GAUSSIAN PROPERTY OF THE RINGS $R(X)$ AND $R\langle X\rangle$

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Abstract. The content of a polynomial f over a commutative ring R is the ideal $c(f)$ of R generated by the coefficients of f. A commutative ring R is said to be Gaussian if $c(fg) = c(f)c(g)$ for every polynomials f and g in $R[X]$. A number of authors have formulated necessary and sufficient conditions for $R(X)$ (respectively $R\langle X\rangle$) to be semihereditary, have weak global dimension at most one, be arithmetical, or be Prüfer. An open question raised by Glaz is to formulate necessary and sufficient conditions that $R(X)$ (respectively $R\langle X\rangle$) have the Gaussian property. We give a necessary and sufficient condition for the rings $R(X)$ and $R\langle X\rangle$ in terms of the ring R in case the square of the nilradical of R is zero.

1. Introduction and Preliminaries

Let R be a commutative ring with identity and X an indeterminate. In this article we are interested in the transference of Prüfer-like conditions between R and its Nagata ring $R(X)$. For a polynomial f in $R[X]$, we let $c_R(f)$ (or simply $c(f)$) be the ideal of R generated by the coefficients of f. Set $S = \{f \in R[X] : c(f) = R\}$, a multiplicatively closed subset of $R[X]$ consists of the regular elements. The Nagata ring over R is the ring $R(X) = R[X]_S$. Another interesting localization of $R[X]$ is given by the multiplicatively closed subset $W = \{f \in R[X] : f \text{ is monic}\}$. We denote $R\langle X\rangle = R[X]_W$. Denote the classical (i.e. total) ring of quotients of a ring R by $q(R)$, we obtain that $R[X] \leq R\langle X\rangle \leq R(X) \leq q(R[X])$. Letting $\mathrm{Max}(R)$ denote the set of maximal ideals of R, we recall that $S = R[X] \smallsetminus \cup_{M\in \mathrm{Max}(R)} M[X]$. Thus, S is a saturated multiplicatively closed subset of $R[X]$. Therefore, the units of $R(X)$ are precisely the fractions f/g with $c(f) = c(g) = R$. The set W is not saturated, so the units of $R\langle X\rangle$ are a bit more complicated to describe (see [14], Theorem 17.10).

A natural question one might ask is what conditions on R ascend to $R\langle X\rangle$ and $R(X)$, and conversely what conditions on $R\langle X\rangle$ and $R(X)$ descend to R. A number of authors in the 1970's and 1980's have given both affirmative and negative answers to many nice properties of domains, such as PID, UFD, Dedekind, Prüfer, etc. We are interested in the following Prüfer-like conditions:

Definition 1.1. Let R be a commutative ring with identity

(1) R is called *semihereditary* if every finitely generated ideal of R is projective.
(2) R is said to have *weak dimension* $\leqslant 1$ if every finitely generated ideal of R is flat.
(3) R is called an *arithmetical* ring if its lattice of ideals is distributive.
(4) R is called a *Gaussian ring* if for every $f, g \in R[X], c(fg) = c(f)c(g)$.

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(5) R is called a *locally Prüfer ring* if R_P is Prüfer for every prime ideal P of R.
(6) R is called a *maximally Prüfer ring* if R_M is Prüfer for every maximal ideal of R.
(7) R is called a *Prüfer* ring if every finitely generated regular ideal is invertible.

It is known that each condition implies the next. In the case of integral domains all the conditions are equivalent. For reduced rings conditions (2), (3) and (4) are equivalent. Furthermore, there are examples showing that the other implications cannot be reversed. For more information on this the reader is advised to consult [2, 4, 15].

Glaz [10] and Le Riche [18] proved the following theorem for semihereditary rings.

Theorem 1.2 ([10], Corollary 3 and [18], Theorem 3.7). *Let R be a commutative ring with identity. Then:*

(1) *$R(X)$ is semihereditary if and only if R is semihereditary.*
(2) *$R\langle X\rangle$ is semihereditary if and only if R is semihereditary and has Krull dimension at most one.*

Le Riche [18] and Anderson, Anderson and Markanda [1] proved the following theorem for arithmetical rings.

