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# Conrad frames

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ABSTRACT

A *Conrad frame* is a frame which is isomorphic to the frame $\mathcal{C}(G)$ of all convex $\ell$-subgroups of some lattice-ordered group $G$. It has long been known that Conrad frames have the disjointification property. In this paper a number of properties are considered that strengthen the disjointification property; they are referred to as the *Conrad conditions*. A particularly strong form of the disjointification property, the *C-frame condition*, is studied in detail. The class of lattice-ordered groups $G$ for which $\mathcal{C}(G)$ is a C-frame is shown to coincide with the class of pairwise splitting $\ell$-groups. The arguments are mostly frame-theoretic and *Choice-free*, until one tackles the question of whether C-frames are Conrad frames. They are, but the proof is decidedly not point-free. This proof actually does more: it shows that every algebraic frame with the FIP and disjointification can be coherently embedded in a C-frame. When the discussion is restricted to normal-valued lattice-ordered groups, one is able to produce examples of coherent frames having disjointification, which are not Conrad frames.



## 1. Introduction

The frame $\mathcal{C}(G)$ of all convex $\ell$-subgroups of an $\ell$-group $G$ has the finite intersection property (abbr. FIP) and disjointification. And so far as anyone knew, any property of a $\mathcal{C}(G)$ could be shown to hold in any algebraic frame with the FIP and disjointification.

The naive question is whether every algebraic frame with the FIP and disjointification is a Conrad frame. And the question is naive indeed, because – as a number of authors, who have asked this or other, closely related questions have found out – these questions tend to be intractable, in the sense that it is often reasonably easy to give a counterexample to the question, but difficult to replace the newly resolved question with a more challenging one. The reader is referred to [4] and [5], which deal with such questions, but with reference to the real spectrum of a commutative ring.

The answer to our naive question is a qualified *no*: we are able to show that disjointification is not enough to make an algebraic frame $L$ with the FIP and disjointification arise as $L = \mathcal{C}(G)$ with $G$ normal-valued. Indeed, Proposition 4.1.2 shows that each Conrad frame arising as $\mathcal{C}(G)$, with $G$ normal-valued, is a $\sigma$-Conrad frame (defined in 4.1.1). The latter have disjointification, yet Example 4.2.2 shows that an algebraic frame with the FIP and disjointification need not be $\sigma$-Conrad.

Early in this investigation we stumbled onto one of these Conrad conditions, and for a time we believed it characterized Conrad frames. Having realized that "being a C-frame" is too strong, we find it interesting enough to make it one of the themes of this exposition.

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There are four parts to the presentation. The first part lays the foundation: Section 1.1 contains the basic frame-theoretical background information, and Section 1.2 introduces C-frames. In Section 1.3 it is established that every regular algebraic frame is a C-frame (Proposition 1.3.3), as are all the algebraic frames with disjointification which satisfy the dual frame law (Proposition 1.3.2).

Part 2 recalls the concept of a pairwise splitting $\ell$-group, and immediately capitalizes on the intuition it evokes, to formulate a version of it for algebraic frames with the FIP. The main theorem of this part (Theorem 2.2.3) shows that the C-frames are precisely the pairwise splitting frames. It then follows easily that $G$ is a pairwise splitting $\ell$-group if and only if its frame $\mathcal{C}(G)$ of convex $\ell$-subgroups is a C-frame.

The third part contains the principal accomplishment of this paper, Theorem 3.1.1, proving that every C-frame is a Conrad frame. It is a satisfying result, because the proof tells us more: that every algebraic frame with the FIP and the disjointification can be embedded in a C-frame. What is left unsettled is the question of whether this theorem cannot be proved *Choice*- and *point-free*. We shall take up this problem elsewhere, along with the growing list of questions regarding the way an algebraic frame with the FIP and the disjointification may be embedded in a C-frame.

We conclude in Part 4 with a discussion of $\sigma$-Conrad frames, leading up to the example mentioned above.

### *1.1. Preliminaries*

We begin with a list of basic frame-theoretic definitions, which the knowledgeable reader ought to be able to skip entirely. For additional information one may refer to [7] and Chapter 2 of [16]. In particular, we feel free to assume that the reader is familiar with algebraic lattices.

For completeness, we stipulate that a *frame* is a complete lattice $L$ in which the following distributive law holds: for each $S \subseteq L$ and $a \in L$,

$$a \wedge \left(\bigvee S\right) = \bigvee\{a \wedge x\colon\ x \in S\}.$$

**Definition 1.1.1.** Throughout, $L$ is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in L$, denote the set of elements of $L$ less than or equal to (resp. greater than or equal to) $x$ by $\downarrow x$ (resp. $\uparrow x$). We denote by $\mathfrak{k}(L)$ the set of all compact elements of $L$.

- The algebraic lattice $L$ has the *finite intersection property* (abbr. *FIP*) if for any pair $a, b \in \mathfrak{k}(L)$, $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema. $L$ is *coherent* if 1 is compact and $L$ has the FIP.
- The *Heyting operation* $a \to b$ (in a frame $L$):

  $$a \to b = \bigvee\{x \in L\colon\ a \wedge x \leqslant b\}.$$

  Also put $x^{\perp} \equiv x \to 0$.
- $p \in L$ a *polar*: one of the form $p = y^{\perp}$, for some $y \in L$. It is well known that the set $\mathcal{P}L$ of all polars forms a complete boolean algebra, in which infima agree with those in $L$.
- In a frame, $a$ is *well below* $b$, written $a \preccurlyeq b$ : $b \vee a^{\perp} = 1$.
- $x \in L$ is *regular*: $x = \bigvee\{a \in L\colon\ a \preccurlyeq x\}$. Let $Reg(L)$ denote the subset of all regular elements of $L$. A frame $L$ is *regular*: each element of $L$ is regular.

We record a brief comment concerning frame homomorphisms and their adjoints. Recall that a *frame homomorphism* is one which preserves arbitrary joins and all finite meets, including the empty one. This forces a frame homomorphism to preserve the top and the bottom of the frame.

**Definition & Remarks 1.1.2.** We start in the category $\mathfrak{Frm}$ of all frames and all frame homomorphisms. If $h : L \to M$ is a $\mathfrak{Frm}$-morphism, then $h_* : M \to L$ denotes its right adjoint; that is, the map defined by

$$x \leqslant h_*(y) \quad \Leftrightarrow \quad h(x) \leqslant y, \quad \text{for all } x \in L,\ y \in M.$$

The following are well known:

1. $h_*$ preserves all infima.
2. $x \leqslant h_* \cdot h(x)$, for each $x \in L$, and $h \cdot h_*(y) \leqslant y$, for each $y \in M$. Thus, $h \cdot h_* \cdot h = h$ and $h_* \cdot h \cdot h_* = h_*$.

