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Commutative weakly nil clean unital rings

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ABSTRACT

We define the concept of a *weakly nil clean* commutative ring which generalizes Diesel's [11] notion of a *nil clean* commutative ring, and investigate this class of rings. We obtain some fundamental properties. In particular, it is proved that these rings are *clean*. We also consider the questions of when the Nagata ring as well as the group ring is weakly nil clean.

1. Introduction and background

Throughout the present paper all rings considered, unless otherwise noted, shall be assumed to be commutative and possess an identity. Our notation and terminology shall follow [10] and [15]. For instance, for such a ring R, $U(R)$ denotes the group of all units

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of R, $N(R)$ is the nil-radical of R, $J(R)$ is the Jacobson radical of R, and $Id(R)$ is the set of all idempotents of R. It is a known fact that $U(R) + N(R) = U(R)$. If $1 - x \in U(R)$, then x is called *quasi-regular*. We denote the set of quasi-regular elements of R by $\mathfrak{Q}(R)$. We denote the set of all maximal ideals of R by $\mathrm{Max}(R)$.

W.K. Nicholson [23] introduced the notion of a clean ring. Over the last ten to fifteen years there has been an explosion of interest in this class of rings as well as the many generalizations and variations (see [14] too). For a comprehensive history of clean rings up until its publication the reader is urged to read [19].

Definition 1.1. A ring R is called *clean* if for every $r \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $r = u + e$.

Recently, A. Diesl [11] modified the definition of a clean ring and obtained an interesting new concept he called *nil clean*. In his article he proved many fundamental properties as well as developed a general theory of nil clean rings; his interest, as was Nicholson's, was in the context of non-commutative rings. For other recent articles related to nil clean rings see [5,7,8].

Definition 1.2. A ring R is said to be *nil clean* if for each $r \in R$ there are $n \in N(R)$ and $e \in Id(R)$ such that $r = n + e$.

By rephrasing Corollary 3.20 of [11], we can give a nice characterization of nil clean rings.

Proposition 1.3. *The ring R is nil clean if and only if $R/N(R)$ is a boolean ring.*

The notion of *uniquely clean rings* was firstly defined in [4] in the commutative case as those rings in which every element is uniquely the sum of a unit and an idempotent. Later on, in [24] the authors study such arbitrary rings, again calling them *uniquely clean*; notice that this uniqueness is tantamount to the existence of a unique idempotent with the given sum property (see [6]). In a subsequent paper, [25], the same authors coined the term *semi-boolean ring* as a (general) ring I satisfying the condition that for all $r \in I$ there is an $e \in Id(R)$ and $j \in J(R)$ such that $r = j + e$; this was done in the context of rings not necessarily commutative nor possessing an identity. In Example 25 of the article they show that a uniquely clean ring is semi-boolean. Then Proposition 26 clarifies the situation for commutative rings with identity. Specifically, such a ring is uniquely clean if and only it is semi-boolean. Thus, we arrive at our first result whose proof follows from the definition of semi-boolean and the equivalence of semi-boolean to uniquely clean in our contexts.

Proposition 1.4. *Let R be a ring. If R is nil clean, then it is uniquely clean and therefore clean. The converse is not true as witnessed by the ring $R = \mathbb{Z}_{(2)}$, the localization of the integers at the prime 2.*

Remark 1.5. At this point we could consider the appropriate definition of a uniquely nil clean ring. However, we would like to demonstrate that every nil clean ring is actually uniquely nil clean (see also Corollary 3.8 from [11] and the surrounding discussion there).

