FUSIBLE RINGS

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ABSTRACT. A ring R is called *left fusible* if every nonzero element is the sum of a left zero-divisor and a non-left zero-divisor. It is shown that if R is a left fusible ring and σ is a ring automorphism of R, then $R[x;\sigma]$ and $R[[x;\sigma]]$ are left fusible. It is proved that if R is a left fusible ring, then $M_n(R)$ is a left fusible ring. Examples of fusible rings are complemented rings, special almost clean rings, and commutative Jacobson semi-simple clean rings. A ring R is called *left unit fusible* if every nonzero element of R can be written as the sum of a unit and a left zero-divisor in R. Full Rings of continuous functions are fusible. It is also shown that if $1 = e_1 + e_2 + ... + e_n$ in a ring R where the e_i are orthogonal idempotents and each e_iRe_i is left unit fusible, then R is left unit fusible. Finally, we give some properties of fusible rings.
Keywords: Fusible rings, unit fusible rings, zero-divisors, clean rings, $C(X)$.
MSC(2010): Primary:16U99; Secondary:16W99, 13A99.

1. INTRODUCTION

It is well-known that the sum of two zero-divisors need not be a zero-divisor. In [18, Theorem 1.12] the authors characterized commutative rings for which the set of zero-divisors is an ideal. The set of zero-divisors (i.e., $Z(R)$) is an ideal in a commutative ring R if and only if the classical ring of quotients of R (i.e., $q_{cl}(R)$) is a local ring if and only if $Z(R)$ is a prime ideal of R and $q_{cl}(R)$ is the localization at $Z(R)$. We remind the reader that a finite commutative ring R is local if and only if $Z(R)$ is an ideal. It is clear that if the set of left zero-divisors in a ring R is not a left ideal, then there exists a left zero-divisor which can be expressed as the sum of a left zero-divisor and a non-left zero-divisor in R. This leads to our investigation of the class of rings in which every element can be written as the sum of a left zero-divisor and a non-left zero-divisor.

Throughout this paper, R will be an associative ring with identity, $U(R)$ its group of units, $\mathfrak{J}(R)$ its Jacobson radical, $Id(R)$ its set of idempotents, and $r(R)$ the set of regular elements of R. For $x \in R$, $Ann_l(x) = \{a \in R : ax = 0\}$ and $Ann_r(x)$ denote the left annihilator and the right annihilator ideals of x in R, respectively. When $Ann_r(x) \neq \{0\}$ we say x is a left zero-divisor; otherwise it is a non-left zero-divisor. Let $Z_l(R)$ (respectively, $Z_l^*(R)$) denote the set of left zero-divisors (respectively, non-left zero-divisors) of R. Similarly, let $Z_r(R)$ (respectively, $Z_r^*(R)$) denote the set of right zero-divisors (respectively, non-right zero-divisors) of R. Note that $Z_l^*(R) \cap Z_r^*(R) = r(R)$. Clearly, for a commutative ring R, $Z_l(R) = Z_r(R) = Z(R)$ and $Z_l^*(R) = Z_r^*(R) = r(R)$. We denote the ring of $n \times n$ matrices over R by $M_n(R)$. For a ring R, $q_{cl}^l(R)$ (respectively, $q_{cl}^r(R)$) denote the classical (or total) left (respectively, right) ring of quotients of R, when it exists. If R is a commutative ring, then $q_{cl}^l(R) = q_{cl}^r(R) = q_{cl}(R)$.

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Our main goal is to study the class of fusible rings. In the next section, we investigate some fundamental properties of this class. In the third section, we consider the commutative case. In the last section, we strengthen the definition of a fusible element and report our results on this subclass of rings.

We end this section by recalling some ring theoretic concepts that will play a pivotal role throughout the article.

Notation 1.1. Let R be a ring with identity.

(1) The Jacobson radical of R is denoted by $\mathfrak{J}(R)$. When $\mathfrak{J}(R) = 0$, R is said to be *Jacobson semi-simple* or *semi-primitive*.

(2) R is said to be reduced if it has no nonzero nilpotent elements.

(3) An element $a \in R$ is a left zero-divisor if there is $0 \neq r \in R$ with $ar = 0$. An element which is not a left zero-divisor is called a non-left zero-divisor. (Right zero-divisors and non-right zero-divisors are defined analogously.) Notice that $0 \in Z_l(R) \cap Z_r(R)$.

(4) An element $a \in R$ is regular if it is neither a left zero-divisor nor a right zero-divisor. The set of regular elements of R is denoted by $r(R)$.

2. FUSIBLE RINGS

We begin with a formal definition of the central concept of the article.

Definition 2.1. We call a nonzero element $a \in R$ *left fusible* if it can be expressed as the sum of a left zero-divisor and a non-left zero-divisor in R. We call a ring R *left fusible* if every nonzero element of R is left fusible. Right fusible rings are defined analogously. A ring which is both right and left fusible is called *fusible*.

A ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Examples of reversible rings are symmetric rings, reduced rings, and commutative rings. Therefore in the class of reversible rings the two concepts of left fusibility and right fusibility coincide.

Remark 2.2. Observe that a regular element is trivially left fusible; similarly for right fusible. Therefore, every domain is fusible, e.g. $\mathbb{Z}$. Also, every nonzero idempotent is right and left fusible: for each $0 \neq e \in Id(R)$, $e = (1 - e) + (2e - 1)$ is a fusible representation of e. Therefore, every Boolean ring is fusible.

We recall that a ring R is said to be almost clean if each of its elements can be written as the sum of an idempotent and a regular element. Almost clean rings, in the commutative context, were introduced by McGovern in [23]. It is clear that an almost clean representation of an element $a = e + r$ where $1 \neq e \in Id(R)$ and $r \in r(R)$, serves as a fusible representation. That $1 \neq e$ is important since not every almost clean ring is fusible. A ring R is said to be special almost clean if each element a can be decomposed as the sum of a regular element r and an idempotent e with $aR \bigcap eR = 0$, see [2]. Thus, every special almost clean ring is fusible. Note that there exist fusible rings which are not special almost clean rings, see [2, Example 4.2] and Remark 5.7. We should remind the reader that there exist almost clean rings which are not special almost clean rings, for example $\mathbb{Z}_4$.

Example 2.3. For any prime integer p and $n \geq 2$, $\mathbb{Z}_{p^n}$ is not fusible. In these rings the powers of p, p^k $(1 \leq k \leq n - 1)$ cannot be expressed as the sum of a zero-divisor and a non-zero-divisor in $\mathbb{Z}_{p^n}$. It follows that the class of fusible rings is not closed under homomorphic images. (Note that it will follow from later work that $\mathbb{Z}_n$ is fusible if and only if n is square free.)

There are some nice properties of rings that force fusibility. We consider a few of these and then move to consider the passage of the condition to polynomial rings.

Proposition 2.4. *Let R be a ring with comparability relation between right annihilators of its elements. Then R is left fusible if and only if R is a domain.*

Proof. As has already been observed a domain is left fusible. Suppose that R is left fusible and $0 \neq a \in Z_l(R)$. There exist $z \in Z_l(R)$ and $s \in Z_l^*(R)$ such that $a = z + s$. Now by assumption $Ann_r(a) \subseteq Ann_r(z)$ or $Ann_r(z) \subseteq Ann_r(a)$. In either case any non-zero element belonging to both will also belong to $Ann_r(s)$. Hence, s is a left zero-divisor, a contradiction. □

Example 2.5. A ring R is said to be right (left) uniserial if and only if its right (left) ideals are totally ordered under inclusion. Commutative uniserial rings are known as *chained* rings. It follows that a right uniserial ring is left fusible if and only if it is a domain.