It follows from the above properties that $h$ is one-to-one if and only if $h_* \cdot h = 1_L$, and that $h$ is surjective if and only if $h \cdot h_* = 1_M$.

Recall that a frame homomorphism is called *coherent* when it carries compact elements to compact elements.

### 1.2. Disjointification

We proceed to strengthen disjointification. The goal is, as explained before, to characterize the frame $\mathcal{C}(G)$ of convex $\ell$-subgroups of an $\ell$-group $G$. Let us anchor the discussion by declaring that a frame $L$ will be called a *Conrad frame* if it is isomorphic to $\mathcal{C}(G)$, for a suitable $\ell$-group $G$. For example, one may reasonably put matters this way, in light of the work of [10,13]: *a frame is a regular Conrad frame if and only if it is algebraic and each compact element is complemented.*

**Definition & Remarks 1.2.1.** Throughout this commentary $L$ stands for an algebraic frame.

1. Call $p < 1$ *prime* if whenever $a \wedge b \leqslant p$, then either $a \leqslant p$ or $b \leqslant p$. The collection of prime elements of $L$ is denoted by $\mathrm{Spec}(L)$.
2. A routine Zorn's Lemma argument shows that if $0 < c$ is compact, then there exists an element, say $z$, that is maximal with respect to $c \nleqslant z$. Such an element is said to be a *value of c*. It is well known that values are always prime, and that the values of $L$ are the meet-irreducible elements – that is, $p$ is a value precisely when $p = \bigwedge S$ implies that $p \in S$. It is then clear that each value $p$ has a *cover* $p^*$, namely the infimum of all the elements that strictly exceed $p$. It is well known that $p$ is a value of $c$ if and only if $c \nleqslant p$ and $c \leqslant p^*$.
3. We say that $L$ has *disjointification* if for every $a, b \in \mathfrak{k}(L)$ there are disjoint $c, d \in L$ such that $a \vee b = c \vee b = a \vee d$. Such a pair $(c, d)$ is called a *splitting* of $(a, b)$. It is easy to see that in a splitting $c \leqslant a$, $d \leqslant b$, and both $c$ and $d$ may be chosen compact.
   If $L$ has disjointification, then for every $p \in \mathrm{Spec}(L)$, $(\uparrow p) \cap \mathrm{Spec}(L)$ is a chain. If $L$ also has the FIP, then the converse is true. This was first proved by Monteiro (see [15], or [17, Lemma 2.1] where a proof is given). A poset with this property is called a *root system*.

Our interest in C-frames is due to the mistaken impression that they would characterize Conrad frames. As we will establish, they characterize frames $\mathcal{C}(G)$ for a very interesting class of $\ell$-groups.

**Definition & Remarks 1.2.2.** Suppose $L$ is algebraic and has the FIP. We say that $L$ is a *C-frame* if for each $a, b \in \mathfrak{k}(L)$ there exist $x, y \in L$ such that:

1. $(x, y)$ is a splitting of $(a, b)$;
2. for any $u, w \in L$ if $a \leqslant u \vee b$, then $x \leqslant u$, and if $b \leqslant a \vee w$, then $y \leqslant w$.

It is easy to see that the pair $(x, y)$ that witnesses that $L$ is a C-frame for $(a, b)$ is unique. We call it the *kernel splitting* of $(a, b)$. We also label $x = a(b)$ and $y = b(a)$.

The reader is encouraged to think of the second defining condition here as a lattice-theoretic "Riesz Interpolation," as it is this property of $\ell$-groups – see 1.2.4(4) – which motivated it in the first place.

A routine compactness argument fine-tunes the above.

**Proposition 1.2.3.** *Suppose that L is an algebraic frame with the FIP. L is a C-frame if and only if for each $a, b \in \mathfrak{k}(L)$ there exist disjoint $x, y \in \mathfrak{k}(L)$ such that*:

1. *$(x, y)$ is a splitting of $(a, b)$;*
2. *for any $u, w \in \mathfrak{k}(L)$ if $a \leqslant u \vee b$, then $x \leqslant u$, and if $b \leqslant a \vee w$, then $y \leqslant w$.*

*Moreover, the kernel splitting $(a(b), b(a))$ of $(a, b)$ consists of compact elements.*

We turn now to a review of basic notions from the theory of lattice-ordered groups. For additional background on the subject we refer the reader to [1,3].

**Definition & Remarks 1.2.4.** For the record, $(G, +, 0, -(\cdot), \vee, \wedge)$ is a *lattice-ordered group* (abbreviated *$\ell$-group*) if $(G, +, 0, -(\cdot))$ is a group with $(G, \vee, \wedge)$ as an underlying lattice, and the following distributive laws holds:

$$a + (b \vee c) + d = (a + b + d) \vee (a + c + d).$$

The above then implies the corresponding distributive law for sum over infimum. If $g \geqslant 0$ in $G$, it is said to be *positive*; the set of positive elements of $G$ is denoted $G^+$.

We recite the information to be used in this article; in the sequel $G$ stands for an $\ell$-group.

1. The underlying lattice of an $\ell$-group is distributive [3, Corollary 3.17], and the group structure is torsion free [3, Propositions 3.15 & 3.16].

2. A subgroup of $G$ is called an $\ell$-*subgroup* if it is a sublattice as well. The $\ell$-subgroup $K$ is *convex* if $a \leqslant g \leqslant b$ with $a, b \in K$ implies that $g \in K$. Let $\mathcal{C}(G)$ denote the lattice of all convex $\ell$-subgroups of $G$. $\mathcal{C}(G)$ is a complete sublattice of the lattice of all subgroups of $G$ [3, Theorem 7.5], and an algebraic frame; the latter is due to G. Birkhoff [3, Proposition 7.10]. $\mathcal{C}(G)$ satisfies the FIP [3, Proposition 7.15], but, in general, fails to be coherent.
In $\mathcal{C}(G)$ the convex $\ell$-subgroup generated by $a \in G$ is denoted $\langle a \rangle_G$. (This is not the customary notation for this object. However, we feel strongly that the notation offered here makes more sense, and it is closer to other symbols representing ideals and the like in the context of lattices.)
Each compact element of $\mathcal{C}(G)$ is of this form [3, Proposition 7.16]. Note that, for $0 \leqslant a, b \in G$, $\langle a \rangle_G \subseteq \langle b \rangle_G$ precisely when $a \leqslant nb$, for a suitable natural number $n$.
3. For every $\ell$-group $G$, $\mathcal{C}(G)$ is a frame with disjointification. Indeed, if $a, b \geqslant 0$ in $G$, let $c = a - (a \wedge b)$ and $d = b - (a \wedge b)$; then $\langle c \rangle_G$ and $\langle d \rangle_G$ witness the disjointification of $\langle a \rangle_G$ and $\langle b \rangle_G$.
4. In the proof of Proposition 4.1.2 we will have occasion to use the *Riesz Interpolation Property* and its proof: if $g, x, y \in G^+$ and $g \leqslant x + y$, then $g = x' + y'$, for suitable positive elements $x' \leqslant x$ and $y' \leqslant y$. The standard argument used to prove this puts $x' = g \wedge x$ and $y' = -x' + g$. It is easy to show that $y' \leqslant y$ [3, Theorem 3.11].