Proposition 1.6. *Let R be a ring. If R is nil clean, then for every $r \in R$ there exists a unique nilpotent, say $n \in N(R)$, such that $r - n \in Id(R)$. In addition, each element in R has a unique nil clean expression.*

Proof. Suppose R is nil clean. We begin by showing that every idempotent has a unique nil clean expression. To that end, let $e \in Id(R)$ and let $n \in N(R)$ and $f \in Id(R)$ be such that $e = n + f$. We claim that $n = 0$ and $f = e$. Since $n \in N(R)$ there is a $j \in \mathbb{N}$ such that $n^j = 0$. By the Binomial Theorem together with the fact that e is an idempotent we have

$$e = e^j = (n+f)^j = n^j + \binom{j}{1} n^{j-1} f + \binom{j}{2} n^{j-2} f^2 + \cdots + \binom{j}{j-1} n f^{j-1} + f^j.$$

Set $n' = \binom{j}{1} n^{j-1} + \binom{j}{2} n^{j-2} + \cdots + \binom{j}{j-1} n \in N(R)$ and observe that, since f is idempotent and $n^j = 0$, it follows that

$$e = \left(n' + 1\right) f.$$

Since $n' \in N(R)$, we know that $u = n' + 1 \in U(R)$. From here one can deduce that $e = f$ because idempotents that are associate are equal. We include a proof for completeness-sake. On the one hand, $ef = uf^2 = uf = e$. On the other hand $f = u^{-1}e$ so that $ef = u^{-1}e^2 = u^{-1}e = f$. It follows that $e = f$ and hence $n = 0$.

Next, suppose $r \in R$ has two nil clean expressions, say $r = n_1 + e_1 = n_2 + e_2$ for $n_1, n_2 \in N(R)$ and $e_1, e_2 \in Id(R)$. Then $e_1 = (n_2 - n_1) + e_2$. Furthermore, from what we have already proved above, we deduce that $e_1 = e_2$ and $n_2 - n_1 = 0$, as wanted. □

Ahn and Anderson [1] generalized the notion of clean ring in the following manner.

Definition 1.7. The ring R is called *weakly clean* if every $r \in R$ can be written as $r = u + e$ or $r = u - e$ for some $u \in U(R)$ and $e \in Id(R)$.

A new interesting criterion for a ring to be weakly clean is the following one (cf. [9]). The ring R is weakly clean if and only if for any $x \in R$ there exists $e \in Id(R)$ such that $e \in xR$ and either $1 - e \in (1 - x)R$ or $1 - e \in (1 + x)R$.

The main instrument in our exploration is the following generalization of Definition 1.2 which uses Ahn and Anderson's idea.

Definition 1.8. The ring R is said to be *weakly nil clean* if each $r \in R$ can be written as $r = n + e$ or $r = n - e$, where $n \in N(R)$ and $e \in Id(R)$.

Recall that an element in a ring, say $r \in R$, is called *unipotent* if it can be written as $1+b$ for some nilpotent $b \in R$. It is patent to check that any ring is weakly nil clean if and only if every element can be written as either the sum of a nilpotent and an idempotent, or of a unipotent and an idempotent.

In [1], an example is given of a weakly clean ring that is not clean. With regards to weakly nil clean rings, $\mathbb{Z}_3$ is an example of a weakly nil clean ring that is not nil clean. Since a reduced ring is nil clean if and only if it is boolean, it follows that the only domain that is nil clean is $R = \mathbb{Z}_2$. We will mention some other properties of the class of weakly nil clean rings.

Proposition 1.9. *Let R be a ring. Then the following statements are true:*

(i) *The class of weakly nil clean rings is closed under homomorphic images. In particular, R is weakly nil clean if and only if R/I is weakly nil clean, provided that I is a nil-ideal of R.*
(ii) *The class of weakly nil clean rings is not closed under finite products; e.g. $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly nil clean.*
(iii) *A reduced indecomposable ring is weakly nil clean if and only if it is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In particular, any weakly nil clean domain is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*
(iv) *A weakly nil clean ring is zero-dimensional. Hence a weakly nil clean ring is clean.*

Proof. The first part of (i) is clear since the homomorphic image of a nilpotent (resp. an idempotent) element is again a nilpotent (resp. an idempotent). As for the second part of (i), let $r \in R$. Write $r + I = n + I + e + I$ or $r + I = n + I - e + I$, where n is a nilpotent and e is an idempotent. Therefore, $r - n - e =\in I$ or $r - n + e =\in I$. It follows immediately that $r - e = n + i$ or $r + e = n + i$, where $i \in I$. Either way $n + i$ is again a nilpotent. So, R is weakly nil clean, as desired.

