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THE YOSIDA SPACE AND REPRESENTATION OF THE PROJECTABLE HULL OF AN ARCHIMEDEAN ℓ-GROUP WITH WEAK UNIT

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Abstract. $\mathbf{W}$ is the category of archimedean ℓ-groups with designated weak order unit. The full subcategory of $\mathbf{W}$ of objects for which the unit is strong unit is denoted by $\mathbf{W}^*$; such ℓ-groups are called bounded. Thus arises a coreflection $\mathbf{W} \xrightarrow{B} \mathbf{W}^*$. For $G \in \mathbf{W}$, YG is the Yosida space, and $G \leq pG$ is the much-studied projectable hull. Recently, in [1], for $G \in \mathbf{W}^*$, YpG is identified as the Stone space of a certain boolean algebra $\mathscr{A}(G)$ of subsets of the minimal prime spectrum $\mathrm{Min}(G)$, and skepticism is expressed about extending this to $\mathbf{W}$. Here, we show that indeed such an extension is possible, using a result from [5] and the following simple facts: in very concrete ways $\mathrm{Min}(G)$ and $\mathrm{Min}(BG)$ are homeomorphic spaces, and $\mathscr{A}(G)$ and $\mathscr{A}(BG)$ are isomorphic boolean algebras; p and B commute.

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1. Introduction. An ℓ-group G is called *projectable* if for each $g \in G$, the polar $g^{\perp} \equiv \{h \in G : |h| \wedge |g| = 0\}$ is a summand. The vector lattice terminology [9] is "principal projection property". Each representable ℓ-group has a projectable hull $G \leq pG$, a unique minimal essential extension to a projectable ℓ-group. See [3], and the many further references for projectable (and other) hulls recorded in [1], [5], [4], and [10].

Henceforth, all ℓ-groups are assume to be archimedean and have a designated weak order unit; $G \in \mathbf{W}$. We record the Yosida representation as $G \leq D(YG)$ and note that $pG \in \mathbf{W}$. Yosida quasi-duality yields an irreducible surjection $YG \xleftarrow{\sigma} YpG$ which realizes $G \leq pG \leq D(YpG)$ as $g \mapsto g \circ \sigma$, see [8].

Knowing YpG one has the description of pG as

Theorem 1.1. ([5], 3.6 and 2.5) *$pG \leq D(YpG)$ is*

$$pG = \{\sum_{\text{finite}} (g_i \circ \sigma)\chi_{U_i} | g_i \in G, \{U_i\} \text{ a clopen partition of } YpG\}.$$

Here the χ_{U_i} are characteristic functions and $(g \circ \sigma)\chi_{U_i}$ is the pointwise multiplication. It is obvious that when G is a vector lattice, so is pG. Everything we say below applies to vector lattices.

The space of minimal prime ideals of G equipped with the hull-kernel topology is denoted $\mathrm{Min}(G)$. Formally, the open base consists of sets of the form

$$U_G(g) = \{P \in \mathrm{Min}(G) : g \notin P\}.$$

These sets are also closed, so $\mathrm{Min}(G)$ is a zero-dimensional space. We let $\mathscr{A}(G)$ denote the sub-boolean algebra of all clopen sets generated by the collection of $U_G(g)$. From basic Stone duality [11] the Stone representation space $\mathcal{S}(\mathscr{A}(G))$ is a compactification of $\mathrm{Min}(G)$, and the Stone map is $\mathscr{A}(G) \ni A \mapsto \bar{A} \in clop(\mathcal{S}(\mathscr{A}(G)))$.

The result from [1] mentioned in the abstract is the novel item that for $H \in \mathbf{W}^*$, $YpH = \mathcal{S}(\mathscr{A}(H))$. This is derived by exhibiting an embedding $H \leq C(\mathcal{S}(\mathscr{A}(H)))$, which produces a map $YH \xleftarrow{\tau} \mathcal{S}(\mathscr{A}(H))$ (Yosida quasi-duality) and subsequently a construct called $\mathcal{P}(H)$ resembling the right side of Theorem 1.1, using τ instead of σ. A proof is then given that $\mathcal{P}(H)$ is indeed pH, and therefore $YpH = \mathcal{S}(\mathscr{A}(H))$. This is for $H \in \mathbf{W}^*$; Theorem 1.1 is not mentioned. (Note then that τ and σ are equivalent.)