### *1.3. Properties of C-frames*

The first objective in this section is to prove that if $L$ is an algebraic frame with the FIP and disjointification, that also satisfies the dual frame law, then it is a C-frame.

Throughout it is assumed that $L$ is an algebraic frame with the FIP. We summarize the necessary background information in the next theorem. There are several accounts of this beyond Conrad's Theorem on finite-valued $\ell$-groups [2]; see, for example, [9,17].

**Theorem 1.3.1.** *Suppose $L$ has disjointification. Then the following are equivalent.*

(a) *$L$ is completely distributive.*
(b) *$L$ is a dual frame; that is, for each $S \subseteq L$,*

$$a \vee \Bigl(\bigwedge S\Bigr) = \bigwedge \{a \vee s \colon s \in S\}.$$

(c) *For each $c \in \mathfrak{k}(L)$, $\downarrow c$ has a finite number of maximal elements.*

In the sequel, we shall use the convention that *$a$ is a component of $b$* to signify that $b = a \vee x$, with $a \wedge x = 0$, for some $x \in L$. Without further ado, one has the following.

**Proposition 1.3.2.** *Suppose $L$ is a dual frame with disjointification. Then $L$ is a C-frame.*

**Proof.** Suppose that $a$ and $b$ are compact, and consider all splittings $(x, y)$ of $(a, b)$. Let $u$ be the meet of all such $x$, that is, all "first components" of splittings of $(a, b)$. By the dual frame law,

$$a \vee b = u \vee b = a \vee y,$$

and so $(u, y)$ is a splitting. Taking the infimum $v$ over all second components, produces $(u, v)$, which is clearly the kernel splitting of $(a, b)$. □

It is easily seen that a C-frame need not be a dual frame.

**Proposition 1.3.3.** *Every regular algebraic frame is a C-frame.*

**Proof.** (Sketch.) Each compact element of $L$ is complemented. Thus, if $a$ and $b$ are compact, then $u \equiv a \wedge b^{\perp}$ and $v \equiv b \wedge a^{\perp}$ are the entries in the kernel splitting $(u, v)$ of $(a, b)$. □

We now consider the behavior of the class of C-frames under frame-homomorphic images. We begin the discussion with a brief review of closed maps.

**Definition & Remarks 1.3.4.** A map of the form $x \mapsto x \vee a$ from $A$ onto $\uparrow a$ is a *closed quotient*. Next, suppose that $h : A \to B$ is a frame homomorphism, and let $q : B \to F$ be a frame surjection. Factor $q \cdot h = m \cdot e$ through the image, as indicated in the square below:

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & B \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ E & \xrightarrow{\;m\;} & F \end{array} \tag{1.3.4.1}$$

We say that $h$ is *closed* if for each closed quotient $q$, $e$ too is a closed quotient.

It is shown in [16, Chapter II, §5] that the following are equivalent:

- $h : A \to B$ is closed.
- $h_*(h(a) \vee y) = a \vee h_*(y)$, for each $a \in A$ and $y \in B$.
- $h(b) \leqslant h(a) \vee y \;\Rightarrow\; b \leqslant a \vee h_*(y)$.

It is well known that if $h : A \to B$ is a frame surjection, $A$ is regular, and $B$ is compact, then $h$ is closed, and hence a closed quotient [7, Chapter III, Proposition 1.2]. Using this fact, and the property that a frame-homomorphic image of a regular frame is regular, it is easy to prove that every frame embedding $m : A \to B$ of compact regular frames is closed.

The following characterization of closed maps among the coherent ones will be put to good use; the proof is straightforward.

**Lemma 1.3.5.** *Suppose $h : L \to M$ is a coherent frame map between algebraic frames. Then $h$ is closed if and only if for each $a, b \in \mathfrak{k}(L)$ and $d \in \mathfrak{k}(M)$ such that $h(a) \leqslant h(b) \vee d$, there is a compact $c \in L$ such that $h(c) \leqslant d$, and $a \leqslant b \vee c$.*

Suppose that $h : L \to M$ is a coherent map to the algebraic frame $M$. Let $a, b \in \mathfrak{k}(L)$ and suppose this pair has kernel splitting $(u, v)$. We will say that $h$ *preserves kernel splittings* if for each pair $a, b \in \mathfrak{k}(L)$, $(h(u), h(v))$ is the kernel splitting for $(h(a), h(b))$.

**Proposition 1.3.6.** *Suppose $L$ is a C-frame and $h : L \to M$ is a coherent map to the algebraic frame $M$. Then $h$ is closed if and only if it preserves kernel splittings.*

*In particular, the image under a closed coherent frame homomorphism of a C-frame is a C-frame.*

**Proof.** (Necessity.) That $(h(u), h(v))$ is a splitting of $(h(a), h(b))$ follows directly from the definition; we leave the details to the reader. So suppose that $h(a) \leqslant y \vee h(b)$. Since $h$ is closed, we may use Lemma 1.3.5: pick $t \in \mathfrak{k}(L)$ such that $a \leqslant t \vee b$ and $h(t) \leqslant y$, whence $c \leqslant t$, and $h(c) \leqslant h(t) \leqslant y$. This proves that $(h(u), h(v))$ is a kernel splitting of $(h(a), h(b))$.

The proof of the sufficiency is similar, and is omitted. The last claim is obvious. □

## 2. Pairwise splitting revisited

The $\ell$-groups $G$ for which $\mathcal{C}(G)$ is a C-frame turn out to be the pairwise splitting $\ell$-groups first studied in [11], and mostly forgotten since. Let us then reintroduce them now.

### 2.1. Infinitesimals

This $\ell$-group-theoretic concept, along with a number of notions extracted from [12], motivate the frame-theoretic definition which follows.

**Definition & Remarks 2.1.1.** An $\ell$-group $G$ is said to be *pairwise splitting* if for each $0 \leqslant x, y \in G$, we may write

$$x = x_1 + x_2, \quad \text{with } x_1 \wedge x_2 = 0,\ x_1 \in \langle y \rangle_G, \text{ and } x_2 \wedge y \ll x_2,$$

where $a \ll b$ denotes that $na < b$ for every positive integer $n$. We say that $x$ *splits by* $y$ when the above decomposition occurs.

The following remark will not be used anywhere in these pages, and so we mention it, but without any further comment: *an archimedean $\ell$-group is pairwise splitting if and only if it is hyper-archimedean.*

The critical technical issue in lifting this idea to frames is defining infinitesimals in frames. For this we will appeal to [12]. Throughout the rest of this section $L$ denotes an algebraic frame with the FIP.