Theorem 1.3 ([1], Theorem 3.1). *Let R be a commutative ring with identity.*

(1) *$R(X)$ is an arithmetical ring if and only if R is an arithmetical ring.*
(2) *$R\langle X\rangle$ is an arithmetical ring if and only if R is an arithmetical ring, $\dim R \leq 1$, and R_P is a field for every non-maximal prime ideal P.*

An interesting property of the ring $R(X)$ is that any finitely generated locally principal ideal is principal ([14], Theorem 15.4). Since arithmetical rings can be characterized as the locally chained rings ([16], Theorem 1), we gather that $R(X)$ is arithmetical if and only if it is a Bézout ring. Recall that a Bézout ring is one in which every finitely generated ideal is principal. A Bézout ring is arithmetical. In [1] the authors introduced a new class of Prüfer rings: the strong Prüfer rings. A ring is called a *strong Prüfer ring* if every finitely generated ideal I with $\operatorname{Ann}_R I = 0$ is locally principal. With the notion of strong Prüfer rings the authors also established a theorem for Prüfer rings analogous to Theorem 1.2 and Theorem 1.3. Note that an arithmetical ring is strong Prüfer ring, while a strong Prüfer ring is a Prüfer ring.

Theorem 1.4 ([1], Theorem 3.2). *Let R be a commutative ring. Then:*

(1) *$R(X)$ is a strong Prüfer ring if and only if R is a strong Prüfer ring.*
(2) *$R\langle X\rangle$ is a strong Prüfer ring if and only if R is a strong Prüfer ring, $\dim R \leq 1$, and R_P is a field for every non-maximal prime ideal P.*

We note that any finitely generated ideal I of $R(X)$ (or $R\langle X\rangle$) with $\operatorname{Ann} I = 0$ is regular. Therefore, $R(X)$ (or $R\langle X\rangle$) is a strong Prüfer ring if and only if it is a Prüfer ring.

In this paper we derive analogous results for the class of Gaussian rings in which the square of the nilradical is zero and also for maximally Prüfer rings. In Section 3 we recall the definition of a p-extension and characterize when each of the extensions $R \leq R(X)$

and $R \leq R\langle X\rangle$ is a p-extension. Finally, in section 4 we apply the notion of p-extension to the Prüfer-like conditions.

Throughout the paper all rings are commutative with identity. For a ring R, we let $Z(R)$ denote the set of zerodivisors of R, and denote the nilradical of R by $\mathfrak{N}(R)$. When useful and unambiguous we shall denote the ideal generated by a finite set using parentheses, e.g. (a, b).

For an extensive treatment of $R(X)$ and $R\langle X\rangle$, the book by Huckaba [14] and the articles [1] and [18] are very informative. We end this section by stating some results that we have found useful.

Theorem 1.5 ([14], Theorem 14.1)**.** *Let R be a commutative ring with identity. The following statements hold.*

1. *There is a one-to-one correspondence between the maximal (resp. minimal) prime ideals of R and the maximal (resp. minimal) prime ideal ideal $R(X)$ given by $P \leftrightarrow PR(X) = P(X)$.*
2. *For an ideal I of R, $I(X) \cap R = I$, and $R(X)/I(X) = R/I(X)$.*
3. *For each prime ideal P of R, $R_P(X) = R[X]_{P[X]} = R(X)_{P(X)}$.*

Theorem 1.6 ([14], Theorem 17.11)**.** *The rings $R(X)$ and $R\langle X\rangle$ coincide if and only if $dim\, R = 0$.*

Theorem 1.7 ([14], Theorem 15.1)**.** *Let R be a ring and $f \in R[X]$. The following conditions are equivalent:*

1. *$c(f)$ is locally principal.*
2. *$fR(X) = c(f)R(X)$.*
3. *$fR(X) = IR(X)$, for some ideal I of R.*
4. *$c(f)R(X)$ is principal.*
5. *$c(f)R(X)$ is locally principal.*

2. Gaussian Property of the rings $R(X)$ and $R\langle X\rangle$

In this section we partially answer a question of Sarah Glaz ([8], Open question 10). For the class of rings in which the square of the nilradical is zero, we prove a theorem similar to Theorem 1.4 for Gaussian rings. We consider such rings for obvious reason, as reduced Gaussian rings are arithmetical ([9], Theorem 2.2), and for arithmetical rings the relation between the ring R and the rings $R(X)$ and $R\langle X\rangle$ is completely understood. We begin with the following known results.