We leave the verification of (ii) to the interested reader, which is not too hard.

For (iii) notice that we are saying that 0 is the only nilpotent element and 0 and 1 are the only idempotents. That a ring is weakly nil clean in this case only leaves us with three possibilities for elements in R: $0, 1, -1$.

As for (iv) let R be a weakly nil clean ring and P a prime ideal of R. Then by (i) and (iii), the quotient R/P is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$, and so P is a maximal ideal. That zero-dimensional rings are clean is well known (cf. [22]). □

Observe that if a ring has characteristic 2 then, trivially, the ring is weakly nil clean if and only if it is nil clean. We can generalize this to the following statement.

Proposition 1.10. *Let R be a ring with identity. Then R is weakly nil clean and $2 \in N(R)$ if and only if R is nil clean.*

Proof. The sufficiency is straightforward as it is known from Proposition 3.14 in [11] that $2 \in N(R)$ in a nil clean ring. Conversely, suppose that R is weakly nil clean and $2 \in N(R)$. Since Proposition 1.9 (i) allows us to deduce that R is weakly nil clean if and only if $R/N(R)$ is weakly nil clean, we may assume that $2 = 0$ in the quotient $R/N(R)$, i.e., $\mathrm{char}(R/N(R)) = 2$. It is therefore obvious that $R/N(R)$ is boolean. Thus, we cite Proposition 1.3 to get that R is nil clean. □

Remark 1.11. Note that 2 being a nilpotent, however, does not imply that weakly nil cleanness coincides with weakly cleanness. In fact, there is even a clean ring of characteristic 2 which is not weakly nil clean. For example, in terms of Section 2 below, such a ring is the group ring $\mathbb{Z}_2[C_3]$ – it is finite and hence clean. In the next section we prove a result (namely Corollary 2.2) from which we can conclude that $\mathbb{Z}_2[C_3]$ is not weakly nil clean.

Proposition 3.14 of [11] states that 2 is nilpotent in any nil clean ring. This is not true in a weakly nil clean ring, but the following does hold.

Proposition 1.12. *If R is a weakly nil clean ring, then $6 \in N(R)$.*

Proof. If $6 = 0$, we are done, so assume that $6 \neq 0$. Write either $2 = n + e$ or $2 = n - e$ where $n \in N(R)$ and $e \in Id(R)$. In the first case, $1 - e = n - 1$ is both an idempotent and a unit, hence $n - 1 = 1$ gives that $2 = n$ is nilpotent, and hence so is 6, as desired.

In the second case, $1 + e = n - 1$, whence $1 + 3e = (1 + e)^2 = (n - 1)^2 = n^2 - 2n + 1$, that is, $3e = n^2 - 2n$ is a nilpotent element. Furthermore, multiplying both sides of $2 = n - e$ by 3, we derive that $6 = 3n - 3e = 5n - n^2$ is again a nilpotent element, as asserted. □

Some authors have been interested in when a ring R is a union of subsets consisting of clean elements. In this way, Theorem 14 of [4] characterizes rings R for which $R = Id(R) \cup U(R)$ as either fields or boolean rings. Likewise, Theorem 3.1 of [16] demonstrates that $R = U(R) \cup \mathfrak{Q}(R)$ if and only if R is a local ring, and shows that $R = Id(R) \cup \mathfrak{Q}(R)$ if and only if R is a division ring. In Theorem 1.12 of [1] the authors looked at the condition of a ring for which $R = Id(R) \cup -Id(R)$. We next recall this last theorem (adding the fourth condition). This places the weakly nil clean condition in the appropriate context.