Remark 2.6. Satyanarayana in [30, Proposition 2.4] proved that if R is a ring with a classical right quotient ring $q^r_{cl}(R)$, then q^r_{cl} is a local ring if and only if $Z(R)$ is an ideal. Hence a ring R with a local classical right quotient ring has no fusible right zero-divisor.

Lemma 2.7. *Let $\{R_i\}_{i\in I}$ be a family of rings. Then $Z_l^*(\prod_{i\in I} R_i) = \prod_{i\in I} Z_l^*(R_i)$ and $Z_r^*(\prod_{i\in I} R_i) = \prod_{i\in I} Z_r^*(R_i)$. Also, $U(\prod_{i\in I} R_i) = \prod_{i\in I} U(R_i)$.*

Proposition 2.8. *Let $\{R_i\}_{i\in I}$ be a family of rings and let $R = \prod_{i\in I} R_i$. R is fusible if and only for each $i \in I$, R_i is fusible.*

Proof. Suppose that k be an arbitrary element in I and $0 \neq a_k \in R_k$. Put $(x_i)_{i\in I} \in \prod_{i\in I} R_i$ such that $x_i = a_k$ for $i = k$, in otherwise $x_i = 0$. Since $\prod_{i\in I} R_i$ is left fusible then there exist $(z_i)_{i\in I} \in Z_l(\prod_{i\in I} R_i)$ and $(r_i)_{i\in I} \in Z_l^*(\prod_{i\in I} R_i)$ such that $(x_i)_{i\in I} = (z_i)_{i\in I} + (r_i)_{i\in I}$. It is clear that $z_i = -r_i$ for every $i \neq k$. Hence for every $i \neq k$, $z_i \in Z_l^*(R_i)$, and $z_k \in Z_l(R_k)$ by Lemma 2.7. Consequently $a_k = z_k + r_k$ where $z_k \in Z_l(R_k)$ and $r_k \in Z_l^*(R_k)$. Hence R_k is left fusible.

Conversely, suppose that $0 \notin (a_i)_{i\in I} \in \prod_{i\in I} R_i$. Since R_i is a left fusible ring for every $i \in I$, then for each $0 \neq a_i$ there exist $z_i \in Z_l(R_i)$ and $r_i \in Z_l^*(R_i)$ such that $a_i = z_i + r_i$. Now we consider $(b_i)_{i\in I}$ and $(c_i)_{i\in I}$ as the following:
(1) If $0 \notin a_i$ then define $b_i := z_i$ and $c_i := r_i$,
(2) If $0 = a_i$ then define $b_i := 1$ and $c_i := -1$.
Hence $(a_i)_{i\in I} = (b_i)_{i\in I} + (c_i)_{i\in I}$. Therefore $(b_i)_{i\in I} \in Z_l(\prod_{i\in I} R_i)$ and $(c_i)_{i\in I} \in Z_l^*(\prod_{i\in I} R_i)$ by Lemma 2.7. This means that $\prod_{i\in I} R_i$ is a left fusible ring. □

Let R be a ring and σ be a ring endomorphism of R. Let $R[x;\sigma]$ denote the skew polynomial ring consisting of the polynomials in x with coefficients in R written on the left, with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. $R[[x;\sigma]]$ denotes the skew formal power series ring. It is clear that if we put $\sigma = 1_R$, then we have $R[x] = R[x;\sigma]$ and $R[[x]] = R[[x;\sigma]]$.

Proposition 2.9. *Let R be a left fusible ring and σ be a ring automorphism of R. Then $R[x;\sigma]$ and $R[[x;\sigma]]$ are left fusible.*

Proof. Let R be a left fusible ring. Suppose that $0 \neq f(x) = \sum_{i=0}^{n} a_i \in Z_l(R[x;\sigma])$ where k $(0 \leq k \leq n)$ is the least integer such that $a_k \neq 0$. Therefore $f(x) = a_k x^k + a_{k+1}x^{k+1} + ... + a_n x^n$. Since R is left fusible, then there exist $z_k \in Z_l(R)$ and $n_k \in Z_l^*(R)$ such that $a_k = z_k + n_k$ and $f(x) = (z_k x^k) + (n_k x^k + a_{k+1}x^{k+1} + ... + a_n x^n)$. Now put $g(x) = z_k x^k$ and $h(x) = n_k x^k + a_{k+1}x^{k+1} + ... + a_n x^n$. First we claim that $g(x) \in Z_l(R[x;\sigma])$. To see this, since $z_k \in Z_l(R)$ there exists $0 \neq d \in R$ such that $z_k d = 0$. But σ is an automorphism so that there exists $0 \neq e \in R$ such that $\sigma^k(e) = d$. Hence it is clear that $g(x)e = 0$ and this means that $g(x) \in Z_l(R[x;\sigma])$. Now we claim that $h(x) \in Z_l^*(R[x;\sigma])$. If $h(x) \in Z_l(R[x;\sigma])$ then there exists $0 \neq b \in R$ such that $n_k x^k b = 0$. Hence $n_k \sigma^k(b) x^k = 0$. But σ is an automorphism therefore $0 \neq \sigma^k(b)$ and n_k is a left zero-divisor, a contradiction. Hence $f(x) = g(x) + h(x)$ such that $g(x) \in Z_l(R[x])$ and $h(x) \in Z_l^*(R[x])$. Consequently, $R[x;\sigma]$ is a left fusible ring. A similar proof yields that $R[[x;\sigma]]$ is a left fusible ring. □

The following example shows that Proposition 2.9 need not be true for any skew polynomial or skew power series rings and shows there exist right fusible elements which are not left fusible.

Example 2.10. Let R be a domain and σ be a ring endomorphism of R which is not injective. Let $0 \neq r \in R$ satisfy $\sigma(r) = 0$. Then $xr = \sigma(r)x = 0$. By definition it is clear that $< x >\subseteq Z_l(R[x,\sigma])$. Now let $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a left zero-divisor element of $R[x,\sigma]$ such that $a_0 \neq 0$. Hence there exists $b_0 + b_1x + b_2x^2 + ... + b_mx^m \neq 0$ such that $(a_0 + a_1x + a_2x^2 + ... + a_nx^n)(b_0 + b_1x + b_2x^2 + ... + b_mx^m) = 0$. Therefore, we have $(a_0b_0 + (a_0b_1 + a_1\sigma(b_0))x + (a_0b_2 + a_1\sigma(b_1) + a_2\sigma^2(b_0))x^2 + ... + (a_n\sigma^n(b_m))x^{n+m} = 0$. Therefore, $a_0b_0 = 0$, $a_0b_1 + a_1\sigma(b_0) = 0$, $a_0b_2 + a_1\sigma(b_1) + a_2\sigma^2(b_0) = 0$,..., $a_n\sigma^n(b_m) = 0$. But by our assumption $a_0 \neq 0$ and R is a domain, thus $b_0 = 0$. Since $a_0b_1 + a_1\sigma(b_0) = 0$ then $b_1 = 0$. By this process we have $b_m = 0$. Hence $b_0 + b_1x + b_2x^2 + ... + b_mx^m = 0$ that is a contradiction. Thus $< x >= Z_l(R[x,\sigma])$ and $Z_l^*(R[x,\sigma]) = R\setminus < x >$. Since $Z_l(R[x,\sigma])$ is a left ideal then $R[x,\sigma]$ is not a left fusible ring. By definition it is clear that x is a non-right zero-divisor element and hence x is a right fusible element while is not a left fusible element.