Theorem 2.1 ([19], Theorem 3.5)**.** *Let R be a local ring. Then R is Gaussian if and only if (i) for all a, $b \in R$, $(a, b)^2$ is principal and generated by either a^2 or b^2 and (ii) for all $a, b \in R$ with $(a, b)^2 = (a^2)$, if $ab = 0$, then $b^2 = 0$.*

Theorem 2.2 ([19], Theorem 3.3)**.** *Let R be a commutative ring such that $\mathfrak{N}(R)^2 = (0)$. Then R is Gaussian if and only if $R/\mathfrak{N}(R)$ is arithmetical and for each finitely generated ideal I not contained in $\mathfrak{N}(R)$ and each nilpotent $b \in \mathfrak{N}(R)$, $bI \subseteq I^2$.*

We are now ready to present our first result.

Theorem 2.3. *Let R be a ring such that $\mathfrak{N}(R)^2 = (0)$. Then R is Gaussian if and only if $R(X)$ is Gaussian.*

Proof. First note that being Gaussian is a local property, i.e., R is Gaussian if and only if R_M is Gaussian for each maximal ideal M of R ([23], Lemma 5). Also, since maximal ideals of R and $R(X)$ are in one-to-one correspondence and $R(X)_{M(X)} = R_M(X)$ by Theorem 1.5, it suffices to consider the case when R is local.

($\Rightarrow$) Assume R is a local Gaussian ring such that $\mathfrak{N}(R)^2 = 0$. Since the prime ideals of a Gaussian ring are linearly ordered by inclusion ([24], Theorem 6.1), $\mathfrak{N}(R)$ is the unique minimal prime ideal of R. Also, the minimal prime ideals of R and $R(X)$ are in one-to-one correspondence ($P \leftrightarrow P(X)$), so the nilradical of $R(X)$ is $\mathfrak{N}(R)(X)$. Moreover, $\mathfrak{N}(R)(X)^2 = 0$.

Set $\bar{R} = R/\mathfrak{N}(R)$ and for $x \in R$ set $\bar{x} = x + \mathfrak{N}(R)$. Since the homomorphic image of a Gaussian ring is Gaussian and a reduced Gaussian ring is arithmetical, $\bar{R}$ is arithmetical, and so is $\bar{R}(X) = R(X)/\mathfrak{N}(R)(X)$ by Theorem 1.3. We prove that the necessary condition of Theorem 2.2 is satisfied by the ring $R(X)$. Note that it suffices to prove the condition for the principal ideals. Let $f = a_0 + a_1X + ... + a_nX^n \in R[X] \setminus \mathfrak{N}(R)[X]$ and $p = p_0 + p_1X + ... + p_mX^m \in \mathfrak{N}(R)[X]$. We claim the following

