{Q}$, by Maschke's Theorem [2, Theorem 3.14], where each field F is generated by the images of the elements of the group G. But these are all roots of unity dividing $|G|$, so each F has the form $\mathbb{Q}[\omega_m]$ for some divisor m of $|G|$, where ω_m is a primitive m-th root of unity in $\mathbb{C}$. Thus, $\mathbb{Q}[G] \cong \bigoplus_m \mathbb{Q}[\omega_m]$, so the proposition follows in case $p = 0$.

If $p \neq 0$, then since p does not divide $|G|$, it follows that $\mathbb{Z}_{(p)}[G]$ is a maximal $\mathbb{Z}_{(p)}$-order in $\mathbb{Q}[G]$ [2, Proposition 27.1]. But each $\mathbb{Q}[\omega_m]$ contains a unique maximal $\mathbb{Z}_{(p)}$-order, the integral closure $\mathbb{Z}_{(p)}[\omega_m]$ of $\mathbb{Z}_{(p)}$ in $\mathbb{Q}[\omega_m]$ [2, Proposition 4.5], so the proposition follows in this case also. □

From Propositions 2.2 and 2.3, for finite abelian group G with order not divisible by p, $R[G] \cong \bigoplus_m (R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega_m])$, so to show that $R[G]$ is semi-clean, it suffices to show that each $R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega_m]$ is semi-clean, where ω_m is a primitive m-th root of unity in $\mathbb{C}$ and $p \nmid m$. We drop the subscript m and assume that ω is a primitive m-th root of unity in $\mathbb{C}$.

Since $\mathbb{Z}_{(p)}[\omega]$ is a free $\mathbb{Z}_{(p)}$-module of rank $\phi(m)$ (Euler ϕ) [2, Proposition 4.5], it follows that $R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega]$ is a free R-module also of rank $\phi(m)$. Changing notation once again, let ξ denote the element $1 \otimes \omega$, so that $R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega] = R[\xi]$ and $\xi^m = 1$, which makes ξ periodic. The goal is to show that $R[\xi]$ is

semi-clean, which we accomplish by finding enough units of the appropriate form and bounding the number of maximal ideals.

Proposition 2.4. *If ω is a primitive m-th root of unity in $\mathbb{C}$, where $p \nmid m$, and $\xi = 1 \otimes \omega$ in $R[\xi] = R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega]$, then $1 - \xi^k$ is a unit in $R[\xi]$ for all k, $1 \leq k < m$.*

Proof. Since $1 + X + \ldots + X^{m-1} = \dfrac{X^m - 1}{X - 1} = \displaystyle\prod_{k=1}^{m-1} (X - \omega^k)$ in the polynomial ring $\mathbb{C}[X]$, evaluating at 1 and noting that m is a unit in $\mathbb{Z}_{(p)}$ shows that $1 - \omega^k$ is a unit in $\mathbb{Z}_{(p)}[\omega]$ for all k, $1 \leq k < m$. The proposition then follows from the identification $\xi = 1 \otimes \omega$ in $R[\xi] = R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega]$. □

Proposition 2.5. *If ω is a primitive m-th root of unity in $\mathbb{C}$, where $p \nmid m$, and $\xi = 1 \otimes \omega$ in $R[\xi] = R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega]$, then the ring $R[\xi]$ is semilocal, with at most $\phi(m)$ maximal ideals.*

Proof. Let M be the maximal ideal of R, and $K = R/M$ the residue field. As noted above, $R[\xi]$ is a free R-module of rank $\phi(m)$. Since $R[\xi]$ is integral over R, its maximal ideals contract to M in R, and hence the maximal ideals of $R[\xi]$ are in one-to-one correspondence with the maximal ideals of $R[\xi]/M[\xi] \cong K[\xi]$, a vector space of dimension $\phi(m)$ over K. Since $\phi(m) \leq m - 1$, the proposition follows. □

Putting it all together yields the result that we need.