A right ideal I of a ring R is said to be essential (or large), denoted by $I \leq_e R_R$, if for every right ideal L of R, $I \bigcap L = 0$ implies that $L = 0$. The right singular ideal of a ring R is defined by $Sing(R_R) = \{x \in R | Ann_r(x) \leq_e R\}$. A ring R is called right nonsingular if $Sing(R_R) = 0$.

Proposition 2.11. *Every left fusible ring is right nonsingular.*

Proof. Let R be a left fusible ring and $0 \neq x \in Sing(R_R)$. Thus there exist $z \in Z_l(R)$ and $r \in Z_l^*(R)$ such that $x = z + r$. Since $x - z = r$ then $Ann_r(x) \bigcap Ann_r(z) \subseteq Ann_r(x - z) = Ann_r(r) = 0$. Hence $Ann_r(x) \bigcap Ann_r(z) = 0$. But $Ann_r(x) \leq_e R$ and $Ann_r(z) \neq 0$ which is a contradiction. □

We next consider the fusibility of the split-null extension $\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ for rings A and B and a bimodule $M = {}_AM_B$. In general, this ring need not be fusible as the next two results demonstrate.

Example 2.12. Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. It is clear that R is a right nonsingular ring. Note that $Z_l(R) = \{\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} n & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2n & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2n & 1 \\ 0 & 1 \end{bmatrix} | n \in \mathbb{Z}\}$ is the set of left zero-divisors of R. For example, $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is not a left fusible element of R. Hence R is not a left fusible ring.

Proposition 2.13. *Let A, B be rings and $M = {}_AM_B$ be a bimodule. Then the split-null extension $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is left fusible if and only if $M = 0$ and A,B are left fusible rings.*

Proof. First, assume that R is a left fusible ring and $M \neq 0$. Now suppose that $C = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$ is a nonzero element of R. Hence there exist $D = \begin{bmatrix} a & l \\ 0 & b \end{bmatrix} \in Z_l(R)$ and $E = \begin{bmatrix} -a & k \\ 0 & -b \end{bmatrix} \in Z_l^*(R)$ such that $C = D + E$. Since $D \in Z_l(R)$ therefore $a \in Z_l(A)$ and $b \in Z_l(B)$. Thus $-a \in Z_l(A)$ and $-b \in Z_l(B)$ and this means that $E = \begin{bmatrix} -a & k \\ 0 & -b \end{bmatrix}$ is a left zero-divisor, a contradiction. (Note, Since $E \in Z_l^*(R)$ then at least one of a or b is not zero. Suppose that $a \neq 0$ therefore $\begin{bmatrix} -a & k \\ 0 & -b \end{bmatrix}\begin{bmatrix} Ann_r(a) & 0 \\ 0 & 0 \end{bmatrix}$=0).

Next, let $a \in A$. By our assumption for $C = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ there exist $D = \begin{bmatrix} a_1 & m \\ 0 & b \end{bmatrix} \in Z_l(R)$ and $E = \begin{bmatrix} a_2 & -m \\ 0 & -b \end{bmatrix} \in Z_l^*(R)$ such that $C = D + E$. Since $E \in Z_l^*(R)$ then $a_2 \in Z_l^*(A)$ and $-b \in Z_l^*(B)$, hence $b \in Z_l^*(B)$. Since $D \in Z_l(R)$, $a_1 \in Z_l(A)$. Therefore $a = a_1 + a_2$ is a fusible element and we conclude that A is left fusible. A similar proof yields that B is left fusible.

Conversely, if $M = 0$ and A,B are fusible rings. Thus $R \cong A \times B$. Since A and B are left fusible we conclude that R is left fusible. □

Corollary 2.14. *Let R be a ring and $n \geq 2$. Then ring of lower (upper) triangular matrices $T_n(R)$ is never a left fusible ring.*

Recall that a ring R is a left p.p. ring if any principal left ideal of R is projective. Left p.p rings are also known as left Rickart rings. In the commutative case being a p.p. ring is equivalent to being a weak Baer ring: if the annihilator of each element is generated by an idempotent. McGovern [23, Proposition 16] proved that every commutative p.p. ring is almost clean. This was strengthened by Akalan and Vas [2, Theorem 3.1] who demonstrated that for an abelian ring R, R is left p.p. ring if and only if R is special almost clean. Thus the following result is immediate by Remark 2.2.

Corollary 2.15. *Let R be an abelian ring. If R is a left (right) p.p. ring, then R is fusible.*

Remark 2.16. Notice that any attempt at removing the hypothesis that R is abelian will not be successful. The ring of upper triangular $n \times n$ matrices over a field is a left p.p. ring (being hereditary) yet is not fusible by Corollary 2.14.

A ring R is said to be left principally quasi-Baer (or simply, left p.q.-Baer) if the left annihilator of a principal left ideal is generated (as a left ideal) by an idempotent. This generalizes the notion of a weak Baer ring to the non-commutative context. Birkenmeier et al. in [7, Corollary 1.15] proved that a ring R is an abelian left p.p. ring if and only if R is a reduced p.q.-Baer ring. Hence every reduced p.q.-Baer ring is fusible by Corollary 2.15. Furthermore, a reduced p.q.-Baer ring is abelian, but not conversely; see [7, Example 3.16]. We do not know whether an abelian p.q.-Baer ring is fusible.

Example 2.17. In [29] Osofsky constructed an example of a semiprimitive ring with nonzero right singular ideal. Hence the Osofsky ring is not left fusible by Proposition 2.11. Lawrence in [22, Theorem 4] showed that the Osofsky ring is prime thus it is a left p.q.-Baer ring. Hence the Osofsky ring is a left p.q.-Baer ring which is not left fusible.

Theorem 2.18. *Let R be a left fusible ring. Then $M_n(R)$ is left fusible ring.*

Proof. Let $0 \neq A = [a_{ij}]_{n\times n}$. Consider two different cases.