$$p \cdot fR(X) \subseteq f^2R(X).$$

That $\bar{R}$ is arithmetical implies $(c(f) + \mathfrak{N}(R))/\mathfrak{N}(R)$ is a principal ideal of $\overline{R}$. We may assume $(c(f) + \mathfrak{N}(R))/\mathfrak{N}(R) = a_kR/\mathfrak{N}(R)$. Then for each i there are $r_i \in R$ and $b_i \in \mathfrak{N}(R)$ such that $a_i = a_kr_i + b_i$. Of course, we can take $r_k = 1$ and $b_k = 0$, and we do so. Therefore, $f = a_kf_0(X) + b(X)$, where $f_0(X) = r_0 + r_1X + ... + 1X^k + ... + r_nX^n$ and $b(X) = b_0 + b_1X + ... + 0X^k + ... + b_nX^n \in \mathfrak{N}(R)[X]$. Thus, $p \cdot f = a_kf_0 \cdot p$, because $b \cdot p \in \mathfrak{N}(R)^2[X] = 0$. Since R is Gaussian, by Theorem 2.2, $p_s \cdot a_k = a_k^2 \cdot t_s$ for some $t_s \in R$. It then follows that, $t_s \in \mathfrak{N}(R)$. Therefore, $a_k \cdot p(X) = a_k^2t(X)$, where $t(X) = t_0+t_1X+...+t_mX^m$, and hence $f \cdot p = f_0 \cdot a_k^2t(X) = (a_k^2f_0^2+2a_kf_0(X)b(X)) \cdot \frac{t(X)}{f_0(X)}$, again because $b(X) \cdot t(X) \in \mathfrak{N}(R)^2[X] = 0$. Thus,

$$p \cdot fR(X) \subseteq f^2R(X)$$

as desired.

($\Leftarrow$) Assume $R(X)$ is a Gaussian ring. Let $a, b \in R$. Then by Theorem 2.1 we may assume that $(a, b)^2R(X) = a^2R(X)$. Therefore, $b^2h_0 = a^2h_1$ and $abg_0 = a^2g_1$ for some $h_0, h_1, g_0, g_1 \in R[X]$ with $c(h_0) = c(g_0) = R$. It follows that $(a, b)^2R = a^2R$. Also, if $ab = 0$, then $b^2 = 0$ in $R(X)$, and hence $b^2 = 0$ in R as well. □

Remark 2.4. Notice that the proof of the sufficiency does not use the extra condition that the square of the nilradical of R is zero.

Theorem 2.5. *Let R be a commutative ring such that $\mathfrak{N}(R)^2 = (0)$. Then $R\langle X\rangle$ is Gaussian if and only if R is Gaussian, dim $R \leq 1$, and R_P is a field for every non-maximal prime ideal P.*

Proof. ($\Rightarrow$) Assume $R\langle X\rangle$ is a Gaussian ring. $R(X)$, being an overring of a Gaussian ring is also a Gaussian ring ([2], Theorem 3.3), and by Theorem 2.3, R is also Gaussian. Since Gaussian rings are Prüfer rings, other conclusions immediately follow from Theorem 1.4.

($\Leftarrow$) Assume R satisfies the stated conditions. As noted in Theorem 2.3, it suffices to prove that $R\langle X\rangle$ is locally Gaussian at each maximal ideal. Let $\mathfrak{m}$ be a maximal ideal of $R\langle X\rangle$. Let $M = \mathfrak{m} \cap R[X]$, and $P = M \cap R$. Then by ([14], Theorem 18.2)

$$R\langle X\rangle_{\mathfrak{m}} = R[X]_M = R_P[X]_{MR_P[X]}.$$

Now we have two cases to consider. If P is not a maximal ideal of R, then $R_P[X]_{MR_P[X]}$ is a DVR or a field, and hence it is a Gaussian ring.

If P is maximal, then either $M = P[X]$ or $M = (P[X], f)$ for some monic polynomial $f \in R[X]$. Since M extends to the maximal ideal $\mathfrak{m}$ of $R\langle X\rangle$, we cannot have the latter case. So,

$$R\langle X\rangle_{\mathfrak{m}} = R[X]_M = R[X]_{P[X]} = R_P(X).$$

Finally, since nilradicals localize nicely, the result follows from Theorem 2.3. □

Remark 2.6. We do not know whether the condition that $\mathfrak{N}(R)^2 = 0$ can be dropped. Neither do we know of an example of a Gaussian ring R such that $\mathfrak{N}(R)^2 \neq 0$ and $R(X)$ is not Gaussian. However, there are Gaussian rings with the square of the nilradical nonzero, for example $k[X]/(X^3)$, but this ring is an arithmetical ring therefore $R(X)$ is arithmetical, and hence also a Gaussian ring. We have been able to construct an example of a nonarithmetical Gaussian ring whose nilradical squared is nonzero, for example the ring $R = k[Y]/(Y^3)\,(+)\,(k \oplus k)$ has the property, but again in this case $R(X)$ is Gaussian.