We note that the above $\mathcal{P}(H)$ is the same as the pH in Theorem 1.1 when $H \in \mathbf{W}^*$, because such H is "local", whereas not all $G \in \mathbf{W}$ are. For $G \in \mathbf{W}$, the construct $\mathcal{P}(G)$ (*mutatis mutandis*) gives what we have called the "weakly projectable" hull of G, which must be followed by localization to get pG. This is thoroughly discussed in [5] and [6], and we need not go into it here. (We mention that "local" is built into Theorem 1.1.)

2. $YpG = \mathcal{S}(\mathscr{A}(H))$.

We denote elements of G (resp., BG) by g (resp., h), and elements of $\mathrm{Min}(G)$ (resp., $\mathrm{Min}(BG)$) by P (resp., Q). We denote the distinguished element of G by 1, which is a strong unit of BG; n stands for $n \cdot 1$.

Definition 2.1. Given a $G \in \mathbf{W}$ we define the functions contraction

$$t_G : \mathrm{Min}(G) \to \mathrm{Min}(BG) \text{ and extension } e_G : \mathrm{Min}(BG) \to \mathrm{Min}(G)$$

as follows. For $P \in \mathrm{Min}(G)$,

$$t_G(P) = P \cap BG.$$

For $Q \in \mathrm{Min}(BG)$,

$$e_G(Q) = \{g \in G : \exists q \in Q^+, |g| \leq q\}.$$

Note that $e_G(Q)$ is the convex ℓ-subgroup of G generated by Q. The embedding of $BG \leq G$ is an example of a rigid extension. The main thrust of this is that the contraction mapping t_G takes minimal primes of G to minimal primes of BG, and the contraction map is a bijection with extension as its inverse. (See Section 2 of [4] for more information on rigid extensions.)

Our next result is needed for our main theorem. The new piece here is part (d).

THEOREM 2.2. *For the extension $BG \leq G$, the following statements are true.*

(a) For each $P \in \mathrm{Min}(G)$, $et(P) = P$, and for each $Q \in Min(BG)$, $te(Q) = Q$.

(b) For each $g \in G$ and $h \in BG$, $t(U_G(g)) = U_{BG}(|g| \wedge 1)$, and $e(U_{BG}(h)) = U_G(h)$.

(c) The maps t and e are inverse homeomorphisms.

(d) $t(\mathscr{A}(G)) = \mathscr{A}(BG)$ and $\mathscr{A}(G) = e(\mathscr{A}(BG))$.

REMARK 2.3. The contraction map can be extended in several different ways. For example, one can extend to the set of all prime subgroups of G not containing the weak order unit. In this case, it is known that contraction is an order bijection between this latter set and the set of prime subgroups of BG. Consequently, the restriction of this bijection to YG (resp., $\mathrm{Min}(G)$) produces homeomorphisms between YG and YBG (resp., $\mathrm{Min}(G)$ and $\mathrm{Min}(BG)$).

The following is asserted in 2.2(5) of [2], and can also be recognized (with some thought) as a special case of [10]. We provide a short proof using Theorem 1.1.

THEOREM 2.4. *For all $G \in \mathbf{W}$, $BpG = pBG$. That is, $pB = Bp$.*

Proof. *Per* Theorem 1.1, the elements of pG are the $f \in D(YpG)$ of the form

$$f = \sum (g_i \circ \sigma)\chi_{U_i} \tag{$*$}$$

where $\{U_i\}$ is a finite clopen partition of YpG. It is obvious that if each g_i is bounded, then so is f. Thus if $G = BG$, then $pG = BpG$. This yields that for each $G \in \mathbf{W}$, $pBG = BpBG$.