(1) There exists i $(1 \leq i \leq n)$ such that $a_{ii} \neq 0$.
Define $B := [b_{ij}]_{n\times n}$ such that;

$$b_{ij} := \begin{cases} a_{ij} & i < j \\ 0 & i > j \end{cases}$$

Note that since R is a left fusible ring then for each $a_{ii} \neq 0$ there exist $z_{ii} \in Z_l(R)$ and $r_{ii} \in Z_l^*(R)$ such that $a_{ii} = z_{ii} + r_{ii}$. Now define;

$$b_{ii} := \begin{cases} -1 & \text{if } a_{ii} = 0 \\ z_{ii} & \text{if } a_{ii} \neq 0 \end{cases}$$

We claim that $B := [b_{ij}]_{n\times n}$ is a left zero-divisor element in $M_n(R)$. To see this, suppose that k is the largest number between $1 \leq k \leq n$ such that array $b_{kk} \neq -1$. Thus there exist $z_{kk} \in Z_l(R)$ and $r_{kk} \in Z_l^*(R)$ such that $b_{kk} = z_{kk} + r_{kk}$. Now given an element in $Ann_r(z_{kk}) \neq 0$ and say z'_{kk}. Define $G := [g_{ij}]_{n\times n}$ such that;

$$g_{ij} := \begin{cases} 0 & i < k \text{ or } j \neq k \\ z'_{kk} & i = k \text{ and } j = k \\ a_{(k+1)k} z'_{kk} & i = k+1 \text{ and } j = k \\ a_{(k+2)k} z'_{kk} + a_{(k+2)(k+1)} g_{(k+1)k} & i = k+2 \text{ and } j = k \\ a_{(k+3)k} z'_{kk} + a_{(k+3)(k+1)} g_{(k+1)k} + a_{(k+3)(k+2)} g_{(k+2)k} & i = k+3 \text{ and } j = k \\ \ldots & i = \ldots \text{ and } j = k \\ a_{nk} z'_{kk} + a_{n(k+1)} g_{(k+1)k} + a_{n(k+2)} g_{(k+2)k} + \ldots + a_{n(n-1)} g_{(n-1)k} & i = n \text{ and } j = k \end{cases}$$

It is clear that $BG = 0$ and thus $B \in Z_l(M_n(R))$. Now define $C := A - B$. It is clear that $C \in Z_l^*(M_n(R))$.This means that A is a left fusible matrix.

(2) There dose not exist i $(1 \leq i \leq n)$ such that $a_{ii} \neq 0$.
Since $0 \neq A = [a_{ij}]_{n\times n}$, there exists $a_{st} \neq 0$ ($s < t$ or $s > t$) such that $a_{st} = z_{st} + r_{st}$ where $z_{st} \in Z_l(R)$ and $r_{st} \in Z_l^*(R)$.

First assume that $s < t$. Define $C := [c_{ij}]_{n\times n}$ such that

$$c_{ij} := \begin{cases} 0 & 1 \le i = j \le n \\ r_{st} & i = s \text{ and } j = t \\ 0 & 2 \le i \le n-1,\ j = 1 \\ a_{it} & 1 \le i \le n,\ i \ne s, j \text{ and } j = t \\ 0 & 1 \le j \le n \text{ and } j \ne t, i \text{ and } i = s \\ 1 & 1 \le j \le n-1,\ i = n \text{ and } j \ne t \end{cases}$$

In each of the rows of matrix C ranging from 1 to $n-1$, we replace the one of unknown arrays with 1 and put the other ones to 0 such that 1 arrays are placed in distinct columns for each pair. It is clear that $C \in Z_l^*(M_n(R))$. Now put $B = A - C$. We claim that $B := [b_{ij}]_{n\times n}$ is a left zero-divisor element in $M_n(R)$. To see this, put $G := [g_{ij}]_{n\times n}$ and given an element in $Ann_r(z_{st}) \ne 0$ and say z'_{kk}. Define $G := [g_{ij}]_{n\times n}$ such that;

$$g_{ij} := \begin{cases} 0 & i \ne t \\ z'_{kk} & i = t \end{cases}$$

It is clear that $BG = 0$ and thus $B \in Z_l(M_n(R))$. This means that A is a left fusible matrix and we are done.
If $a_{st} \ne 0$ $(s > t)$, then the proof is similar. □

In view of Theorem 2.18 it does follow that a left fusible ring R need not be a reduced ring. We do not know whether every abelian left fusible ring is reduced. If we strengthen the condition slightly we get a positive result. Recall that a ring R is said to be a right (left) duo ring if every right (left) ideal is two-sided. This is equivalent to the condition that $Ra \subseteq aR$ $(aR \subseteq Ra)$ for all $a \in R$. It is known that every right (left) duo ring is abelian. A ring which is both right and left duo is known as a duo ring.

Lemma 2.19. *Every left fusible, right duo ring is reduced. In particular, every commutative fusible ring is reduced.*

Proof. Let R be a left fusible, right duo ring. Suppose that $0 \ne a \in R$ and $a^2 = 0$. Since R is a left fusible ring, then there exist a left zero-divisor $z \in R$ and a non-left zero-divisor $r \in R$ such that $a = z + r$. Then $0 = a^2 = (z+r)^2 = z^2 + zr + rz + r^2 = z^2 + r'z + rz + r^2 = z^2 + sz + r^2$ by hypothesis. Therefore $(z + s)z = -r^2$. Hence r is a left zero-divisor which is a contradiction. □

Lemma 2.20. [17, Lemma 4.2] *Let R be an Artinian duo ring. Then R is a finite direct product of Artinian local duo rings.*

The following result is in order.

Proposition 2.21. *An Artinian duo ring is fusible if and only if it is isomorphic to a finite direct product of division rings.*

Proof. It is clear by Lemma 2.19 and Lemma 2.20. □

A left R-module M is called *cohopfian* if any injective R-endomorphism of M is an automorphism. Birkenmeier [8, Remark in page 102] shown that the left regular module ${}_RR$ is cohopfian if and only if every non-right zero-divisor in R is a unit. Examples of cohopfian rings are self injective rings, Dedekind finite regular rings, strongly π-regular rings, left or right perfect rings and left or right Artinian rings.

Proposition 2.22. *Let R be a right cohopfian ring. If R is left fusible, then R is Jacobson semi-simple.*

Proof. Suppose that $0 \neq a \in \mathfrak{J}(R)$. Since R is a right cohopfian ring then a is not a non-left zero-divisor. Hence a is a left zero-divisor and write $a = z + r$ where $z \in Z_l(R)$ and $r \in Z_l^*(R)$. Since R is a right cohopfian ring then $r \in Z_l^*(R) = U(R)$. Since z is a left zero-divisor it belongs to some left maximal ideal. Then $r = a - z$ also belongs to that left maximal ideal which is a contradiction. □

A ring R is called *left perfect* if $R/J(R)$ is semisimple and $J(R)$ is left T-nilpotent.

Corollary 2.23. *A left perfect ring R is left fusible if and only if R is a semisimple ring.*

Proof. It is clear that every semisimple ring is fusible. Conversely, let R be a left perfect ring. If R is a left fusible then $J(R) = 0$ by Proposition 2.22. Hence R is semisimple. □

We end this section with a discussion of left fusible rings with the property that each nonzero element has a unique left fusible representation.

Definition 2.24. We call a ring R *uniquely left fusible* if for any $0 \neq a \in R$ there exists a unique left zero-divisor $z \in Z_l(R)$ such that $a - z \in Z_l^*(R)$ is non-left zero-divisor. Notice that this is equivalent to saying that for each $0 \neq a \in R$ there is a unique $r \in Z_l^*(R)$ such that $a - r \in Z_l(R)$. We define uniquely right fusible rings in a similar vein.

It is clear that if either $Z_l(R) = \{0\}$ or $Z_l^*(R) = \{1\}$ are trivial, then R is left fusible if and only if it is uniquely left fusible. The equation $Z_l(R) = \{0\}$ means that there are non-trivial left zero-divisors and hence no non-trivial right zero-divisors. This is equivalent to saying that R is a domain.

Our last result of this section gives a partial characterization of uniquely left fusible rings.

Theorem 2.25. *Let R be a ring with $2 \neq 0$. The following statements are equivalent.*

(1) R is a uniquely left fusible ring.
(2) R is a domain.
(3) $Z_l(R) = \{0\}$.
(4) $Z_r(R) = \{0\}$.
(5) R is a uniquely right fusible ring.