We now give a similar result for maximally Prüfer rings. A commutative ring R is said to be *maximally strong Prüfer* (*locally strong Prüfer*) if R_M is a strong Prüfer ring for every maximal (prime) ideal M of R [15].

Theorem 2.7. *Let R be a commutative ring with identity. Then:*

(1) *$R(X)$ is maximally Prüfer if and only if R is maximally strong Prüfer.*
(2) *$R\langle X\rangle$ is maximally Prüfer if and only if R is maximally strong Prüfer, dim $R \leq 1$, and R_P is a field for every non-maximal prime ideal P of R.*

Proof. (1) Since maximal ideals of the rings R and $R(X)$ are in one-to-one correspondence, $R(X)$ is maximally Prüfer if and only if $R(X)_{M(X)} = R_M(X)$ is Prüfer for every maximal ideal M of R. By Theorem 1.4, $R_M(X)$ is Prüfer if and only if R_M is strong Prüfer. Now the result follows.

(2)($\Rightarrow$) $R(X)$, being an overring of a maximally Prüfer ring, is also a maximally Prüfer ring ([15], Corollary 10). Therefore, by (1) R is maximally strong Prüfer. Since maximally Prüfer rings are Prüfer, other results follow from Theorem 1.4.

($\Leftarrow$) Let $\mathfrak{m}$ be a maximal ideal of $R\langle X\rangle$. Then $\mathfrak{m} = MR\langle X\rangle$ for some prime ideal M of $R[X]$ that is disjoint from the set of monic polynomials of $R[X]$. Let $P = M \cap R$. Again we have,

$$R\langle X\rangle_{\mathfrak{m}} = R[X]_M = R_P[X]_{MR_P[X]}.$$

If P is a maximal ideal of R then either $M = P[X]$ or $M = (P[X], f)$ for some monic polynomials f of R. But latter case cannot occur. Thus, $R\langle X\rangle_{\mathfrak{m}} = R[X]_M = R[X]_{P[X]} = R_P(X)$ is Prüfer. If P is a not a maximal ideal of R, then $R\langle X\rangle_{\mathfrak{m}} = R_P[X]_{MR_P[X]}$ is the localization of a PID, and hence Prüfer. □

Unfortunately, we have been unable to characterize when $R(X)$ is locally Prüfer. On the positive side, we note that a similar result of Theorem 2.7 (2) also holds for the locally Prüfer rings, and the proof is also the same.

Theorem 2.8. *Let R be a commutative ring with identity. Then: $R\langle X\rangle$ is locally Prüfer if and only if R is locally strong Prüfer, dim $R \leq 1$, and R_P is a field for every non-maximal prime ideal P.*

3. When $R(X)$ and $R\langle X\rangle$ are p-extensions of R

In this section we recall the notion of a p-extension of rings, and determine when the extensions $R \subset R(X)$ and $R \subset R\langle X\rangle$ are such.

Definition 3.1. Let R and S be commutative rings with identity; denote the identities of R and S by 1_R and 1_S respectively. Formally, by an *extension of rings* we mean that there is injective morphism of rings, say $\phi : R \mapsto S$ for which $\phi(1_R) = 1_S$. In this manner we assume that R is a subring S, and we say that S is a p-extension of R if, for each $s \in S$, there exists $r \in R$ such that $sS = rS$ [3]. For example, localizing a ring at a regular multiplicatively closed set is a p-extension. Whereas $R \subseteq R[X]$ is never a p-extension.

Theorem 3.2. *Let R be a commutative ring. $R \hookrightarrow R(X)$ is a p-extension if and only if R is a Bézout ring.*

Proof. Assume $R \hookrightarrow R(X)$ is a p-extension. For $a, b \in R$, we have $(aX + b)R(X) = rR(X)$ for some $r \in R$. Write $r = (aX + b) \cdot \frac{f}{g}$, which implies $rg = (aX + b)f$. Since g has unit content, $rR \subseteq c((aX + b)f) \subseteq (aR + bR)c(f) \subseteq aR + bR$. On the other hand, $(aX+b) = r \cdot \frac{h}{k}$ implies $(aX+b)k = rh$. Now content formula $(c(f)^n c(fg) = c(f)^{n+1}c(g)$, where $n = \deg\ (g))$ yields that $c((aX + b)k) = aR + bR$. Therefore, $aR + bR \subseteq rR$, and so $aR + bR = rR$. Hence R is a Bézout ring.