Theorem 1.13. *Let R be a reduced ring. The following statements are equivalent.*

(i) $R = Id(R) \cup -Id(R)$.
(ii) *R is either boolean, isomorphic to $\mathbb{Z}_3$, or isomorphic to $B \times \mathbb{Z}_3$ for some boolean ring B.*
(iii) *For all $x \in R$, either $x^2 = x$ or $x^2 = -x$.*
(iv) *The ring R is weakly nil clean.*

Proof. For a reduced ring R, being a weakly nil clean is easily seen to be equivalent to $R = Id(R) \cup -Id(R)$. □

Corollary 1.14. *Suppose R is a reduced weakly nil clean ring. Any subring of R is also reduced weakly nil clean.*

Proof. Let S be a subring of R, a reduced weakly nil clean ring. By Theorem 1.13 for any $s \in S$, and hence $s \in R$, we know that either $s \in Id(R)$ or $-s \in Id(R)$. But then either $s \in Id(S)$ or $-s \in Id(S)$. Consequently, $S = Id(S) \cup -Id(S)$. □

Our next result will be used repeatedly throughout the rest of the article.

Proposition 1.15. *If R is a reduced weakly nil clean ring, then $U(R)$ is a group of at most two elements.*

Proof. Given $u \in U(R)$ we may write $u = e$ or $u = -e$. In the first case $u^2 = u$, i.e., $u(u-1) = 0$ which by multiplying both sides with u^{-1} forces that $u = 1$. In the second situation $u^2 = -u$, that is, $u(u+1) = 0$. Once again multiplying by u^{-1} allows us to deduce that $u = -1$, as desired. □

As a valuable consequence, we obtain a complete characterization of units in weakly nil clean rings.

Corollary 1.16. *Let R be a ring. If R is weakly nil clean, then $U(R) = N(R) \pm 1$.*

Proof. By Proposition 1.15 we have $U(R/N(R)) = \{1+N(R), -1+N(R)\}$, it now easily follows that for any $r \in U(R)$ it must be that $r \pm 1 \in N(R)$, as required. □

We can now characterize weakly nil clean rings in general.

Theorem 1.17. *Let R be a ring. The following statements are equivalent:*

(i) *R is a weakly nil clean ring.*
(ii) *R is zero-dimensional and there is at most one maximal ideal of R, say M, which satisfies $R/M = \mathbb{Z}_3$ while for all other maximal ideals N of R we have $R/N = \mathbb{Z}_2$.*
(iii) *$R/N(R)$ is isomorphic to either a boolean ring, or $\mathbb{Z}_3$, or the product of two such rings.*
(iv) *$J(R)$ is nil and $R/J(R)$ is isomorphic to either a boolean ring, or $\mathbb{Z}_3$, or the product of two such rings.*

Proof. That (i) and (iii) are equivalent follows directly from Proposition 1.9 (i) combined with Theorem 1.13 (ii).

That (iii) and (iv) are equivalent follows from the fact that in either case $J(R) = N(R)$ because $J(R/N(R)) = J(R)/N(R)$.

Thus we are left showing that (i) and (ii) are equivalent. For the necessity direction let R be a weakly nil clean ring. By Proposition 1.9 (iv), R is zero-dimensional. For any maximal ideal M, we know that either $R/M \cong \mathbb{Z}_2$ or $R/M \cong \mathbb{Z}_3$. By the Chinese Reminder Theorem, for any two maximal ideals M and N of R, we know that $R/(M \times N) \cong (R/M) \times (R/N)$ is weakly nil clean. We can apply Proposition 1.9 (ii) to finish the proof.

Next, suppose that R is zero-dimensional (hence $J(R) = N(R)$) and that there is at most one maximal ideal of R, say M, which satisfies $R/M \cong \mathbb{Z}_3$ while for all other maximal ideals N of R we have $R/N = \mathbb{Z}_2$. It follows that $R/N(R) = R/J(R)$ is embeddable inside of $\Pi_{M\in \mathrm{Max}(R)}(R/M)$; which is isomorphic to either a product of copies of $\mathbb{Z}_2$ or a product of copies of $\mathbb{Z}_2$ and one copy of $\mathbb{Z}_3$. In both cases we know that $R/N(R)$ is a subring of a reduced weakly nil clean ring and hence weakly nil clean by Corollary 1.14. □

As an immediate consequence, we yield:

Corollary 1.18. *A ring R is weakly nil clean if and only if $R/N(R)$ is weakly nil clean if and only if $R/J(R)$ is weakly nil clean and $J(R)$ is nil.*

By considering Theorem 1.12 and Corollary 1.13 of [1], we are able to say something interesting.