**Definition 2.1.2.** Let $a, b \in \mathfrak{k}(L)$, with $a \leqslant b$; we say that $a$ *is infinitesimal to* $b$ – write $a \ll b$ – if $b = a \vee y$ implies that $y = b$. Imitating $\ell$-groups, this notion has the following interpretation, assuming the Axiom of Choice. We leave the proof, using Zorn's Lemma, to the reader.

**Proposition 2.1.3.** *Suppose $L$ is an algebraic frame with the FIP, and $a < b$ are compact. Then $a \ll b$ if and only if each value of $a$ is strictly below a value of $b$.*

We state as a formal lemma, a very practical observation. The proof is immediate, and is left to the reader. Assuming *Choice* once more, Proposition 2.1.3 and this lemma give us the subsequent corollary.

**Lemma 2.1.4.** *Let $a$ and $b$ be compact elements in an algebraic frame $L$. Then $a \wedge b \ll a$ precisely when $a \leqslant y \vee b$, implies that $a \leqslant y$.*

**Corollary 2.1.5.** *Suppose $L$ is an algebraic frame with the FIP. Then for $a, b \in \mathfrak{k}(L)$, $a \wedge b \ll a$ if and only if no value of $a$ is bounded above by a value of $b$.*

### *2.2. Pairwise splitting in frames*

We may now formulate the pairwise splitting of elements in a frame. In doing that, we borrow liberally from $\ell$-group terminology employed in the definitions of 2.1.1.

**Definition 2.2.1.** Let $a, b \in \mathfrak{k}(L)$. We say that $a$ *splits by* $b$, if $a = a_1 \vee a_2$, with $a_1 \wedge a_2 = 0$, $a_1 \leqslant b$, and $a_2 \wedge b \ll a_2$. $L$ has *pairwise splitting* if any two compact elements split by one another.

The next lemma is false without the assumption of disjointification. Consider the finite frame $L = \{0, x, a, b, 1\}$, with

$$0 < x = a \wedge b < a, b < a \vee b = 1.$$

Note that $x$ is infinitesimal to both $a$ and $b$, but not to their meet.

**Lemma 2.2.2.** *Suppose $L$ is an algebraic frame with the FIP and disjointification, with $x$ and $y$ compact. If $x \wedge y \ll x$ and $x \wedge y \ll y$, then $x \wedge y = 0$.*

**Proof.** Disjointify $x$ and $y$ with $u \leqslant x$ and $v \leqslant y$. Then $x \leqslant u \vee y$ implies that $x = u$, and, likewise, $y = v$, whence we have $x \wedge y = 0$. □

We are ready to prove the main theorem of this section.

**Theorem 2.2.3.** *Suppose $L$ is an algebraic frame with the FIP and disjointification. Then $L$ is a C-frame precisely when it has pairwise splitting.*

**Proof.** Assume first that $L$ is a C-frame, and let $a$ and $b$ be compact elements. Consider the kernel splitting $(a(b), b(a))$ of $(a, b)$. Now observe that $a = (a \wedge b) \vee a(b)$, a join we disjointify with $(x, y)$, so that

$$a = (a \wedge b) \vee a(b) = x \vee a(b) = (a \wedge b) \vee y,$$

with $x \leqslant a \wedge b \leqslant b$ and $y \leqslant a(b)$. However, since $a \leqslant b \vee y$, it follows that $y = a(b)$, and so the join $a = x \vee y$ witnesses the splitting of $a$ by $b$, as it is clear that $b \wedge y \ll y$.

Conversely, suppose $L$ has pairwise splitting, and $a, b \in \mathfrak{k}(L)$. Split $a$ by $b$ and vice-versa: $a = a_1 \vee a_2$, with $a_1 \leqslant b$ and $a_2 \wedge b \ll a_2$, and likewise with $a$ and $b$ interchanged. It is easily checked that $a_2 \wedge b_2$ is infinitesimal to both $a_2$ and $b_2$; invoking Lemma 2.2.2, we conclude that $a_2 \wedge b_2 = 0$, which shows that $(a_2, b_2)$ is a splitting of $(a, b)$. It is, in fact, the kernel splitting, because it easily follows from the inequality $a \leqslant z \vee b$ that $a_2 \leqslant z$.

This suffices to establish the theorem. □

Lemma 3 in [11] leads to a corollary of Theorem 2.2.3. We will use the following notation: if $a$ compact, $n_a$ denotes the join of all compact elements $b \ll a$; evidently, $n_a < a$.

**Corollary 2.2.4.** *Suppose L is an algebraic frame with the FIP and disjointification. Then L is a C-frame if and only if, for each $a \in \mathfrak{k}(L)$, the frame*

$$L_a \equiv (\downarrow a) \cap (\uparrow n_a)$$

*is regular.*

**Proof.** First suppose that $L$ is a C-frame. Observe that a compact element of $L_a$ is of the form $b \vee n_a$, with $b \leqslant a$. Split $a$ by $b$: $a = a_1 \vee a_2$, disjointly, with $a_1 \leqslant b$ and $b \wedge a_2 \ll a_2$. It then follows that $b \wedge a_2 \ll a$, whence $a = (a_1 \vee n_a) \vee (a_2 \vee n_a)$ in $L_a$, with $a_1 \vee n_a \leqslant b \vee n_a$ and $b \wedge a_2 \leqslant n_a$; that is, $a_2 \vee n_a \in (b \vee n_a)^{\perp}$ (in $L_a$). This proves that $L_a$ is regular.

Conversely, suppose each $L_a$ is regular, and pick $a$ and $b$ compact in $L$. Write $a = x \vee y$, with $x, y \in \mathfrak{k}(L)$ and such that $x \vee n_a \leqslant (a \wedge b) \vee n_a$ and $y \wedge b \ll a$. Without loss of generality one may, in fact, simplify, by assuming that $x \leqslant b$. We claim that $y \wedge b \ll y$: for suppose that $y \leqslant b \vee z$; then $a \leqslant x \vee b \vee z = b \vee z$, which implies that $y \leqslant a \leqslant z$, proving the claim. Then also $y \wedge x \ll y$, which we will leave for the reader to verify. Finally, observe that if $x \leqslant y \vee s$, then

$$x = x \wedge b \leqslant (y \wedge b) \vee s,$$

which shows that $y \wedge b \ll x$, and hence also $y \wedge x \ll x$. Invoking Lemma 2.2.2, one concludes that $x \wedge y = 0$.

The reader will observe that it is shown that $a$ splits by $b$, and thus that $L$ is a C-frame. □

Finally, in this interplay between the C-frame condition and pairwise splitting in frames, there is the following observation. This proposition will be used in the proof of Theorem 3.1.1.