Conversely, assume that R is a Bézout ring. Let $f \in R[X]$. By Theorem 1.7, $fR(X) = c_R(f)R(X)$ if and only if $c_R(f)$ is locally principal. Since R is Bézout $c(f)$ is principal, and hence locally principal. Therefore $fR(X) = rR(X)$ for some generator r of $c_R(f)$. □

The argument of Theorem 3.2 also proves that a necessary condition for the extension $R \hookrightarrow R\langle X\rangle$ to be a p-extension is that R is Bézout. But the condition is far from being sufficient. In fact, if $R\langle X\rangle \subsetneq R(X)$, which is the case when dim $R \neq 0$ ([14], Theorem 17.11), then $R \hookrightarrow R\langle X\rangle$ is never a p-extension, which we now prove. First, we need a few lemmas.

Lemma 3.3. *Let R be a ring. If $R \hookrightarrow R\langle X\rangle$ is a p-extension, then R is a total quotient ring.*

Proof. Let a be a regular element of R. Assume that

$$(aX+1)R\langle X\rangle = rR\langle X\rangle$$

for some r in R. Therefore

$$(aX+1) = r\cdot\frac{f}{g} \text{ and } r = (aX+1)\cdot\frac{h}{k}$$

which implies

$$(aX+1)\cdot g = r\cdot f \text{ and } r\cdot k = (aX+1)\cdot h$$

where f and h are polynomials over R, and g and k are monic polynomials over R. Writing $f = a_nX^n + ... + a_1X + a_0$ and $h = b_mX^m + ... + b_1X + b_0$ and comparing the coefficients of leading terms we get $a = ra_n$ and $r = ab_m$. It follows that r is a regular element of R, and a_n and b_m are units of R. Regularity of r implies that degree of g is $n-1$. Again, writing $g = X^{n-1}+c_{n-2}X^{n-2}+...+c_1X+c_0$ and comparing the coefficients of X^{n-1} gives $1 + ac_{n-2} = ab_ma_{n-1}$, which implies that a is a unit of R. Thus, every regular element of R is a unit. Therefore, R is a total quotient ring. $\square$

The following two lemmas prove that a p-extension is stable under localization and reducing modulo an ideal for the inclusion $R \hookrightarrow R\langle X\rangle$.

Lemma 3.4. *If $R \hookrightarrow R\langle X\rangle$ is a p-extension, then $R_P \hookrightarrow R_P\langle X\rangle$ is also a p-extension, for any prime ideal P of R.*

Proof. Let P be a prime ideal of R, and let f be a polynomial in $R_P[X]$. We can write $f = \frac{g}{t}$ where $g \in R[X]$ and $t \in R - P$. Since $R \hookrightarrow R\langle X\rangle$ is a p-extension, we have $gR\langle X\rangle = rR\langle X\rangle$ for some $r \in R$. Therefore, $g\cdot h = r.k$ for some polynomials h and k in $R[X]$, with h monic. Dividing by t gives $f\cdot h = \frac{r}{t}\cdot k$, an equation in $R_P[X]$. It follows that $fR_P\langle X\rangle \subseteq rR_P\langle X\rangle$.