Proof. First, that (2), (3), and (4) has already been noted. Clearly, these imply (1). We show that (1) implies (2). The rest of the proof is patent.

Suppose R is uniquely left fusible $0 \neq 2$. We show that let R is a domain. Let $x \neq 1$ be a non-left zero-divisor of R. Therefore $x = 0 + x$ is the unique fusible representation of x by hypothesis. Since 1 is a regular element different than x, the expression $x = (x - 1) + 1$ is not a fusible representation of n. Therefore, we conclude that $x - 1 \in Z_l^*(R)$. Since $1 \neq -1$ are both non-left zero-divisors we gather that $-2 = -1 - 1$ and hence 2 are also non-left zero-divisors.

Now suppose that $0 \neq z \in Z_l(R)$ and $z = z_1 + n$ is the unique fusible representation of z with $0 \neq z_1 \in Z_l(R)$ and $n \in Z_l^*(R)$. In the case that $n = 1$ then the equation $z = (z + 1) - 1$ is not a fusible representation and so $z + 1$ is a non-left zero-divisor, and so by the previous paragraph so is z. In the case that $n \neq 1$ we write $z = (z+1)+(n-1)$ and again gather that $z+1$ and hence z are non-left zero-divisors. Both cases lead us to a contradiction. Consequently, R has no (non-zero) left zero-divisors, i.e. R is a domain. □

To get a complete characterization of the uniquely left fusible rings of we would need to consider the case when the ring has characteristic 2. We notice that a boolean ring is uniquely fusible ring. However, we leave consideration of this case for another time.

3. Commutative Rings

In this section we investigate fusibility in the context of commutative rings. When we want to stress that a ring is commutative we shall denote the ring by A. When we use the symbol R then we are stressing that the ring is not necessarily commutative. Recall that if A is a fusible ring, then A is reduced. There are many well-known properties of commutative rings that force it to be reduced; fusibility adds to this list.

Cohn [11] introduced the term '0-ring' for commutative rings with 1, in which every element different from 1 is a zero-divisor. Examples of 0-rings are Boolean rings. Observe that 0-rings are fusible: for each $a \neq 0, 1$, $a = 1 + (a-1)$ is a fusible representation of a.

Proposition 3.1. *Let A be a potent ring, that is, for each $x \in A$ there exists an integer $1 < n$ for which $x^n = x$. Then A is a fusible commutative ring.*

Proof. That a potent ring is commutative was proved by ([16], p. 217) in a more general context. Next, let $0 \neq x \in Z(A)$ and $1 < n \in \mathbb{N}$ such that $x^n = x$. Since $(x + 1 - x^{n-1})(x^{2n-3} + 1 - x^{n-1}) = 1$ then $x + 1 - x^{n-1} \in U(A)$. Clearly, $x^{n-1} - 1 \in Z(A)$. Hence $x = (x^{n-1} - 1) + (x + 1 - x^{n-1})$ and we are done. □

A periodic ring is a ring R satisfying for every element $a \in R$ there are $n < m \in \mathbb{N}$ such that $a^n = a^m$. A commutative periodic ring is potent if and only if it is reduced. Periodic rings need not be commutative. Herstein [19] showed that a periodic ring with central nilpotent elements is commutative. We thus have our next result.

Corollary 3.2. *The periodic ring A is fusible if and only if A is potent.*

A natural class of rings to consider are the von Neumann regular rings; R is von Neumann regular if for all $a \in R$ there is an $x \in R$ such that $axa = a$. Commutative von Neuamann regular rings are fusible. We can say more. Recall that a commutative ring A is complemented (also known as quasi-regular) if for each $x \in A$, there is a $y \in A$ such that $xy = 0$ and $x + y \in r(A)$. Commutative von Neumann regular rings are complemented since they are characterized by the statement for each $x \in R$, there is a $y \in R$ such that $xy = 0$ and $x + y \in U(R)$. Commutative complemented rings are reduced and can be characterized via their classical ring of quotients: A is complemented if and only if $q_{cl}(A)$ is von Neumann regular, see [21, Theorem 2.5] for some references. For each $a \in A$, let P_a be the intersection of all minimal prime ideals containing a. We can add that A is complemented if and only if for each $a \in Z(A)$ there exists a $b \in R$ such that $Ann(a) = P_b$ [5, Lemma 1.25]. The weak Baer rings previously defined are complemented. In Theorem 3 of

[26] the author generalizes the above result involving the classical ring of quotients to the non-commutative case. This leads to us a definition.

Definition 3.3. Suppose R is a ring. We call R a *right complemented ring* if for each $a \in R$ there is a $b \in R$ such that $ab = 0$ and $a + b$ is regular. Notice that in a right complemented ring every left regular element is regular. Left complemented ring are defined analogously.

Proposition 3.4. *Suppose R is right (left) complemented. Then R is right (left) fusible and reduced.*

Proof. Let $0 \neq a \in R$ and choose $b \in R$ such that $ab = 0$ and $a + b$ is regular. Then $a = (a + b) - b$ is a right fusible representation.

To see that it is reduced for $a \in R$ such that $a^2 = 0$ choose d regular such that $ad = a^2$ forces $a = 0$. □

Theorem 3.5. [26, Theorem 3] *Suppose R is a ring and Q is a right classical quotient ring of R. The following statements are equivalent.*

(1) *Q is strongly regular, that is, for each $q \in Q$ there is a $u \in Q$ such that $q^2u = q$.*
(2) *Q is reduced and von-Neumann regular.*
(3) *Q is abelian and von Neumann regular.*
(4) *For each $a \in R$ there is a regular element $d \in R$ such that $ad = a^2$.*
(5) *R is right complemented.*

Proof. The only new thing here is the last condition. We also point out that the fourth and fifth conditions are equivalent without mention of a right classical quotient ring. We do not know if a right complemented ring must possess a right classical quotient ring.

Let $a, d \in R$ such that $ad = a^2$. Then $a(d - a) = 0$. Setting $b = d - a$ we find that $ab = 0$ and $a + b = d$.

Conversely, let $a \in R$ and choose $b \in R$ such that $ab = 0$ and $a + b$ is regular. Set $d = b + a$ and observe that $ad = a(b + a) = a^2$. □

Corollary 3.6. *Let R be a von Neumann regular ring. R is right complemented if and only if R is strongly regular if and only if R is left complemented.*

For commutative rings we can reword fusibility in such a way that we are saying for $0 \neq x \in A$ there is a zero-divisor y such that $x + y \in r(A)$ and $x = (x + y) - y$. It clearly possible that $xy \neq 0$. Thus, it is not surprising that fusibility does not imply complementedness. We shall see in the next section that it definitely does not.

Let A be a commutative ring with identity. The set of all minimal prime ideals of A is denoted by $\mathrm{Min}(A)$. It is known that $Z(A) = \bigcup \mathrm{Min}(A)$. Also, for A to be reduced it is equivalent to saying that $\bigcap \mathrm{Min}(A) = \{0\}$. Our next result is interesting in that it shows certain conditions on $\mathrm{Min}(A)$ imply fusibility within the class of reduced rings.