**Proposition 2.2.5.** *Suppose L is a C-frame, and a and b are compact elements of L. Then a and b admit disjoint decompositions*

$$a = a_{\uparrow} \vee a_{\downarrow} \vee a_{\oplus}, \quad \textit{and} \quad b = b_{\uparrow} \vee b_{\downarrow} \vee b_{\oplus},$$

*such that*

(a) $a_{\oplus} = b_{\oplus}$;
(b) $a_{\downarrow} \ll b_{\uparrow}$ *and* $b_{\downarrow} \ll a_{\uparrow}$.

**Proof.** Split $a$ by $b$, as $a = a_1 \vee a_2$, disjointly, with $a_1 \leqslant b$ and $a_2 \wedge b \ll a_2$. Now split $b$ by $a_1$, obtaining $b = b_1 \vee b_2$, disjointly, with $b_1 \leqslant a_1$ and $a_1 \wedge b_2 \ll b_2$. Since $a_1 \leqslant b$, we get $a_1 = b_1 \vee (a_1 \wedge b_2)$, also disjointly. Put $a_{\downarrow} = a_1 \wedge b_2$, and define $b_{\downarrow}$ similarly, reversing the roles of $a$ and $b$. Further, set $a_{\oplus} = b_{\oplus} = b_1$ and $a_{\uparrow} = a_2$, with $b_{\uparrow}$ defined analogously, and the remaining details are easy to verify. □

### *2.3. When $\mathcal{C}(G)$ is a C-frame*

We have shown, for algebraic frames with the FIP and disjointification, that a C-frame is nothing more or less than one with pairwise splitting. But the realization that this would be so came from $\ell$-groups. The theorem that follows came first, albeit *Choice dependent*, driven by the intuition derived from [11], its original formulation given for normal-valued $\ell$-groups. Theorem 2.2.3 is *Choice-free*, and the version of Theorem 2.3.2 the reader gets is too. Theorem 2.3.2 is ultimately a consequence of Theorem 2.2.3, resolving the relationship between infinitesimals in frames and in $\ell$-groups. The reader ought to take note that the argument proving the lemma which follows is *Choice-free* and valid for all $\ell$-groups.

**Lemma 2.3.1.** *In any $\ell$-group G, and for each pair of elements $0 \leqslant a \leqslant b \in G$, $a \ll b$ if and only if $\langle a\rangle_G \ll \langle b\rangle_G$.*

**Proof.** Suppose that $\langle a\rangle_G \ll \langle b\rangle_G$. From the identity

$$\langle b\rangle_G = \langle a\rangle_G \vee \big\langle b - (b \wedge na)\big\rangle_G$$

it follows that $\langle b\rangle_G = \langle b - (b \wedge na)\rangle_G$. Further, since

$$na - (b \wedge na) \in \langle a\rangle_G \subseteq \langle b\rangle_G = \big\langle b - (b \wedge na)\big\rangle_G,$$

there is a positive integer $m$ such that

$$(b - na)^{-} = na - (b \wedge na) \leqslant m\big(b - (b \wedge na)\big) = m(b - na)^{+}.$$

But this means that $(b - na)^{-} = 0$, which shows that $na \leqslant b$ and that $a \ll b$.

In the other direction the argument is easier, and we leave it to the reader. □

Proving Theorem 2.3.2 is now straightforward.

**Theorem 2.3.2.** *Suppose G is an ℓ-group. $\mathcal{C}(G)$ is a C-frame if and only if G is pairwise splitting.*

**Proof.** By Theorem 2.2.3, $\mathcal{C}(G)$ is a C-frame if and only if it is pairwise splitting. The latter occurs precisely when, for each $0 \leqslant a, b \in G$, $\langle b\rangle_G = \langle b_1\rangle_G \langle b\rangle_G \vee \langle b_2\rangle_G$, with $b_1 \wedge b_2 = 0$, $b_1 \leqslant na$ for some natural number $n$, and (owing to Lemma 2.3.1) $a \wedge b_2 \ll b_2$. Thus, $\mathcal{C}(G)$ is a C-frame if and only if $b = x_1 + x_2$, with $x_1$ and $x_2$ disjoint and $0 \leqslant x_i \in \langle b_i\rangle_G$ $(i = 1, 2)$, with the $b_i$ as before. Then it follows that $\langle b_i\rangle_G = \langle x_i\rangle_G$ $(i = 1, 2)$, and so $x_1$ and $x_2$ witness the splitting of $b$ by $a$ as prescribed in 2.1.1.

This suffices to prove the theorem. □

## 3. Conrad conditions

It is shown in [11, Theorem 2] that a pairwise splitting ℓ-group is normal-valued. This would seem to signal that pairwise splitting has group-theoretic consequences. Let us look more closely.

We begin by reminding the reader of the discussion in 1.2.1 concerning values. Here we shall use that material in the context of $\mathcal{C}(G)$. Each value $M$ has a cover $M^*$, the intersection of all the convex ℓ-subgroups that properly contain $M$. When $M$ is always a normal subgroup of $M^*$, $G$ is said to be *normal-valued*. For our purposes, the important fact to highlight is that $G$ is normal-valued precisely when $A + B = B + A$, for any two convex ℓ-subgroups $A$ and $B$ [3, Theorem 41.1]. Thus, the join operation in $\mathcal{C}(G)$ is the sum of subgroups.

### 3.1. C-frames are Conrad frames

Before we go any further, let us agree on a convention. If the Conrad frame $L$ is $L = \mathcal{C}(G)$, and $G$ has property $\mathcal{X}$, we shall call $L$ an $\mathcal{X}$ *Conrad frame*.

Without any preliminary fuss, here is the main theorem, made more interesting because of the proof.

**Theorem 3.1.1.** *Suppose that L is an algebraic frame with the FIP and disjointification. If L is a C-frame, then it is an abelian, pairwise splitting Conrad frame.*

**Proof.** We imitate the (idea of) the proof of [10, Theorem 3.2]. To make the proof easier to follow, we divide it, formally, into parts.

1. *Hahn groups.* Let $V$ denote the set of values of $L$, and consider the group $H$ of all integer-valued functions $f$ on $V$ which have finite range and such that

$$\mathrm{coz}(f) = \{x \in V:\ f(x) \neq 0\}$$

satisfies the ascending chain condition. Order $H$ by the so-called "Hahn" ordering; that is $f > 0$ if $f \neq 0$ and $f(m) > 0$, for each maximal $m \in \mathrm{coz}(f)$.

2. *The connection.* For each $a \in \mathfrak{k}(L)$, let $Y_a$ denote the set of values of $a$. Define $\gamma(a)$ to be the characteristic function of $Y_a$. It is easily seen that $a \leqslant b$ in $\mathfrak{k}(L)$ if and only if $\gamma(a) \leqslant \gamma(b)$; in particular, $\gamma$ is one-to-one. Finally, let $G$ be the ℓ-subgroup of $H$ generated by the $\gamma(a)$. We will prove that $L$ is isomorphic to $\mathcal{C}(G)$. To that end it is enough to show that the map defined by $\Gamma(c) = \langle\gamma(c)\rangle_G$ is an isomorphism of $\mathfrak{k}(L)$ onto $\mathfrak{k}(\mathcal{C}(G))$.