On the other hand, writing $r\cdot k' = h'\cdot g$ with k' monic polynomial in $R[X]$ and dividing by t gives $\frac{r}{t}\cdot k' = h'\cdot\frac{g}{t}$. Therefore, $\frac{r}{t}R_P\langle X\rangle \subseteq fR_P\langle X\rangle$. Thus, $fR_P\langle X\rangle = rR_P\langle X\rangle$ $\square$

Lemma 3.5. *If $R \hookrightarrow R\langle X\rangle$ is a p-extension, then $R/I \hookrightarrow R/I\langle X\rangle$ is also a p-extension for any ideal I of R.*

Proof. Let $\bar{f} \in R/I[X]$. Since $R \hookrightarrow R\langle X\rangle$ is a p-extension, we have $fR\langle X\rangle = rR\langle X\rangle$, for some $r \in R$. It follows that $f\cdot g = r\cdot h$ for some polynomials h and g in $R[X]$ with g monic. Reducing the polynomials modulo I, we get $\bar{f}\cdot\bar{g} = \bar{r}\cdot\bar{h}$. Since monic polynomials stay monic under reducing modulo an ideal we have $\bar{f}R/I\langle X\rangle \subseteq \bar{r}R/I\langle X\rangle$.

A similar argument shows that, $\bar{r}R/I\langle X\rangle \subseteq \bar{f}R/I\langle X\rangle$. Thus, $\bar{r}R/I\langle X\rangle = \bar{f}R/I\langle X\rangle$ $\square$

Theorem 3.6. *$R \hookrightarrow R\langle X\rangle$ is a p-extension if and only if R is a zero-dimensional Bézout ring.*

Proof. ($\Rightarrow$) The discussion after Theorem 3.2 shows that R is a Bézout ring. Let P be a prime ideal of R. By Lemma 3.5 and Lemma 3.4, we have $R_P/\mathfrak{N}(R_P) \hookrightarrow R_P/\mathfrak{N}(R_P)\langle X\rangle$ is a p-extension. Now by Lemma 3.3 $R_P/\mathfrak{N}(R_P)$ is a total quotient ring. On the other hand since R is Bézout, the ring $R_P/\mathfrak{N}(R_P)$, being a reduced chained ring, is an integral domain. Therefore, it is a field. It follows that dim $R_P = 0$. Since P is an arbitrary prime ideal of R, dim $R = 0$.

($\Leftarrow$) Since R is zero-dimensional, $R\langle X\rangle = R(X)$ by ([14], Theorem 17.11). Now the result follows from Theorem 3.2. □

4. p-extensions and Prüfer-like conditions

In this section we prove that, of the Prüfer-like conditions discussed in the introduction, all except maximally Prüfer ring ascend through p-extension, i.e. if $R \subseteq S$ is a p-extension and R satisfies the condition n for $n = 1, 2, 3, 4, 5, 7$ of the introduction, then S satisfies condition n as well. First we need a couple of lemmas.

Lemma 4.1. *Suppose $R \hookrightarrow S$ is a p-extension and Q is a prime ideal of S. If $P = Q \cap R$, then $R_P \hookrightarrow S_Q$ is also a p-extension.*

Proof. We first prove that the natural map $R_P \longrightarrow S_Q$ is one-to-one. Let $\frac{a}{t} \in R_P$, and assume $\frac{a}{t} = 0$ in S_Q. Then $au = 0$ for some $u \in S - Q$. But $Su = Sr$ for some $r \in R$, so $ra = 0$. Also, since $u \notin Q$, we have $r \notin P$. Therefore, $\frac{a}{t} = 0$ in R_P. Now consider a principal ideal $\frac{a}{w}S_Q$ where $a \in S$, $w \in S - Q$. Again $aS = bS$ for some $b \in R$. Therefore, $\frac{a}{w}S_Q = \frac{b}{1}S_Q$ and hence $R_P \hookrightarrow S_Q$ is a p-extension. □

Lemma 4.2. *Suppose $R \hookrightarrow S$ is a p-extension. Then following hold:*

(1) *If R is a field then so is S.*
(2) *If R is a von Neumann regular ring then so is S.*
(3) *If R is a chained ring then so is S.*
(4) *If R is reduced then so is S.*

Proof. (1) Let s be a nonzero element of S. Then $sS = rS$ for some nonzero element r in R. Therefore, since r is invertible in R we have $r^{-1}sS = S$. Thus $r^{-1}s$, and hence also s, is a unit of S.