Proposition 1.19. *Let R be a ring. Then $R = N(R) \cup Id(R) \cup -Id(R)$ if and only if either $R = Id(R) \cup -Id(R)$ or $R = \mathbb{Z}_4$.*

Proof. For the sufficiency we simply need to show that $\mathbb{Z}_4 = N(\mathbb{Z}_4) \cup Id(\mathbb{Z}_4) \cup -Id(\mathbb{Z}_4)$. But this is clear by a direct calculation.

As to the necessity, let R be a ring for which $R = N(R) \cup Id(R) \cup -Id(R)$. If R is reduced, then $R = Id(R) \cup -Id(R)$ and we are done. So, we treat the case when R is not reduced. Let $0 \neq n \in N(R)$. We consider the three possible cases for the unit $1 - n \in N(R) \cup Id(R) \cup -Id(R)$. The unit cannot be nilpotent. If the unit is idempotent then $1 - n = 1$ and thus $n = 0$. In the last case we have that the negative of the unit, that is again a unit, is idempotent and so $1 - n = -1$. It follows now that $n = 2$. Since n was assumed to be any nonzero nilpotent element, we conclude that $N(R) = \{0, 2\}$ and $0 \neq 2$. Since $2^2 = 4 \in N(R)$ we deduce that either $4 = 0$ or $4 = 2$. The only viable option is that $4 = 0$. Moreover, we conclude that the characteristic of R is 4.

Next, let $u \in U(R)$. Then $u \notin N(R)$ and so either u is an idempotent, whence $u = 1$, or otherwise $u = -e$ for some $e \in Id(R)$. But then $-u$ is both an idempotent and a unit and hence $-u = 1$, i.e., $u = -1$. It follows that $U(R) = \{1, -1\}$. Since $\mathrm{char}(R) = 4$, we infer that $1 \neq -1$. So we have produced at least four elements in R that are $0, 1, 2, 3$.

If $x \in R \setminus \{0, 1, 2, 3\}$ then $x \in Id(R)$ or $x \in -Id(R)$. Since $\{0, 1, 2, 3\}$ is closed under negation, it follows that if R has more than 4 elements, then there is an idempotent $e \in Id(R)$ such that $e \neq 0, 1$.

Consider $1 + e$ for any idempotent $e \neq 0, 1$. If $1 + e = 0$, then $e = -1$ is idempotent which has already been established not to be the case. If $1 + e = 2$, then $e = 1$, which we are assuming is not the case. If $1 + e \in -Id(R)$, then there exists $f \in Id(R)$ such that $1 + e = -f$. Therefore, $3 - e = f = f^2 = (3 - e)^2 = 1 - 2e + e^2 = 1 - e$. This yields that $1 = 3$ which is not the case. Thus, for any idempotent e different that 0 and 1, one must have $1 + e \in Id(R)$. Also, notice that in this case $1 + e \neq 0, 1$. So we can make the same argument to conclude that $1 + (1 + e) = 2 + e$ is an idempotent. This last statement implies that $2 + e = (2 + e)^2 = 4 + 4e + e = e$ and so $0 = 2$, which gives the desired contradiction. Consequently, $R = \{0, 1, 2, 3\} = \mathbb{Z}_4$, as promised. □

Remark 1.20. Notice that the only difference between the equalities $R = Id(R) \cup -Id(R)$ and $R = N(R) \cup Id(R) \cup -Id(R)$ is the inclusion of $R = \mathbb{Z}_4$; all of which are weakly nil clean. It follows that not every weakly nil clean ring satisfies this equation, e.g., $R = \mathbb{Z}_8$.