3. *Γ is a lattice embedding.* To show that this map preserves the lattice operations, observe that if $a, b \in \mathfrak{k}(L)$, then $m \in L$ is a value of $a \wedge b$ precisely when it is a value of one of the two, which is below a value of the other. Thus, it is clear that $\gamma$ preserves finite meets, and since

$$\langle g\rangle_G \cap \langle h\rangle_G = \langle g \wedge h\rangle_G,$$

for any two positive elements of an ℓ-group, $\Gamma$ too preserves finite infima. The argument for finite joins is similar, since $m$ is a value of $a \vee b$ $(a, b \in \mathfrak{k}(L))$ if and only if $m$ is a value of one of them which is not strictly less than a value of the other. The details are left to the reader.

Now assume $a$ and $b$ are compact in $L$, with $a \nleqslant b$. Then, there is a value $q$ of $a$, such that $b \leqslant q$. This shows that $\gamma(a)(q) = 1$, while $\gamma(b)(q) = 0$, and therefore $\Gamma(a) \nleqslant \Gamma(b)$, proving that $\Gamma$ is one-to-one.

What remains is to be shown $\Gamma$ is surjective.

4. *Surjectivity of Γ.*

(a) A typical element of $G$ has the form

$$\bigvee_A \bigwedge_B \sum_i m(\alpha, \beta, i)\gamma\big(c(\alpha, \beta, i)\big), \tag{†}$$

the indicated joins and meets being taken over finite sets; the $m(\alpha, \beta, i)$ are integers, and the $c(\alpha, \beta, i)$ are compact. If $0 \leqslant g \in G$, and $g$ is expressed as in (†), we may alter it – by taking the join with 0 and distributing – to read:

$$\bigvee_A \bigwedge_B \left( \sum_i m(\alpha, \beta, i)\gamma\big(c(\alpha, \beta, i)\big) \right) \vee 0, \tag{‡}$$

using the fact that the underlying lattice of $G$ is distributive. What must be shown is that $\langle g \rangle_G = \langle \gamma(a) \rangle_G$, for some $a \in \mathfrak{k}(L)$, and since $\mathfrak{k}(L)$ is a sublattice, it suffices to do this for each expression $(\sum_i m(\alpha, \beta, i)\gamma(c(\alpha, \beta, i))) \vee 0$. We may therefore assume that $g = (\sum_{i=1}^n m_i \gamma(c_i)) \vee 0$, with each $m_i \neq 0$. From here we proceed by induction on $n$.

(b) If $g = (m_1\gamma(c_1) + m_2\gamma(c_2)) \vee 0$, use Proposition 2.2.5 (and the notation employed there) on $c_1$ and $c_2$ to obtain a disjoint decomposition of $c_1 \vee c_2 = c_{1,\uparrow} \vee c_\oplus \vee c_{2,\uparrow}$, and rewrite $m_1\gamma(c_1) + m_2\gamma(c_2)$ as

$$m_1\gamma(c_{1,\uparrow}) + m_2\gamma(c_{2,\downarrow}) + (m_1 + m_2)\gamma(c_\oplus) + m_1\gamma(c_{1,\downarrow}) + m_2\gamma(c_{2,\uparrow}),$$

and we note that

$$\big|m_1\gamma(c_{1,\uparrow}) + m_2\gamma(c_{2,\downarrow})\big|, \qquad \big|(m_1 + m_2)\gamma(c_\oplus)\big|, \quad \text{and} \quad \big|m_1\gamma(c_{1,\downarrow}) + m_2\gamma(c_{2,\uparrow})\big|$$

are pairwise disjoint. Thus, $g$ may be written as

$$\big[\big(m_1\gamma(c_{1,\uparrow}) + m_2\gamma(c_{2,\downarrow})\big) \vee 0\big] + \big[\big((m_1 + m_2)\gamma(c_\oplus)\big) \vee 0\big] + \big[\big(m_1\gamma(c_{1,\downarrow}) + m_2\gamma(c_{2,\uparrow})\big) \vee 0\big],$$

and in this expression the first and last term in square brackets, respectively, generate the same convex $\ell$-subgroup as $\gamma(c_{1,\uparrow})$ (if $m_1 > 0$) and $\gamma(c_{2,\uparrow})$ (if $m_2 > 0$), or else equal zero. As to the middle term in brackets, it generates the same convex $\ell$-subgroup as $\gamma(c_\oplus)$, if $m_1 + m_2 > 0$, and is zero otherwise. In any of the possible cases, it is now clear that $g$ generates the same convex $\ell$-subgroup as $\gamma(a)$, for a suitable compact element $a$, as promised.

(c) Now we apply induction. Given $g = (\sum_{i=1}^n m_i\gamma(c_i)) \vee 0$, with each $m_i \neq 0$, we rewrite $d = c_2 \vee \cdots \vee c_n$. The inductive hypothesis serves to insure that $h = \sum_{i=2}^n m_i\gamma(c_i)$ may be rewritten as $h = a_1 + a_2 + \cdots + a_t$, such that the $|a_i|$ are pairwise disjoint elements of $G$, each $a_i$ is either strictly positive or strictly negative, and the join of the convex $\ell$-subgroups of $G$ that they generate coincides with $\langle \gamma(d') \rangle_G$, for a suitable component of $d$.

(d) Next, one applies Proposition 2.2.5 – and its proof! – to $c_1$ and $d'$. Leaving the details to the reader, we observe that this produces a revised expression:

$$m_1\gamma(c_1) + h = f_1 + f_2 + f_3,$$

in $G$, where
- the $|f_i|$ are pairwise disjoint;
- for each maximal $x \in \mathrm{coz}(f_1)$, we have $f_1(x) = m_1$, and $\langle f_1 \rangle_G = \langle \gamma(e) \rangle_G$, for some component $e$ of $c_1$;
- $\langle f_3 \rangle_G = \langle \gamma(d'') \rangle_G$, for some component $d''$ of $d'$;
- finally, either $f_2$ is (like $f_1$) the constant $m_1$ on the maximal points of $\mathrm{coz}(f_2)$, or else it can, in turn, be written as a sum $f_2 = b_1 + \cdots + b_r$, with $|b_j|$ pairwise disjoint, $r \leqslant s$, and (by induction once more, since fewer than $n$ compact elements of $L$ are involved) each $b_i$ is either strictly positive or strictly negative, and the join of the convex $\ell$-subgroups of $G$ that they generate coincides with $\langle \gamma(v) \rangle_G$, for a suitable component of $d$.

