THE GROUP RING $\mathbb{Z}_p C_q$ AND YE'S THEOREM

WARREN WM. MCGOVERN

ABSTRACT. We generalize Ye's Theorem which states that the group ring $\mathbb{Z}_{(p)}[C_3]$ is a semi-clean ring [13]. The proof provided here is more efficient; it is less algorithmic but has the feature that the following statement is evident: for distinct primes p, q, the group ring $\mathbb{Z}_{(p)}[C_q]$ is feebly clean if and only if the order of p modulo q is at least $\frac{q-1}{2}$.

1. Introduction

Throughout the article, R shall denote a commutative ring with identity. We are interested in the problem of studying the class of clean rings and its many variations and generalizations. In particular, we are interested in the study of certain commutative group rings RG and determining when they have certain properties. Throughout, we let $U(R)$, $Id(R)$, and $\mathfrak{J}(R)$ denote the group of units, the set of idempotents, and the Jacobson radical of R, respectively. Recall that an element $a \in R$ is called *periodic* if there are $k < n \in \mathbb{N}$ such that $a^k = a^n$; we denote the set of periodic elements by $\mathrm{Per}(R)$. Obviously, an idempotent is periodic. A brief history is in order.

Nicholson [12] defined a ring R to be *clean* if for each $r \in R$ there is an idempotent $e \in R$ such that $r - e \in U(R)$. Since their inception, clean rings have been a topic of research in both the commutative case as well as the non-commutative case. In the commutative context, clean rings can be seen as a bigger class of rings containing semi-perfect rings (e.g. local rings) as well as zero-dimensional rings. In fact, the local rings are precisely the indecomposable clean rings. Some essential references for commutative clean rings are [2] and [11].

For a given ring R and group G, we will denote the group ring by RG. Throughout the article, when G is cyclic of order n we shall write $G = C_n =< g >$. Also, $\mathbb{Z}_{(p)}$ denotes the localization of the integers at the prime ideal (p), while $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. We shall use $R[x]$ to denote the polynomial ring with indeterminate x.

In one of the earlier papers studying clean rings, Han and Nicholson [5] examined clean group rings. Some of the main results of [5] include the following: i) if R is boolean and G is locally finite, then RG is clean, ii) if R is semi-perfect, then RC_2 is clean, and iii) the ring $\mathbb{Z}_{(7)}C_3$ is not clean. This last result showed that it is not sufficient that R be clean and G to be finite for RG to be clean. In [10], it is shown that if the commutative group ring RG (equivalently, R is commutative and G is abelian) is a clean ring, then R is clean and G is torsion. (This is also discussed in [4].) The example $R = \mathbb{Z}_{(7)}C_3$ has been a driving force behind many of the results in this area. In [6] and [7], the classification of when $\mathbb{Z}_{(p)}C_n$ is provided. We recall the main theorem and its corollary, which puts the example $\mathbb{Z}_{(7)}C_3$ in perspective.

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Theorem 1.1. [7, Proposition 2.7] *Let $p \in \mathbb{N}$ be prime and $n \in \mathbb{N}$ with $n = p^k m$ where $p \nmid m$. The group ring $\mathbb{Z}_{(p)}C_n$ is a clean ring if and only if p is a primitive root of d for each positive divisor d of m.*

Corollary 1.2. [7, Corollary 2.8] *Let $p \in \mathbb{N}$ be prime. The group ring $\mathbb{Z}_{(p)}C_3$ is a clean ring if and only if $p \not\equiv 1$ modulo 3.*

Ye [13], in also studying rings of the form $\mathbb{Z}_{(p)}C_3$, generalized the notion of a clean ring by looking at representations of elements as sums of units and periodic elements calling such elements and rings *semi-clean*. Ye showed that for any prime p, $\mathbb{Z}_{(p)}C_3$ is a semi-clean ring. The proof involved showing that for each non-unit $\alpha \in \mathbb{Z}_{(p)}C_3$, one of the following elements is a unit: $\alpha \pm 1, \alpha \pm g$, $\alpha \pm g^2$ is a unit. The proof is very arithmetic and does not lend itself to an easy generalization to other primes; hence the goal here. On a side note, Ye proved as a separate case that $\mathbb{Z}_{(2)}C_3$ is semi-clean, but by Corollary 1.2 we know that the ring is clean.