Proposition 2.6. *If ω is a primitive m-th root of unity in $\mathbb{C}$, where $p \nmid m$, and $\xi = 1 \otimes \omega$ in $R[\xi] = R \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\omega]$, then the ring $R[\xi]$ is semi-clean.*

Proof. Let $\alpha \in R[\xi]$; we must show that α is semi-clean. Since $0 = -1 + 1$ is semi-clean, we can assume that α is nonzero. Consider the m elements $\alpha, \alpha - \xi, \alpha - \xi^2, \ldots, \alpha - \xi^{m-1}$. We claim that no two of these elements can be in the same maximal ideal of $R[\xi]$. For $1 \leq k < m$, the difference $\alpha - (\alpha - \xi^k) = \xi^k$ is a unit, so α and $\alpha - \xi^k$ cannot be in the same maximal ideal. For $1 \leq i < j < m$, the difference $\alpha - \xi^j - (\alpha - \xi^i) = \xi^i(1 - \xi^{j-i})$ is a unit by Proposition 2.4, so $\alpha - \xi^i$ and $\alpha - \xi^j$ cannot be in the same maximal ideal, completing the proof of the claim.

Now by Proposition 2.5 the ring $R[\xi]$ has at most $m - 1$ maximal ideals, so from the claim it follows that at least one of the m elements $\alpha, \alpha - \xi, \alpha - \xi^2, \ldots, \alpha - \xi^{m-1}$ must be a unit. If α is a unit, then $\alpha = \alpha + 0$ is semi-clean, while if $\alpha - \xi^k$ is a unit, then $\alpha = \alpha - \xi^k + \xi^k$ is still semi-clean, completing the proof of the proposition. □

Finally, we obtain the sufficient conditions that $R[G]$ be semi-clean.

Theorem 2.7. *Let R be a local ring with residue field of characteristic p. If G is a torsion abelian group, then the group ring $R[G]$ is semi-clean.*

Proof. For any $x \in R[G]$, there is a finite subgroup $H \subseteq G$ such that $x \in R[H]$. If $p \neq 0$, then we can write $H = H_1 H_2$ where H_1 is a p-group and p does not divide $|H_2|$. By Proposition 2.1, $R[H_1]$ is a local ring whose residue field has characteristic p, and $R[H] = R[H_1][H_2]$. Let $R' = R[H_1]$, so that $R[H] = R'[H_2]$. In case $p = 0$, simply take $R' = R$ and $H_2 = H$, so that $R[H] = R'[H_2]$ in this case also.

By Propositions 2.2 and 2.3, the group ring $R'[H_2]$ is a direct sum of rings of the form $R'[\xi] = R' \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p[\omega]$, where ω is a primitive m-th root of unity in $\mathbb{C}$ and $p \nmid m$, so each $R'[\xi]$ is semi-clean by Proposition 2.6. Therefore, $R'[H_2]$ is a semi-clean ring, and hence x is semi-clean in $R[H] \cong R'[H_2]$ and thus also in $R[G]$, so that $R[G]$ is a semi-clean ring. □

We conclude with a couple of open questions. The astute reader will notice that our Theorem 2.7 is not the converse of our Theorem 1.3. Indeed, for abelian group G, necessary conditions that the group ring $R[G]$ be semi-clean are that R be semi-clean and G be torsion (Theorem 1.3), while sufficient conditions that the group ring $R[G]$ be semi-clean are that R be local and G be torsion (Theorem 2.7). There is some space between local and semi-clean; for example, local implies clean, and clean implies semi-clean.

We know that R local and G torsion does not imply that $R[G]$ is clean (e.g., $\mathbb{Z}_{(7)}[C_3]$ is not clean [4, Example 1]), so certainly R clean and G torsion does not imply that $R[G]$ is clean. But does it imply that $R[G]$ is semi-clean? If so, does also R semi-clean and G torsion imply that $R[G]$ is semi-clean (which would be, of course, the converse to Theorem 1.3)? We have neither proofs nor counterexamples.

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