(2) By ([5], Theorem 1), it suffices to prove that S_M is a field for all maximal ideals M of S. Let M be a maximal ideal of S and let $P = M \cap R$. By Lemma 4.1, $R_P \hookrightarrow S_M$ is also a p-extension. Let Q be a maximal ideal of R that contains P, then we have the isomorphism $R_P \cong (R_Q)_{PR_Q}$. Since R_Q is a field by ([5], Theorem 1) we have $PR_Q \subseteq QR_Q = 0$. Therefore $R_P = R_Q$. Thus S_M is a field by (1).

(3) To prove that S is a chained ring, it suffices to prove that any two principal ideals are comparable. Let cS and dS be two principal ideals of S. Then there exist p and q in R such that $cS = pS$ and $dS = qS$. Without loss of generality, assume $pR \subseteq qR$. It follows that, $cS \subseteq dS$. Thus S is a chained ring.

(4) Let s be a nilpotent element of S. We have $sS = rS$ for some r in R. This implies that $r = st$ for some t in S. Therefore, r is a nilpotent element of R, so $r = 0$, and hence $s = 0$. □

Theorem 4.3. *Suppose $R \hookrightarrow S$ is a p-extension. If R satisfies the condition (n) for $n = 1, 2, 3, 4, 5, 7$ of the introduction, then S satisfies the same condition (n).*

Proof. (7) It suffices to prove that every two-generated regular ideal of S is invertible ([17], Theorem 10.18). Let $I = Ss_1 + Ss_2$ be a regular ideal of S. Let $a \in I$ be a regular element of I. There exists r_1 and r_2 in R such that $Ss_1 = Sr_1$ and $Ss_2 = Sr_1$, and $Sa = Sr$. Then r is also a regular element of R.

Let $I_0 = Rr_1 + Rr_2 + Rr$. Clearly, I_0 is a regular ideal of R and $I_0S = Sr_1 + Sr_2 + Sr = I$. Since R is a Prüfer ring, there exists an ideal J_0 of R such that $I_0J_0 = Rc$ for some regular element $c \in R$. Therefore, $(I_0S)(J_0S) = Sc$. i.e., I is invertible in S. Thus S is also Prüfer.

(5) Follows from (7) and Lemma 4.1.

(4) Since a commutative ring R is Gaussian if and only if R_P is Gaussian for every prime ideal of R. We may assume that R and S are local rings. By Tsang's characterization of local Gaussian rings, we have to prove that given two elements a and b in S there exists $d \in \mathrm{Ann}_S(Sa + Sb)$ such that $Sa + Sb = Sa + Sd$ or $Sa + Sb = Sb + Sd$. There are r_1 and r_2 in R such that $Sa = Sr_1$ and $Sb = Sr_2$. Since R is a local Gaussian ring we can assume without loss of generality that $Rr_1 + Rr_2 = Rr_1 + Rd$ where $d \in \mathrm{Ann}_R(Rr_1 + Rr_2)$. Now it is easy to verify that $Sa + Sb = Sa + Sd$ and $d \in \mathrm{Ann}_S(Sa + Sb)$.

(3) It suffices to prove that the lattice of ideals of S_Q at any prime ideal Q of S is linearly ordered. Let $P = Q \cap R$. Then P is a prime ideal of R. Since R is arithmetical, the lattice of ideals of R_P are linearly ordered ([16], Theorem 1), and so is true for S_Q by Lemma 4.1 and Lemma 4.2 (3).

(2) By ([9], Theorem 2.2) it suffices to prove that S is a reduced Gaussian ring, and this immediately follows from (3) Lemma and 4.2 (4).

(1) By ([9], Theorem 2.3) it suffices to prove that S is Gaussian and $Q(S)$ is a von Neumann regular ring. Since R is semihereditary, it is Gaussian and $Q(R)$ is von Neumann regular. It is easy to check that $Q(R) \hookrightarrow Q(S)$ is a p-extension. Therefore, S is Gaussian and $Q(S)$ is a von Neumann regular ring. □

Remark 4.4. It is an open problem whether a p-extension of a maximally Prüfer ring is also a maximally Prüfer ring.

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