A class of rings in between the classes of clean rings and semi-clean clean rings, is the class of weakly clean rings defined by Ahn and Anderson [1]. An element $a \in R$ is called *weakly clean* if there is a an idempotent $e \in R$ such that either $a - e \in U(R)$ or $a + e \in U(R)$. Since both $e, -e \in \operatorname{Per}(R)$ a weakly clean ring is semi-clean. Recall the following interesting characterization of weakly clean indecomposable rings. The following results will be useful in our proofs.

Theorem 1.3. [1, Theorem 1.3][2, Proposition 2(1)] *Suppose R is an indecomposable ring. R is weakly clean if and only if it is either local or it has exactly two maximal ideals and $2 \in U(R)$.*

Theorem 1.4. [1, Theorem 1.7] *Suppose $R = \Pi_i R_i$. Then R is weakly clean if and only if each R_i is weakly clean and at most one R_i is clean.*

Recently, Arora and Kundu [3] investigated another variant of clean rings.. The authors study a generalization of clean rings. They call the element $a \in R$ *feebly clean* if there are orthogonal idempotents $e_1, e_2 \in Id(R)$ such that $a-(e_1-e_2) \in U(R)$. Clearly, a weakly clean ring is feebly clean. Observe that since e_1 and e_2 are orthogonal, then $(e_1 - e_2)^3 = e_1 - e_2$ and so a feebly clean element is semi-clean. The authors mention that "all the examples of semi-clean rings we have seen in the literature turn out to be feebly clean rings." It is not presumptuous to say that this includes the group rings $\mathbb{Z}_{(p)}C_3$ as Ye's paper is listed in the references. Our main theorem in this short article will characterize when $\mathbb{Z}_{(p)}C_q$ is feebly clean, including the case when $q = 3$.

Lemma 1.5. [3] *Let R be an indecomposable ring, e.g. a domain. R is weakly clean if and only if R is feebly clean.*

Theorem 1.6. *Suppose $R = \Pi_i R_i$. Then R is feebly clean if and only if each R_i is feebly clean.*

Recall that for a natural $d \in \mathbb{N}$, $\Phi_d(x)$ denotes the d^{th}-cyclotomic polynomial; a polynomial with integer coefficients and degree $\phi(d)$, the Euler Totient function. Over $\mathbb{Q}$, each $\Phi_d(x)$ is irreducible and therefore also irreducible over any subring of $\mathbb{Q}$, in particular, over $\mathbb{Z}$ and $\mathbb{Z}_{(p)}$. We let ξ_d denote a primitive d^{th}-root of unity. Next, some needed lemmas.

Lemma 1.7. *For $n \in \mathbb{N}$, $x^n - 1 = \Pi_{d|n}\Phi_d(x)$ over $\mathbb{Q}$.*

Lemma 1.8. *Let R be a commutative ring. Then $R[x]/(x^n - 1) \cong RC_n$.*

Proposition 1.9. *For distinct primes $p, q \in \mathbb{N}$, $\mathbb{Z}_{(p)}C_q \cong \mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}[\xi_q]$.*

Proof. The ideals $(x-1)$ and $(x^{q-1}+\ldots+x+1)$ are co-maximal in $\mathbb{Z}_{(p))}[x]$, and so the result follows by the Chinese Remainder Theorem and Lemma 1.8. □

Corollary 1.10. *For distinct primes $p, q \in \mathbb{N}$, $\mathbb{Z}_{(p)}C_q$ is clean if and only if $\mathbb{Z}_{(p)}[\xi_q]$ is a local domain.*

Proof. So $\mathbb{Z}_{(p)}C_q \cong \mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}[\xi_q]$ is clean if and only if $\mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}[\xi_q]$ is clean, which in turn is clean if and only $\mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}[\xi_q]$ is clean. But this ring is a subring of $\mathbb{C}$ and hence a domain. Since domains are indecomposable, $\mathbb{Z}_{(p)} \times \mathbb{Z}_{(p)}[\xi_q]$ is clean if and only if is local. □

Combining Theorem 1.4, Theorem 1.6, and Theorem 1.5 yields the following result.

Theorem 1.11. *Let $p, q \in \mathbb{N}$ be distinct primes. When p is odd, the following statements are equivalent.*

1. *$\mathbb{Z}_{(p)}C_q$ is weakly clean.*
2. *$\mathbb{Z}_{(p)}C_q$ is feebly clean.*
3. *$\mathbb{Z}_{(p)}[\xi_q]$ has at most 2 maximal ideals.*

When $p = 2$, we gather that $\mathbb{Z}_{(2)}C_q$ is feebly clean if and only if it is weakly clean if and only if it is clean.

