WHEN $\mathrm{Max}_d(G)$ IS ZERO-DIMENSIONAL

Ricardo CARRERA, Ramiro H. LAFUENTE-RODRIGUEZ,
Warren Wm. McGOVERN

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Abstract. Our setting here is **W**. A **W**-object is an archimedean lattice-ordered group G together with distinguished weak order unit $u \in G^+$. R. H. Lafuente-Rodriguez, W. Wm. McGovern (2021) classified when the space of maximal d-subgroups under the hull-kernel topology of such an object has a clopen π-base. In this article, we characterize when the space has the stronger property of zero-dimensionality.

Keywords: archimedean lattice-ordered group; d-subgroups; zero-dimensional compact Hausdorff space

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1. Introduction and preliminaries

Those familiar with [16], feel free to proceed to the next section. In general, we assume that the reader is familiar with the theory of lattice-ordered groups and with **W**-objects. For anyone needing a refresher, our two suggestions for references are [10] and [8].

Recall that **W** is the category whose objects are pairs (G, u) such that G is an archimedean ℓ-group and $u \in G$ is a positive weak order unit. We use $\mathcal{C}(G)$ to denote the collection of convex ℓ-subgroups of G. When ordered by inclusion, it is well known that this object is an example of what is known today as an algebraic frame with the finite intersection property. (Algebraic frames with FIP are also known as arithmetic frames [3], and M-frames [4], [5]).

In $\mathcal{C}(G)$, there are convex ℓ-subgroups that are maximal with respect to not containing u (Zorn's Lemma). Such subgroups are called *values of* u, and the set of these is called the *Yosida space of* (G, u); it is denoted by YG. When equipped with the hull-kernel topology, YG is a compact Hausdorff space. The collection of sets of the form

$$\operatorname{coz}(g) = \{V \in YG\colon\ g \notin V\}$$

for some $g \in G$ forms a base for the open subsets; any set of the form $\operatorname{coz}(g)$ is called a *G-cozeroset* of YG. The complement of $\operatorname{coz}(g)$ is denoted by $Z(g)$ and is called the *zeroset of g*. The collections of G-cozerosets of YG and G-zerosets of G will be denoted by $\operatorname{Coz}(G)$ and $Z(G)$, respectively. Values are examples of *prime* subgroups: a convex ℓ-subgroup P with the property that if $a, b \in G^+$ and $a \wedge b = 0$, then either $a \in P$ or $b \in P$. Thus, the collection of G-cozerosets is closed under finite union and finite intersection. Namely, for $g, h \in G^+$, $\operatorname{coz}(g) \cap \operatorname{coz}(h) = \operatorname{coz}(g \wedge h)$ and $\operatorname{coz}(g) \cup \operatorname{coz}(h) = \operatorname{coz}(g+h) = \operatorname{coz}(g \vee h)$. Similarly, $Z(g) \cup Z(h) = Z(g \wedge h)$ and $Z(g) \cap Z(h) = Z(g+h) = Z(g \vee h)$.

Recall the Yosida representation. Let $\overline{R} = \mathbb{R} \cup \{\pm\infty\}$, the two-point compactification of the reals. For a compact Hausdorff space X, a continuous function $f\colon\ X \to \overline{R}$ is said to be *almost real-valued* if $f^{-1}(\mathbb{R})$ is a dense subset of X. The collection of all almost real-valued continuous functions on X is denoted by $D(X)$. In general, $D(X)$ need not be a group as addition might not be well defined; it is always a lattice. However, we do speak of ℓ-subgroups of $D(X)$, by which we mean a sublattice of $D(X)$, which is in fact closed under addition making it into a lattice-ordered group. For example, $C(X)$, the collection of continuous real-valued functions defined on X, is an ℓ-subgroup of $D(X)$. For any real number $r \in \mathbb{R}$ we let $\mathbf{r}$ denote the constant function in $D(X)$ mapping each point in X to the r.

Theorem 1.1 (The Yosida Representation)**.** *Let $(G, u) \in \mathbf{W}$. There is an ℓ-group $\widehat{G}$ of continuous almost real-valued functions on YG and an ℓ-isomorphism of G onto $\widehat{G}$ ($g \mapsto \hat{g}$) such that $\hat{u} = \mathbf{1}$. Furthermore, for each closed subset V of YG and $p \in YG \smallsetminus V$ there is some $g \in G$ such that $\hat{g}(p) = 1$ and $\hat{g}(v) = 0$ for each $v \in V$. YG is the unique (up to homeomorphism) compact Hausdorff space with this property.*

We identify any (G, u) with $(\widehat{G}, \mathbf{1})$. The property mentioned in the theorem that states for each closed subset V of YG and each $p \in G \smallsetminus V$ there is some $g \in G$ such that $g(p) = 1$ and $g(v) = 0$ for all $v \in V$, together with the following consequences, will be referred to as the *separation property.* One consequence of the Yosida Representation is that given two disjoint closed sets, say V_0, V_1, then there is some $g \in G^+$ such that $0 \leqslant g \leqslant \mathbf{1}$ and for each $v \in V_i$, $g(v) = i$. A second consequence of the Yosida Representation is that by considering the closed sets $g^{-1}([0, \frac{1}{3}])$ and $g^{-1}([\frac{2}{3}, 1])$ and separating these, yields that given disjoint subsets V_0, V_1 there is an $h \in G^+$ such that $V_0 \subseteq \operatorname{int} Z(h)$ and $V_1 \subseteq \operatorname{int} Z(\mathbf{1} - h)$. For a more detailed discussion of the Yosida Representation and its consequences we urge the readers to peruse Section 1 of [11].