Theorem 3.7. *A commutative ring A with only finitely many minimal prime ideals is reduced if and only if A is fusible.*

Proof. First assume that A is reduced. Then $Z(A) = \bigcup_{i=1}^{n} P_i$ where P_i $(1 \leq i \leq n)$ are all minimal prime ideals of A. Suppose that $x \in Z(A)$. We can assume that P_i's have been indexed in such a way that there exists an integer m where $1 \leq m \leq n$ such that $x \in \bigcap_{i=1}^{m} P_i$ and $x \notin \bigcup_{i=m+1}^{n} P_i$. Since P_i's are minimal prime ideals then $\bigcap_{i=1}^{m} P_i \neq 0$ and by Prime avoidance lemma, we have $\bigcap_{i=1}^{m} P_i \not\subset \bigcup_{i=m+1}^{n} P_i$. Hence there exists $y \in \bigcap_{i=m+1}^{n} P_i \setminus \bigcup_{i=1}^{m} P_i$. It is clear that $x + y$ is regular. This means that A is fusible. The converse is clear. □

Remark 3.8. The previous theorem cannot be generalized to reduced rings with infinitely many minimal primes, e.g. the infinite direct product of a reduced non-fusible ring. See Example 4.5 and observe that this is an example of a local reduced ring which is not fusible.

Next, we consider another generalization of von Neumann regular rings. Recall that a ring R is called a clean ring if every element of R is the sum of a unit and an idempotent. This class of rings was originally defined by Nicholson [27] and has since been a focus of many investigations. All local rings are clean. Since it is not the case that a local ring is fusible we cannot conclude that clean rings are fusible. However, we can say something interesting with an added condition. (By the way complemented rings need not be clean, nor are clean rings complemented.)

Theorem 3.9. *Suppose A is a Jacobson semi-simple commutative clean ring. Then A is fusible.*

Proof. Let $0 \neq x \in A$ and without loss of generality we assume that $x \notin U(A)$ and so $V(a) \neq \emptyset$.

Case 1. $V(a-1) \neq \emptyset$. Then cleanliness produces an idempotent $e \in A$ such that $a-e \in U(A)$. It follows that $e \neq 1$ and so $a = (a-e)+e$ is a fusible representation.

Case 2. $V(a-1) = \emptyset$. Since A is Jacobson semi-simple there is a maximal ideal, say $M \in \mathrm{Max}(A)$, such that $a \notin M$. Choose an idempotent $e \in A$ which separates M from $V(a)$. In this manner we assume that $V(a) \subseteq U(e)$ and $M \in V(e)$. Clearly e is a zero-divisor since $e \neq 1$. Now $a - e \in U(A)$ otherwise there is some $N \in \mathrm{Max}(A)$ such that $a - e \in N$. If $e \in N$, then $a \in N$, a contradiction. So $e \notin N$. But then $a+N = e+n = 1+N$ and so $V(a-1) \neq \emptyset$, another contradiction. Therefore, $a - e \in U(A)$ and so $a = (a - e) + e$ is a fusible representation. □

Remark 3.10. The above theorem can be generalized to abelian rings using the work laid out in [10].

At this point it is a natural question whether a Jacobson semi-simple ring is fusible. We show in the following that this is not the case. We recall two needed facts as well as a definition. A commutative ring A is said to satisfy *Property A* if each finitely generated ideal contained in $Z(A)$ has a nonzero annihilator.

Theorem 3.11. [3, Theorem 1] *If A is a reduced commutative ring, then $A[x]$ is a Jacobson-semi-simple ring.*

Theorem 3.12. [24, Theorem 3.3] *For a commutative ring A, $Z(A[x])$ is an ideal of $A[x]$ if and only if A satisfies Property A such that $Z(A)$ is an ideal of A.*

Example 3.13. Consider the domain $D = F[X]_{(X)}$ where F is a field and $X = \{x_n\}$ is a countably infinite set of indeterminates. Assume $\mathcal{P}$ denote the primes of D that are generated by finite subsets of X. The set $\mathcal{P}$ includes $P_0 = 0$, the prime generated by the empty subset of X. Also for $n \geq 1$, let $P_n = (x_1, x_2, \ldots, x_n)D$. Note that given a prime $P_\alpha \in P$, there is an integer n such that $P_\alpha \subset P_k$ for each $n \leq k$. For each $P_\alpha \in P$, we let $Q_\alpha = M/P_\alpha$. Put $C = \sum Q_n$. Let $A = D + C$ be the ring formed from the product $D \times C$ by setting $(r,b)+(s,c) = (r+s, b+c)$ and $(r,b)(s,c) = (rs, rc+sb+bc)$. Then A is a reduced ring satisfying Property A such that $Z(A)$ is an ideal. Hence $J(A[x]) = 0$ by Theorem 3.11 and $Z(A[x])$ is an ideal of $A[x]$ by Theorem 3.12. Therefore $A[x]$ is a Jacobson semi-simple ring which is not fusible.

Proposition 3.14. *If A is fusible, then so is $q(A)$.*

Proof. Let A be a fusible ring and $0 \neq x = \frac{a}{m} \in q(A)$.Since $a \neq 0$ we can write $a = r + z$ for some $r \in r(A)$ and $z \in Z(A)$. Then $x = \frac{a}{m} = \frac{r+z}{m} + \frac{r}{m} + \frac{z}{m}$ where $\frac{r}{m} \in U(q(A))$ and $\frac{z}{m} \in Z(q(A))$. ☐

4. Full Rings of Continuous Functions

Throughout this section $\mathbb{A}$ denotes a (unital) subring of $\mathbb{R}$. For a topological space X, we let $C(X, \mathbb{A})$ denote the ring of all continuous $\mathbb{A}$-valued functions with domain X. When $\mathbb{A} = \mathbb{R}$ we instead write $C(X)$. An excellent source for information on $C(X)$ is [14]. Recall that $U(\mathbb{A})$ denotes the group (under multiplication) of units of $\mathbb{A}$. When $\mathbb{A}$ is a field we instead shall write $\mathbb{A}^* = \mathbb{A} \smallsetminus \{0\}$. The three most well-studied examples of $C(X, \mathbb{A})$ are when $\mathbb{A} = \mathbb{R}, \mathbb{Q}$, and $\mathbb{Z}$.

In general, we shall assume that X is a Tychonoff space, that is, completely regular and Hausdorff. When $\mathbb{A}$ is not connected in its subspace topology we can further assume that X is a zero-dimensional space, that is, has a base of clopen sets.

Definition 4.1. Let $f \in C(X, \mathbb{A})$. The *zeroset of f* is the set $Z(f) = \{x \in X : f(x) = 0\}$. The set-theoretic complement of $Z(f)$ is called the cozeroset of f and is denoted by $coz(f)$. A subset of X, say V, is known as a *zeroset* (resp. *cozeroset*) if $V = Z(f)$ (resp. $V = coz(f)$). Zerosets (resp. cozerosets) are always closed (resp. open) subsets of X, and it is known that a Hausdorff space is Tychonoff precisely when the cozerosets form a base for the open sets of X.

For $a \in \mathbb{A}$ we denote the function with constant value a, by $\mathbf{a}$. For example, $\mathbf{1}$ is the ring identity of $C(X, \mathbb{A})$. The ring $C(X, \mathbb{A})$ is a lattice when ordered pointwise: $f \leq g$ means for all $x \in X$, $f(x) \leq g(x)$. We need the least upper bound of the pair f, g by $f \vee g$. Similarly, $f \wedge g$ denotes the greatest lower bound. For $f, g \in C(X, \mathbb{A})$, $f \vee g, f \wedge g \in C(X, \mathbb{A})$.