We next consider the generalization to $R = J(R) \cup Id(R) \cup -Id(R)$. The result is somewhat surprising.

Proposition 1.21. *Let R be a ring. Then $R = J(R) \cup Id(R) \cup -Id(R)$ if and only if $R = N(R) \cup Id(R) \cup -Id(R)$.*

Proof. The sufficiency is trivial. We prove the necessity. To that aim, we first observe that $R \neq J(R)$ and $J(R) \cap \pm Id(R) = \{0\}$; in fact, given $j \in J(R) \cap Id(R)$, we have that $1 - j.j = 1 - j \in U(R) \cap Id(R) = \{1\}$, so that $j = 0$. In the other case, for $j \in J(R) \cap -Id(R)$, we have $j = -e$ where $e \in Id(R)$, whence $1 - j.(-j) = 1 + j = 1 - e \in U(R) \cap Id(R) = \{1\}$ and thus $j = 0$, as required.

To begin, letting $0 \neq j \in J(R)$, we obtain $1 - j \in U(R)$ and so $1 - j \in J(R) \cup Id(R) \cup -Id(R)$ which may be distributed into three possible cases:

Case 1. $1 - j \in J(R)$ gives that $1 \in J(R)$ and hence $R = J(R)$ with $1 - 1.1 = 0 \in U(R)$ which forces that $1 = 0$, a contradiction.

Case 2. $1 - j \in Id(R)$ whence $1 - j \in U(R) \cap Id(R) = \{1\}$ yields $j = 0$, contrary to our assumption.

Case 3. $1 - j \in -Id(R)$ whence $j - 1 \in U(R) \cap Id(R) = \{1\}$ implies $j - 1 = 1$, that is, $j = 2$ and so $J(R) = \{0, 2\}$. Since $4 = 2^2 \in J(R)$, one sees that $4 = 0$ or $4 = 2$, i.e., $4 = 0$ or $2 = 0$ (or respectively $-2 \in J(R)$ ensures that either $-2 = 2$ or $-2 = 0$, that is, $4 = 0$ or $2 = 0$). Anyway, the only possible case is $4 = 0$ because $2 \neq 0$. That is why $\text{char}(R) = 4$.

Next, assuming that $u \in U(R)$, we observe that $u \notin J(R)$ (for if $u \in J(R)$, it follows that $1 - u.u^{-1} = 1 - 1 = 0 \in U(R)$ which is manifestly untrue). Therefore, either $u \in Id(R)$, hence $u = 1$, or $u \in -Id(R)$. In the latter situation we write $u = -e$, where $e \in Id(R)$. Then $-u \in Id(R) \cap U(R)$ and $-u = 1$, i.e., $u = -1$. Consequently, $U(R) = \{1, -1\}$. Since $4 = 0$ we have $1 \neq -1$ and $U(R) = \{1, 3\}$ where $-1 = 3$. Finally, $R \supseteq \{0, 1, 2, 3\}$. Now, given $x \in R \setminus \{0, 1, 2, 3\}$, we see that $x \notin J(R) = \{0, 2\}$, hence $x \in Id(R)$ or $x \in -Id(R)$. The rest of the proof follows similarly to that of Proposition 1.19. We observe that $J(R) = \{0, 2\} = N(R)$. □

2. Weakly nil clean group rings

Throughout the current section, we assume that G is a multiplicative abelian group and R is a commutative ring with identity. We let $R[G]$ denote the group ring of G over R. Let C_p denote the multiplicative cyclic group of order p. For a fixed prime p, we denote the p-component of G by G_p.

Commutative clean group rings were investigated in [20,16,17], respectively. On the other hand, it was proved in [21] that $R[G]$ is nil clean if and only if R is nil clean and $G = G_2$, that is, R is nil clean and G is a 2-group. We now characterize when $R[G]$ is a weakly nil clean commutative group ring.