The upshot of this, as in the initial stage of the induction, is that

$$g = \left( \sum_{i=1}^n m_i\gamma(c_i) \right) \vee 0$$

is a sum of disjoint positive elements $g_i \in G$, each of which generates the same convex $\ell$-subgroup as $\gamma(z_j)$, for some compact $z_j$. Then, obviously,

$$\langle g \rangle_G = \big\langle \gamma(z_1 \vee \cdots \vee z_n) \big\rangle_G,$$

which completes the proof that $\Gamma$ is surjective. With that, the canonical extension of $\Gamma$ to $L$ is an isomorphism onto $\mathcal{C}(G)$.

That $G$ is itself pairwise splitting follows from Theorem 2.3.2; alternately, one can apply [11, Lemma 3], since the functions in $G$ have finite range. □

The proof of Theorem 3.1.1 deserves a closer look, and, in particular, the properties of the map $\Gamma$. The careful reader will observe readily enough that until one gets to the argument showing that $\Gamma$ is surjective, all that has been used is the fact that $L$ is an algebraic frame with the FIP and the disjointification property. That is, one almost has the following corollary.

**Corollary 3.1.2.** *Every algebraic frame with the FIP and disjointification admits a coherent embedding into a pairwise splitting abelian Conrad frame.*

**Proof.** The lattice embedding $\Gamma$ in the proof of Theorem 3.1.1 extends to a one-to-one coherent frame homomorphism – also denoted $\Gamma$ – of the algebraic frame $L$ with the FIP and disjointification into an abelian Conrad frame $\mathcal{C}(G)$, where $G$ is the group constructed in the proof of Theorem 3.1.1. As noted there, $G$ and $\mathcal{C}(G)$ are pairwise splitting. □

**Remark 3.1.3.** Observe that the map $\Gamma$ in the preceding proofs is onto $\mathcal{C}(G)$ precisely when $L$ is a C-frame.

*3.2. Abelian Conrad frames*

The proof of Theorem 3.1.1 shows that the group which supplies the witness showing that a C-frame is a Conrad frame is abelian.

Let us remind the reader that the $\ell$-group $G$ is *finite-valued* if each element of $G$ has at most a finite number of values. It is well known that $G$ is finite-valued precisely when $\mathcal{C}(G)$ is a dual frame. (Refer as well to Theorem 1.3.1.)

It is well known that a finite-valued $\ell$-group is normal-valued.

Theorem 3.1.1 has the expected corollary for frames which are also dual frames. But together with Proposition 1.3.2, one gets a sharper formulation. We use the notation of the proof of Theorem 3.1.1.

**Proposition 3.2.1.** *Suppose that $L$ is an algebraic frame with the FIP and disjointification. If $L$ is also a dual frame, then it is an abelian, finite-valued Conrad frame.*

*Moreover, if $L$ is algebraic with the FIP, and it is a dual frame, then the following are equivalent.*

(a) *$L$ is an abelian Conrad frame.*
(b) *$L$ is a Conrad frame.*
(c) *$L$ has disjointification.*

**Remark 3.2.2.** A fair question at this point is to what extent the group structure, of an $\ell$-group $G$, on the one hand, and the frame structure of $\mathcal{C}(G)$, on the other, play a role in determining one another. For example, Proposition 3.2.1 and Theorem 3.1.1 seem to indicate that the frame structures involved are represented as Conrad frames by abelian groups.

Alternatively, as McCleary shows in [14, Theorem 1], if the $\ell$-group $G$ is (a) a free $\ell$-group on uncountably many generators, or (b) the group of all order-preserving permutations $A(T)$ of a chain $T$ which is *doubly homogeneous* and in which each point has *uncountable point character*, then $\mathcal{C}(G)$ is not a normal-valued Conrad frame. (We rely on the interested reader to investigate McCleary's paper, and, in particular, attach meaning to the italicized words in the preceding sentence.)

Proposition 4.1.2 will illustrate such a dependence of the Conrad frame on group-theoretic features of the underlying group.

## 4. More Conrad conditions

In this part we focus on normal-valued $\ell$-groups. The reader should keep in mind what motivated this work in the first place. Consider the list that follows:

1. C-frames.
2. Pairwise splitting Conrad frames.
3. Normal-valued Conrad frames.
4. Conrad frames.
5. Algebraic frames with the FIP and disjointification.

Theorem 3.1.1 demonstrates that (1) is contained in (2), and the converse is guaranteed by Theorem 2.3.2. That (2) is contained in (3) – and obviously properly – is the content of [11, Theorem 2].

The current paper set about to show that the inclusion of (4) in (5) is strict. We introduce a new Conrad condition which – almost – does that.

*4.1. $\sigma$-Conrad frames*

The notion we are about to introduce seemed contrived, at first, but upon further reflection, it comes through as a rather natural consequence of interpreting the Riesz Interpolation Property in frames.

**Definition & Remarks 4.1.1.** An algebraic frame $L$ with the FIP is a *$\sigma$-Conrad frame* if for each $a, b \in \mathfrak{k}(L)$ there exist sequences of disjoint pairs $a_n, b_n \in \mathfrak{k}(L)$ ($n = 1, 2, \ldots$), such that the following properties hold: for each $n$,

1. $(a_n, b_n)$ is a splitting of $(a, b)$;
2. for any $u, w \in L$ if $a \leqslant u \vee b$, then $a_m \leqslant u$, for some integer $m$, and if $b \leqslant a \vee w$, then $b_p \leqslant w$, for some integer $p$.

In particular, a $\sigma$-Conrad frame has disjointification.

We digress slightly to formulate this notion more formally. Let $\mathcal{S}(a, b)$ denote the set of splittings of $(a, b)$. To say that $L$ has disjointification is to say each $\mathcal{S}(a, b) \neq \emptyset$, for each pair $(a, b)$ of compact elements.

Let us assume $L$ has disjointification, and suppose $B \subseteq \mathcal{S}(a, b)$ such that for any $u \in L$,

$$a \leqslant u \vee b \quad \Rightarrow \quad \exists (p, q) \in B \quad \text{such that } p \leqslant u, \qquad \dagger(a, b)$$

and condition $\dagger(b, a)$ also holds.

Suppose $(x, y)$ is any pair for which $a \leqslant x \vee b$ and $b \leqslant a \vee y$. There is a $(u, v) \in B$, such that $u \leqslant x$. Repeating, with $a$ and $b$ reversing roles, one has a $(p, q) \in B$, with $q \leqslant y$, and observe that $(u, q)$ is a splitting of $(a, b)$. We emphasize that $(u, q) = (u, v) \wedge (p, q)$, in the coordinatewise product on $\mathcal{S}(a, b)$. In view of this, we may extend $B$ to the subsemilattice $B'$ it generates. $B'$ satisfies both $\dagger(a, b)$ and $\dagger(b, a)$ *with the same pair* $(p, q)$, for a given pair of inequalities $a \leqslant x \vee b$ and $b \leqslant a \vee y$, with $|B| = |B'|$.