Some important elements in $C(X, \mathbb{A})$ are as follows. For $f \in C(X, \mathbb{A})$ we let $f^+ = f \vee 0$ and $f^- = -f \vee 0$. The element f^+ (resp. f^-) is called the positive (resp. negative) part of f. Observe that $f = f^+ - f^-$. Next, the absolute value of f, denoted by $|f|$ is the function defined by $|f|(x) = |f(x)|$. For each $f \in C(X, \mathbb{A})$, $|f| \in C(X, \mathbb{A})$ since $|f| = f^+ + f^-$. Lastly, for a clopen subset K of X, the characteristic function χ_K is continuous and therefore, $\chi_K \in C(X, \mathbb{A})$.

Lemma 4.2. *The following statements are true.*

1. *For $f \in C(X, \mathbb{A})$, $f \in U(C(X, \mathbb{A}))$ if and only if f is $U(\mathbb{A})$-valued.*
2. *For $f \in C(X, \mathbb{A})$, $f \in r(C(X, \mathbb{A}))$ if and only if $coz(f) \subseteq X$ is dense.*
3. *For $f \in C(X, \mathbb{A})$, $f \in r(C(X, \mathbb{A}))$ if and only if $Z(f) \subseteq X$ has empty interior.*

Theorem 4.3. *For any $\mathbb{A} \leq \mathbb{R}$, the ring $C(X, \mathbb{A})$ is fusible.*

Proof. Let $0 \neq f \in C(X, \mathbb{A})$ be a zero-divisor. Then $Z(f)$ has non-empty interior. Notice that one of the following happens: $\mathbf{0} \neq f^+$ is bounded away from 0, $\mathbf{0} \neq f^-$ is bounded away from 0, or one of f^+ or f^- is not bounded away from 0. When we say the function g is bounded away from 0 we mean there is some $\epsilon > 0$ such that for all $x \in coz(g)$, $|g(x)| > \epsilon$.

In the case $\mathbf{0} \neq f^+$, the positive part of f, is bounded away from 0, then $K = f^{-1}((-\infty, 0])$ is a proper clopen subset of X. Thus, $r = f - \chi_K$ is regular since it has an empty zeroset. Also, since $K \subset X$, χ_K is a zero-divisor of $C(X, \mathbb{A})$. Consequently, $f = r + \chi_K$ is a fusible representation of f. A similar argument works for when $0 \neq f^-$ is bounded away from 0.

Next we consider the case where f^+ is not bounded away from 0. This means we can choose positive outputs say $0 < r_1 < r_2 \in \mathbb{A}$. Consider the function $r = f - f \wedge \mathbf{r_1} + \mathbf{r_1} \in C(X, \mathbb{A})$. Notice that on $f^{-1}([r_1, \infty))$, $r(x) \geq r_1$. On $f^{-1}((-\infty, r_1))$ we have that $f(x) = r_1$. Therefore, r is regular. Since for all $x \in f^{-1}(r_1, \infty)$, $(f - r)(x) = (f \wedge \mathbf{r_1} - \mathbf{r_1})(x) = 0$, it follows that $f - r$ is a zero-divisor. Therefore, $f = r + (f - r)$ is a fusible representation of f. A similar argument works for when f^- is not bounded away from 0. □

Remark 4.4. The above proof works to show that the rings of the form $C_c(X, \mathbb{A}) = \{f \in C(X, \mathbb{A}) : |f(X)| \leq \aleph_0\}$ are also fusible. Similarly, if you restrict to the ring of $\mathbb{A}$-valued continuous functions with finite range then each function is bounded away from 0 and the proof is even quicker. We shall leave the study of archimedean f-rings as well as the adaptation of fusibilty to the theory of archimedean lattice-ordered groups with weak order unit to another time.

For those familiar with semi-prime f-rings with bounded inversion we would look to point out that it is not the case that such rings are fusible. For a Tychonoff space X and $p \in X$ recall the following ideals of $C(X)$: $M_p = \{f \in C(X) : f(p) = 0\}$ and $O_p = \{f \in C(X) : f$ vanishes on a neighborhood of $p\}$. The ideal M_p is a maximal ideal of $C(X)$ and O_p is realized as the intersection of all prime ideals of $C(X)$ contained in M_p. Equivalently, O_p is the intersection of all minimal prime ideals of $C(X)$ contained in M_p. Furthermore, the only maximal ideal containing O_p is M_p. In terms of ring theory, the localization of $C(X)$ at the maximal ideal M_p is ring isomorphic to the quotient ring $C(X)/O_p$. For any $p \in X$, $C(X)/O_p$ is a reduced (local) semiprime f-ring with bounded inversion.

Example 4.5. Let $X = D \cup \{\alpha\}$ be the one-point compactification of an uncountable discrete space D. The ring $C(X)/O_\alpha$ is not a fusible ring. In this example it is known that the union of the minimal prime ideals (infinitely many) contained in M_α is all of M_α (see [6]). It follows that in $C(X)/O_\alpha$, the maximal ideal $M = M_\alpha/O_\alpha$ is the set of zero-divisors of $C(X)/O_\alpha$. Furthermore, $C(X)/O_\alpha$ is a local reduced ring which is classical and not fusible.

5. Unit fusible rings

In view of Proposition 2.22, every element in a left fusible right cohopfian ring can be expressed as the sum of a left zero-divisor and a unit in R. Proposition 3.1 shows that every element in a potent ring can be written as the sum of a zero-divisor and a unit. This leads to our investigation of the class of rings in which every element can be written as the sum of a left zero-divisor and a unit. We call such rings *left unit fusible*. Unit right fusible rings are defined analogously.

As we did before we consider an element-wise definition.

Definition 5.1. A nonzero element a of a ring R is called *left unit fusible* if it can be expressed as the sum of a left zero-divisor and a unit in R. A ring R is called *left unit fusible* if every nonzero element of R is left unit fusible. A ring which is both right unit and left unit fusible is called *unit fusible*.

Remark 5.2. Observe that a unit element is trivially unit fusible. Therefore, every division ring is unit fusible. Moreover, a domain is left unit fusible if and only if it is a field. This produces the statement that there are fusible rings which are not left unit fusible. Next, every nonzero idempotent is unit fusible: for each $e \neq 0$; $e = (1 - e) + (2e - 1)$ is a unit fusible representation of e. Moreover, a non-trivial clean representation of an element $a = e + u$ where $1 \neq e \in Id(R)$ and $u \in U(R)$, serves as a unit fusible representation too.

Recall that a ring R is called unit regular if for each $a \in R$ there exists $u \in U(R)$ such that $aua = a$. Examples of unit regular rings are semisimple rings and strongly regular rings including the commutative von Neumann regular rings. A characterization by Camillo and Khurana [9, Theorem 1] states that a ring R is unit regular if and only if every element a of R can be written as $a = e + u$ such that $aR \bigcap eR = 0$, where e is an idempotent and u a unit in R. Therefore, every element in a unit regular ring R can be expressed as a left (right) zero-divisor and a unit.

Proposition 5.3. *Every unit regular ring is unit fusible.*

An element $a \in R$ is said to be strongly π-regular in R if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$. A ring R is called strongly π-regular if every element is strongly π-regular. Examples of strongly π-regular are left (right) perfect rings, infinite direct product of fields and zero-dimensional commutative rings.