Theorem 2.1. *Let R be a ring and G a group. The group ring $R[G]$ is weakly nil clean if and only if exactly one of the following three conditions is satisfied:*

(i) *R is nil clean and G is a non-trivial torsion 2-group;*
(ii) *$R/N(R) \cong \mathbb{Z}_3$ and G is a non-trivial torsion 3-group;*
(iii) *R is weakly nil clean and G is trivial.*

Proof. *Necessity.* Suppose $R[G]$ is weakly nil clean. Thus, $R[G]$ is clean by Proposition 1.9 (iv) and in view of Proposition 1.9 (i) the ring R is weakly nil clean being a homomorphic image of $R[G]$. Therefore, G is a torsion group by virtue of Proposition 2.7 of [20]. Consider $R/N(R)$. Either $R/N(R)$ is boolean, $R/N(R) \cong \mathbb{Z}_3$, or $R/N(R) \cong B \times \mathbb{Z}_3$ for some non-trivial boolean ring B. We prove that these three mutually exclusive cases lead to the three conditions in the theorem.

Case 1. Suppose $R/N(R)$ is boolean; whence R is nil clean. Thus there is a maximal ideal of R, say M, such that $R/M \cong \mathbb{Z}_2$. Since $R[G]$ is weakly nil clean, with Proposition 1.9 (i) at hand so is $\mathbb{Z}_2[G/G_2]$. If $G_2 \neq G$, then this group ring possesses more than two units (since $|G/G_2| \geq 3$) even though it is reduced, a contradiction with Proposition 1.15. Therefore, $G = G_2$ and so G is a torsion 2-group.

Case 2. Suppose $R/N(R) \cong \mathbb{Z}_3$. Since $R[G]$ is weakly nil clean, again an appeal to Proposition 1.9 (i) assures that so is $\mathbb{Z}_3[G/G_3]$, a reduced ring with more than two units. Therefore, Proposition 1.15 implies that $G = G_3$ is a torsion 3-group.

Case 3. $R/N(R) \cong B \times \mathbb{Z}_3$ for some non-trivial boolean ring B. It follows that both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are homomorphic images of R. Since $R[G]$ is weakly nil clean, in view of Proposition 1.9 (i), so are $\mathbb{Z}_2[G/G_2]$ and $\mathbb{Z}_3[G/G_3]$. In both cases we arrive that $G = G_2 = G_3$, which means G is trivial.

Sufficiency. We go case by case and show that in each case $R[G]$ is weakly nil clean. The third case is trivial. The first case follows from Theorem 2.11 of [21]; in particular $R[G]$ is nil clean. So assume that $R/N(R) \cong \mathbb{Z}_3$ and G is a torsion 3-group. It is known that in this case R is local with unique maximal ideal $N(R)$ and $\mathrm{char}(R/N(R)) = 3$. By Theorem 19.1 of [13], it follows that $R[G]$ is a local ring. In the proof it was shown that the unique maximal ideal of $R[G]$ is the ideal $N(R)[G]+I$ where I is the augmentation ideal (that is, I is generated by the set $\{1-g : g \in G\}$). Moreover, it was demonstrated there that $R[G]/(N[G]+I) \cong \mathbb{Z}_3$. Notice that the ideal $N[G]+I$ is a nil ideal (see Theorem 9.1 of [13]) and hence it follows that $R[G]$ is weakly nil clean by Proposition 1.9 (i) (see also Theorem 1.17). □

As an immediate consequence, we yield:

Corollary 2.2.