Example 1.12. The ring $\mathbb{Z}_{(2)}C_7$ is not feebly clean since 2 is not a primitive root of 7.

As one can see the goal is to characterize the number of maximal ideals of

$$\mathbb{Z}_{(p)}[\xi_q] \cong \mathbb{Z}_{(p)}[x]/(\Phi_q(x)).$$

Recall the following theorem that will help us understand the number of maximal ideals in $\mathbb{Z}_{(p)}[x]/(\Phi_q(x))$.

Theorem 1.13. *For distinct primes $p, q \in \mathbb{N}$, the Jacobson radical of $\mathbb{Z}_{(p)}C_q$ equals MC_q where $M = p\mathbb{Z}_{(p)}$ is the maximal ideal of $\mathbb{Z}_{(p)}$. Consequently, the maximal ideals of $\mathbb{Z}_{(p)}C_q$ are in 1-1 correspondence with the maximal ideals of $\mathbb{Z}_p C_q$.*

Proof. Karpilovsky's Theorem [8, Theorem] states that for an abelian group G and a (unital) commutative ring R

$$J(RG) = \begin{cases} \mathfrak{J}(R)G + < r(g-1) | g \in G_q, r \in \mathfrak{J}_q(R) \text{ for some } q \in P(G), & \text{if } G \text{ is torsion} \\ \mathfrak{N}(R)G + < r(g-1) | g \in G_q, r \in \mathfrak{N}_q(R) \text{ for some } q \in P(G), & \text{otherwise.} \end{cases}$$

In this theorem, $P(G)$ is the set of primes which are orders of elements in G, e.g. for the prime q, $P(C_q) = \{q\}$. Next, G_q is the q-primary component of G, $\mathfrak{J}_q(R) = \{r \in R | qr \in \mathfrak{J}(R)\}$, and $\mathfrak{N}_q(R) = \{r \in R | qr \in \mathfrak{N}(R)\}$. For distinct primes p, q, the element $q \in \mathbb{Z}_{(p)}$ is invertible and so for the ring $\mathbb{Z}_{(p)}C_q$,

$$\mathfrak{J}_q(\mathbb{Z}_{(p)}) = \mathfrak{J}(\mathbb{Z}_{(p)}) = M \text{ and } \mathfrak{N}_q(\mathbb{Z}_{(p)}) = \mathfrak{N}(\mathbb{Z}_{(p)}) = \{0\}.$$

Therefore, $\mathfrak{J}(\mathbb{Z}_{(p)}C_q) = MC_q$. Then, since $\mathbb{Z}_{(p)}C_q/MC_q \cong \mathbb{Z}_p C_q$, so the remaining statements follow. □

So to understand the maximal ideals of $\mathbb{Z}_p C_q$ it suffices to understand the maximal ideals of $\mathbb{Z}_p[x]/(\Phi_q(x))$, a principal ideal domain. The number of maximal ideals of $\mathbb{Z}_p[x]$ containing $\Phi_q(x)$ corresponds to the number of irreducible factors of $\Phi_q(x)$ in $\mathbb{Z}_p[x]$; we are using that $\Phi_q(x)$ is separable over $\mathbb{Z}_p$. Here we take a theorem from number theory. Recall that $\text{ord}_q(p)$ denotes the order of p modulo q.

Theorem 1.14. [9, Theorem 2.47] *Let p, q be distinct primes. The q^{th}-cyclotomic polynomial can be factored into $\frac{q-1}{d}$ distinct monic irreducible polynomials of the same degree d over $\mathbb{Z}_p$, where $d = \operatorname{ord}_q(p)$.*

Corollary 1.15. *The number of maximal ideals of $\mathbb{Z}_p[x]/\Phi_q(x)$ equals $\frac{q-1}{d}$ where $d = \operatorname{ord}_q(p)$.*

The following theorem is a consequence of the previous results and is the main characterization of when $\mathbb{Z}_{(p)}C_q$ is a feebly clean ring. One particular application is that $\mathbb{Z}_{(p)}C_3$ is a feebly clean ring for all primes p.

Theorem 1.16. *Let $p, q \in \mathbb{Z}$ be distinct odd primes. The ring $\mathbb{Z}_{(p)}C_q$ is feebly clean if and only if $\operatorname{ord}_q(p) \geq \frac{q-1}{2}$.*

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(Warren Wm. McGovern) H. L. Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458

E-mail address: warren.mcgovern@fau.edu