Thus, when we have a subsemilattice $B \subseteq \mathcal{S}(a, b)$ satisfying both $\dagger(a, b)$ and $\dagger(b, a)$ with the same pair $(p, q)$, we call $B$ a *splitting base*. The definition of a $\sigma$-Conrad frame may now be rephrased, by saying that it is one which has disjointification and for which each pair $(a, b)$ has a countable splitting base.

Here is the reason why we are interested in $\sigma$-Conrad frames.

**Proposition 4.1.2.** *Every normal-valued Conrad frame is a $\sigma$-Conrad frame.*

**Proof.** Suppose $G$ is a normal-valued $\ell$-group, and consider $\mathcal{C}(G)$. Suppose that $\langle a\rangle_G \subseteq K \vee \langle b\rangle_G$, with $K \in \mathcal{C}(G)$; then $a \leqslant x + mb$, where $0 \leqslant x \in K$, and for some natural number $m$. Next, put $a_n = a - (a \wedge nb)$ and $b_n = nb - (a \wedge nb)$; we know that $(\langle a_n\rangle_G, \langle b_n\rangle_G)$ is a splitting of $(\langle a\rangle_G, \langle b\rangle_G)$.

Now observe that $a = a_m + (a \wedge mb)$, and using the remarks of 1.2.4(4) we conclude that $a_m \leqslant x$, whence $a_m \in K$ and $\langle a_m\rangle_G \subseteq K$, which suffices, although it is, perhaps, worth stressing that the countable splitting base we have in mind is the subsemilattice generated by the pairs $\langle a_n\rangle_G, \langle b_n\rangle_G$ above, together with the $\langle a'_n\rangle_G, \langle b'_n\rangle_G$, where $b'_n = b - (b \wedge na)$ and $a'_n = na - (b \wedge na)$. $\square$

### *4.2. An example*

We give an example of a coherent frame which has disjointification, but is not $\sigma$-Conrad. It is typical of the kind of examples that populate [6], which examines the Conrad conditions in frames of filters. We should first highlight some results from [6]. Throughout, let $L$ stand for a coherent frame. $\mathbb{F}(L)$ denotes the *opposite* frame, namely, the frame of filters of the distributive lattice $\mathfrak{k}(L)$, or – equivalently – the frame of ideals of $\mathfrak{k}(L)$ with the dual ordering. A thorough account of frames having disjointification for both $L$ and its opposite is given in [6].

Dependence on the Axiom of Choice is indicated by (AC).

**Theorem 4.2.1.**

(a) (AC) $\mathbb{F}(L)$ *has disjointification if and only if no two incomparable primes of $L$ have a common upper bound.*
(b) $\mathbb{F}(L)$ *is a C-frame if and only if for each pair of compact elements $a$ and $b$,*
   1. *$a \to b$ is compact, and*
   2. *$(a \to b) \vee (b \to a) = 1$.*
(c) $\mathbb{F}(L)$ *is $\sigma$-Conrad if and only if for each pair of compact elements $a$ and $b$, there exist a pair of sequences of compact elements $(a_n)_n$ and $(b_n)_n$ such that*
   1. *for each $n$, $a_n \vee b_n = 1$, and*
   2. *$a \to b = \bigvee_{n<\omega} b_n$ and $b \to a = \bigvee_{n<\omega} a_n$.*

**Example 4.2.2.** A coherent frame with the disjointification property, which is not $\sigma$-Conrad.

Let $D$ be an uncountable set and $S(D)$ stand for the group of all integer-valued functions on $D$ that are constant except on a finite subset of $D$. We shall refer to such functions as being *eventually constant*.

Next $G = S(D) \oplus \mathbb{Z}$ with the following lattice ordering: $(f, m) \geqslant 0$ provided $f(x) \geqslant 0$, for every $x \in D$, and if $f$ is eventually 0, then $m \geqslant 0$. The reader will have no trouble verifying the following features of $G$:

1. The prime spectrum $\mathrm{Spec}(\mathcal{C}(G))$ consists of
   - for each $x \in D$, $M_x$, the set of pairs $(f, m)$ such that $f(x) = 0$; these primes are both maximal and minimal;
   - $M$, consisting of all $(f, m)$, with $f$ eventually zero; $M$ is maximal, but not minimal, as it contains:

- $P = \{(f, m)\colon\ f$ is eventually 0 & $m = 0\}$, which is the only minimal prime which is not maximal.

  Thus, Spec($\mathcal{C}(G)$) is stranded, in the sense that no two incomparable elements have a common upper or lower bound.

2. $P$ is a polar; indeed, $P = (0, 1)^{\perp}$. Moreover, $P$ cannot be countably generated, as a convex $\ell$-subgroup.

As a consequence of Theorem 4.2.1, $\mathbb{F}(\mathcal{C}(G))$ has disjointification, but is not $\sigma$-Conrad.

### 4.3. Work points

This concluding section presents some of the open questions regarding Conrad frames that we find the most intriguing.

**Question 4.3.1.** *Is the converse of Proposition* 4.1.2 *true?*

We doubt it, but lack any counterexamples.

It is surely not unreasonable to consider the following a companion question to 4.3.1.

**Question 4.3.2.** *Is every Conrad frame a $\sigma$-Conrad frame?*

That is, can the qualifying "normal-valued" be removed from Proposition 4.1.2? We do not know the answer.

**Question 4.3.3.** *If $L$ is a normal-valued Conrad frame, is it also abelian?*

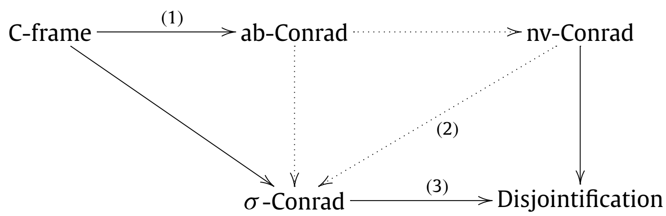
Under the assumption that $L$ is also pairwise splitting, the answer is *yes.*

However, one should also recall Kenoyer's example [8], which shows that there are $\ell$-groups $H$ which are not normal-valued – in fact, without any finite-valued elements whatsoever, yet $\mathcal{C}(H)$ is an abelian Conrad frame, and the witnessing group is even archimedean.

### 4.4. Summarizing

Briefly and schematically, we tell what we know about the relationship between the various Conrad conditions. In the diagram below a solid arrow indicates a strict implication, while a dotted arrow stands for a qualified implication. In such an instance the note by the arrow refers to an explanation below, unless the claim is obvious.

The labels in the diagram designate well defined classes of frames. Just in case the reader is stumped, we observe that the abbreviations 'nv' and 'ab' – hyphenated with "Conrad" – in the diagram denote the classes of normal-valued and abelian Conrad frames, respectively.




1. See Theorem 3.1.1.
2. See Proposition 4.1.2.
3. See Example 4.2.2.

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