(a) *Both rings, $\mathbb{Z}_3[G_2]$ and $\mathbb{Z}_2[G_3]$, are not weakly nil clean.*
(b) *Both rings, $\mathbb{Z}_2[G_2]$ and $\mathbb{Z}_3[G_3]$, are weakly nil clean.*

3. The Nagata ring

As usual, $R[X]$ denotes the polynomial ring. For $f \in R[X]$ the content ideal of f, $c(f)$, is the (finitely generated) ideal of R generated by the coefficients of f. The *Nagata ring* of R, denoted as $R(X)$, is the localization of $R[X]$ at the multiplicative set $U = \{f \in R[X] : c(f) = R\}$. Another interesting localization of $R[X]$ is the one induced by the multiplicative set of monic polynomials. We denote this latter ring by $R\langle X\rangle$. It is known that $R[X] \subseteq R\langle X\rangle \subseteq R(X)$ (see, for instance, [3]). It is also well known that $R[X]$ is never clean (cf. [14]), and therefore $R[X]$ is never weakly nil clean. In [22] the authors classified when the rings $R(X)$ and $R\langle X\rangle$ are clean. Interestingly, we show that neither ring $R(X)$ nor $R\langle X\rangle$ is weakly nil clean.

Proposition 3.1. *For any ring R, neither $R(X)$ nor $R\langle X\rangle$ is weakly nil clean.*

Proof. We first show that $R(X)$ cannot be weakly nil clean. By Proposition 1 of [2] the nil radical of $R(X)$ is $N(R)R(X)$. Moreover, it is not hard to check that $R(X)/N(R)R(X) \cong (R/N(R))(X)$. Without loss of generality, we assume that R is reduced. The elements $f_n(X) = X^n$ are all invertible elements of $R(X)$ for each $n \in \mathbb{N}$ and, therefore, by Proposition 1.15, $R(X)$ is not weakly nil clean.

Next, by way of contradiction assume that $R\langle X\rangle$ is weakly nil clean. Then, by Proposition 1.9 (i), so is R being a homomorphic image of $R\langle X\rangle$. It follows now that R is zero-dimensional. By Theorem 17.11 of [15] or Theorem 8 of [22], $R(X) = R\langle X\rangle$ so that $R(X)$ is a weakly nil clean ring, a contradiction. □

It is a well-known fact that $R[[X]]$ is (weakly) clean if and only if R is (weakly) clean (see, for example, [1]). However, this equivalence is not preserved for weakly nil cleanness. Even much more, the following is true:

Proposition 3.2. *Let R be a ring. The power series ring $R[[X]]$ is never weakly nil clean.*

Proof. It can be easily checked that the element X is never weakly nil clean. In fact, it is well known that $Id(R[[X]]) = Id(R)$. Moreover, it follows from [12] that if $a_0 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots \in N(R[[X]])$, then $a_0, a_1, a_2, \cdots, a_n, \cdots \in N(R)$. So, writing $X = \pm b_0 + a_1X$, where b_0 is an idempotent and a_1X is a nilpotent, it follows that $b_0 = 0$ while $1 = a_1 \in N(R)$ which is impossible. □

4. Left-open problems

In closing, we list some unanswered questions.

Problem 1. Is an arbitrary reduced weakly nil clean ring necessarily commutative?

Problem 2. Find a criterion for when the full matrix ring $M_n(R)$ is a weakly nil clean ring over an arbitrary (possibly noncommutative) ring R.

For a classification of when $M_n(R)$ is nil clean the interested reader can see in [5] or [18].

Problem 3. Characterize *uniquely weakly nil clean rings*, that is, those weakly nil clean rings in which the existing idempotent is unique. Does it follow that uniquely weakly nil clean rings are (uniquely) nil clean?

Problem 4. Characterize those rings for which every proper homomorphic image is (weakly) nil clean.

Problem 5. Characterize those rings R for which each element is a sum or a difference of elements from $J(R)$ and $Id(R)$, respectively.

Clearly these rings form a class that (properly) contains both the classes of semi-boolean rings introduced in [25] and weakly nil clean rings explored above. Such rings could be called *weakly semi-boolean*.

Problem 6. How much of what is now known about reduced weakly clean rings extends to reduced rings for which $U(R)$ has at most two elements?

Added in proof

Actually, Problem 1 obviously holds in the affirmative taking into account Theorem 1.12 from [1] which is valid even for arbitrary (and hence noncommutative) rings.

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