Proposition 5.4. *Let R be a ring. If a is a strongly π-regular and non-nilpotent, then a is unit fusible.*

Proof. Let $a \in R$. Since $0 \neq a$ is strongly π-regular, there exists $n \geq 0$ such that $a^n = eu$ where $e^2 = e$, $u \in U$ and e, u and a all commute by [28, Proposition 1]. Suppose that a is a zero-divisor, then $e \neq 1$. Now put $z = 1 - e$ and $r = a - (1 - e)$. Hence $z \in Z(R)$ and $r \in U(R)$ by [28, Theorem 1]. This means that $a = r + z$ is unit fusible. □

Proposition 5.5. *Let R is a unit left fusible ring. Then $Sing(R_R) = \mathfrak{J}(R) = 0$.*

Proof. Since every left unit fusible is left fusible then $Sing(R_R) = 0$ by Proposition 2.11. Now suppose that that $0 \neq a \in \mathfrak{J}(R)$. By our assumption there exist $z \in Z_l(R)$ and $u \in U(R)$ where $a = z + u$. Since z is not a unit it belongs to a left maximal M ideal of R. But $J(R)$ is the intersection of all left maximal ideals and this means that $a - z \in M$. Hence $u \in M$, a contradiction. □

Corollary 5.6. *Let R be an abelian clean ring. Then R is Jacobson semi-simple if and only if R is unit fusible.*

Proof. It is clear by [10, Theorem 3.1] and Theorem 3.9. □

Remark 5.7. We have been unable to determine whether an arbitrary von Neumann regular ring is unit fusible. Note that for a von Neumann regular ring R, strongly regular is equivalent to complementedness, while unit regular implies unit fusible. Nicholson and Varadarajan have shown that the endomorphism ring of a countable dimensional vector space over a division ring is a clean, von Neumann regular ring which is not unit regular. This has been generalized to continuous modules ([20]): if M is a continuous R-module, then the ring $S = \mathrm{End}_R(M)$ is a clean ring.

The divergence between unit fusible rings and clean rings is that a clean representation might involve the idempotent $e = 1$. Set $S = \mathrm{End}_D(V)$ where D is a division ring and V is any D-vector space. According to Theorem 4.5 of [20], using that $S = \mathrm{End}_D(V)$ is Jacobson semi-simple when D is a division ring, the only uniquely clean elements of $\mathrm{End}_D(V)$ are the central idempotents. Since non-zero idempotents are unit fusible, it follows that every uniquely clean element is unit fusible in S. The remaining elements have at least two clean representations, one of which is unit fusible. (This generalizes to any continuous module M_R for which $\mathrm{End}_R(M)$ is Jacobson semi-simple).

Also, by Theorem 4.4 of [20] if R is a von Neumann regular ring for which every corner ring of R is clean, then R is unit fusible.

It is clear that the class of left unit fusible rings is not closed under homomorphic images. As for products it should be clear (an appeal to Proposition 2.8) that the product of rings is unit fusible if and only if each factor is unit fusible.

Theorem 5.8. *If $1 = e_1 + e_2 + ... + e_n$ in a ring R where the e_i are orthogonal idempotents and each e_iRe_i is left unit fusible , then R is left unit fusible.*

Proof. By similar techniques used by Han and Nicholson in [15]. □

Corollary 5.9. *If R is a left unit fusible ring, then $M_n(R)$ is a left unit fusible ring.*

We now consider unit fusible commutative rings.

Proposition 5.10. *Let A be a commutative ring.*

(1) The rings $A[x]$ and $A[[x]]$ are never unit fusible.

(2) The ring of formal Laurent power series $A((x))$ is unit fusible if and only if A is unit fusible.

Proof. (1) Letting $\mathfrak{N}(A)$ denote the nilradical of A, i.e. the set of nilpotent elements, we recall that

$$U(A[x]) = \{r_0 + r_1x + ... + r_nx^n | r_0 \in U(A), r_i \in \mathfrak{N}(A) \text{ for } i = 1, 2, ..., n\}.$$

and

$$U(A[[x]]) = \{r_0 + r_1x + ... | r_0 \in U(A)\}.$$

Thus, if $x = f(x) + g(x)$ is a fusible representation in $A[[x]]$ with $f(x) \in U(A[[x]])$, then so is $g(x)$, a contradiction. Furthermore, since every zero-divisor of $A[x]$ is annihilated by an element of A we conclude that x is not fusible in $A[x]$. Hence both $A[x]$ and $A[[x]]$ are not unit fusible rings.

(2) Sufficiency: Recall that the units of $A((x))$ are those nonzero power series whose leading coefficient is a unit. So if $0 \neq f(x) \in A((x))$ then let $n \in \mathbb{Z}$ be its least

nonzero coefficient and write $f_n = u_n + z_n$ where $u_n \in A$ and $z_n \in Z(A)$. Construct the power series $g(x)$ by replacing f_n with u_n, which yields $g(x) \in U(A((x)))$. Since $f(x) - g(x) = z_n x^n \in Z(A((x)))$ we conclude that $A((x))$ is unit fusible.

Necessity: Suppose $A((x))$ is unit fusible and let $0 \neq z \in Z(A)$. Write $z = f(x) + g(x)$ with $f(x) \in U(A((x)))$ and $g(x) \in Z(A((x)))$. Write $f(x) = f_n x^n + f_{n+1}x^{n+1} + \cdots$ and $g(x) = g_m x^m + g_{m+1}x^{m+1} + \cdots$ with $n = \deg(f(x))$ and $m = \deg(g(x))$. If $n < 0$, then $m = n$ and $g_n = -f_n \in U(A)$. This argument shows more, that $m \leq n$. Thus, $0 \leq m \leq n$. If $n = 0$, then $z = f_0 + g_0$ is a fusible representation of z in A. So we are left with the case that $n > 0$ and $m = 0$. Observe then that $g(x) = z - f_n x^n - f_{n+1}x^{n+1} - \cdots$. Since $g(x)$ is a zero-divisor there is a $t(x) = t_0 + t_1 x + t_2 x^2 + \cdots$ such that $t_0 \neq 0$ and $g(x)t(x) = 0$. Expanding this product yields that $zt_n - f_n t_0 = 0$. Multiplying both sides by t_0 and using that $f_n \in U(A)$ produces that t_0 is a nilpotent element of $A((x))$, which as a consequence of the assumption is reduced. It follows that $t_0 = 0$, a contradiction. □

Proposition 5.11. *Let A be a commutative reduced ring. Then:*

(1) If A is local, then A is unit fusible if and only if A is a field.
(2) If A is an integral domain, then A is unit fusible if and only if A is a field.
(3) If A is classical, then A is fusible if and only if A is unit fusible.
(4) If A is a fusible ring, then $q_{cl}(A)$ is unit fusible.
(5) The maximal ring of quotients $Q_{max}(A)$ is unit fusible, since it is von Neumann regular.

Unsolved Questions.

(1) Is every abelian p.q. Baer ring fusible?
(2) Is every abelian feebly projectable ring fusible?
(3) Does a right complemented ring have a right classical quotient ring?
(4) Is the notion of complemented is left-right symmetric?
(5) If R has a right classical quotient ring which is von Neumann regular, is R fusible?
(6) If A is commutative and $q(A)$ is fusible, is A fusible?

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