Next, observe that since $BG \leq G$ we also have that $pBG \leq pG$ and thus $pBG = BpBG \leq BpG$. Therefore, $pBG \leq BpG$. Conversely, if $f \in BpG$, then writing

$$f = \sum (g_i \circ \sigma)\chi_{U_i}$$

with f bounded, we can easily modify the g_i to be bounded: if $|f| \leq n \in \mathbb{N}$, then set $g_i' = (-n \vee g) \wedge n$. Then

$$f = \sum (g_i' \circ \sigma)\chi_{U_i}$$

so that $f \in pBG$. □

Now, as mentioned before:

THEOREM 2.5. ([1], Theorem 5.1) *For $H \in \mathbf{W}^*$, $YpH = \mathcal{S}(\mathscr{A}(H))$. Thus, for $G \in \mathbf{W}$, $YpBG = \mathcal{S}(\mathscr{A}(BG))$.*

Combining the previous results leads to the following extension of Theorem 2.5 to $\mathbf{W}$.

THEOREM 2.6. *For $G \in \mathbf{W}$, $YpG = \mathcal{S}(\mathscr{A}(G))$.*

Proof. For any $E \in \mathbf{W}$, $YE = YBE$ [8]. So

$$YpG = YBpG = YpBG = \mathcal{S}(\mathscr{A}(BG)) = \mathcal{S}(\mathscr{A}(G)),$$

the second equality stems from Theorem 2.4, the third equality is Theorem 2.5, and the last equality is Theorem 2.2. □

In [7], we shall give a mostly topological proof of Theorem 2.6, not using Theorem 2.5.

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REFERENCES

1. R.N. BALL, V. MARRA, D. MCNEILL AND A. PEDRINI, From Freudenthal's Spectral Theorem to projectable hulls of unital archimedean lattice-groups, through compactification of minimal spectra, arXiv:1406.3152v2.
2. R.E. CARRERA AND A.W. HAGER, Bounded equivalence of hull classes in archimedean lattice-ordered groups with unit, *Appl. Categor. Struct.* **24** (2016), 163–179.
3. P. CONRAD, The hulls of representable ℓ-groups and f-rings, Collection of articles dedicated to the memory of Hanna Neumann, IV, *J. Austral. Math. Soc.* **16** (1973), 385–415.
4. P. CONRAD AND J. MARTINEZ, Complemented lattice-ordered groups, *Indag. Math. (N.S.)* **1**(3) (1990), 281–297.
5. A.W. HAGER, C.M. KIMBER AND W.WM. MCGOVERN, Weakly least integer closed groups, *Rend. Circ. Mat. Palermo (2)* **52**(3) (2003), 453–480.
6. A.W. HAGER AND J. MARTINEZ, α-projectable and laterally α-complete archimedean lattice-ordered groups, In: *Proc. Conf. on Mem. of T. Retta,* S. Bernau (ed.), Temple U., PA/Addis Ababia, 1995, *Ethiopian J. Sci.* **19** (1996), 73–84.
7. A.W. HAGER AND W.WM. MCGOVERN, The projectable hull of an archimedean ℓ-group with weak unit, submitted.
8. A.W. HAGER AND L. ROBERTSON, Representing and ringifying a Riesz space, *Symposia Mathematica* **21** (1977), 411–431.

9. W.A.J. LUXEMBURG AND A.C. ZAANEN, *Riesz spaces, Vol. I,* North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam/London; American Elsevier Publishing Co., New York, 1971.

10. J. MARTÍNEZ, *Polar functions I, The summand-inducing hull of an Archimedean l-group with unit,* Ordered Algebraic Structures, pp. 275–299, Dev. Math., Vol. 7, Kluwer Acad. Publ., Dordrecht, 2002.

11. R. SIKORSKI, *Boolean algebras,* Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 25 Springer-Verlag New York Inc., New York, 1969.

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