We view $C(X)$ as a $\mathbf{W}$-object by taking the constant function $\mathbf{1}$ as its distinguished weak order unit: $(C(X), \mathbf{1})$. Recall that if X is a compact Hausdorff space, then a consequence of the uniqueness provision in the Yosida Representation is that the Yosida space of $C(X)$ is in fact homeomorphic to X and the values of $(C(X), \mathbf{1})$ are precisely the sets of the form $M_p = \{f \in C(X)\colon\ f(p) = 0\}$ for each $p \in X$. In this case the $C(X)$-zerosets of $(C(X), \mathbf{1})$ are precisely the usual zerosets of X. Because of these comments, we typically view the values of u as V_p for some $p \in YG$; namely those elements $g \in G$ for which p is in the G-zeroset of g.

The collection of all prime subgroups is known as the prime spectrum of G and is denoted by $\mathrm{Spec}(G)$. Lattice-ordered groups have the feature that the collection of primes containing a given prime form a chain, i.e., $\mathrm{Spec}(G)$ is a root system. $\mathrm{Spec}(G)$ is also equipped with the hull-kernel topology: the collection of sets of the form $U(g) = \{P \in \mathrm{Spec}(G)\colon\ g \notin P\}$ forms a base for the open sets. The collection of minimal prime subgroups of G is denoted by $\mathrm{Min}(G)$ and is equipped with the subspace topology.

Investigating topological properties of $\mathrm{Min}(G)$ has been an important part of the theory of $\mathbf{W}$-objects. One could argue that investigating when the hull-kernel topology on $\mathrm{Min}(G)$ is compact, as done in both [13] and [9], has been instrumental in the understanding of complemented ℓ-groups as well as cozero-complemented spaces. Investigations along these lines have led to other discoveries as well. For example, in [1] and [12] the authors use the minimal prime ideal space to characterize the Yosida space of the projectable hull of a $\mathbf{W}$-object.

Besides the hull-kernel topology, $\mathrm{Min}(G)$ can also be equipped with the inverse topology, which is the topology generated by the collection of sets of the form $\mathrm{Min}(G) \smallsetminus U(g)$; we use $\mathrm{Min}(G)^{-1}$ to denote the space of minimal prime subgroups equipped with the inverse topology. In some sense, this article is about classifying other spaces of prime subgroups as we now explain.

Next, given $S \subseteq G$, the *polar* of S is denoted by $S^\perp$ and defined as

$$S^\perp = \{h \in G\colon\ |h| \wedge |g| = 0 \text{ for all } g \in S\}.$$

A *polar subgroup* is a subgroup of the form $S^\perp$ for a subset $S \subseteq G$. When $S = \{g\}$, we instead write $g^\perp$. A *principal polar* is a set of the form $g^{\perp\perp}$ for some $g \in G$. If $g^\perp = 0$, then g is called a *weak order unit.* The set of polars of G is denoted by $\mathcal{P}(G)$, and when ordered by inclusion, it is a complete boolean algebra. It can be identified with the collection of regular closed subsets of YG which will be discussed at length in the next section.

A convex ℓ-subgroup H of G is called a *d-subgroup* if whenever $h \in H$, then $h^{\perp\perp} \subseteq H$. No proper d-subgroup can contain a weak order unit, thus, Zorn's Lemma ensures that maximal d-subgroups exist; denote the set of all maximal d-subgroups by $\mathrm{Max}_d(G)$. Proposition 4.3 of [7] classifies maximal d-subgroups as being those convex ℓ-subgroups which are maximal with respect to not containing any weak order units.

As we do for the Yosida space, we equip $\mathrm{Max}_d(G)$ with the hull-kernel topology. For $g \in G$, let $U_d(g) = \{M \in \mathrm{Max}_d(G) \colon\ g \notin M\}$. Clearly, $U_d(g) = U_d(|g|)$. The set-theoretic complement of $U_d(g)$ in $\mathrm{Max}_d(G)$ shall be denoted by $V_d(g)$. The following is stated in [7, Lemma 4.4] with the comment that the proof is analogous to the proof for $\mathrm{Spec}(G)$, which can be found in [6, Theorem 2.1].

Lemma 1.2. *Let $(G, u) \in \mathbf{W}$. The following hold for all $g, h \in G^+$.*

(a) $U_d(g) = \mathrm{Max}_d(G)$ *if and only if g is a weak order unit.*
(b) $U_d(g) = \emptyset$ *if and only if $g = 0$.*
(c) $U_d(g) \cup U_d(h) = U_d(g \vee h)$.
(d) $U_d(g) \cap U_d(h) = U_d(g \wedge h)$.
(e) *The subset $T \subseteq \mathrm{Max}_d(G)$ is open in the hull-kernel topology if and only if there is a d-subgroup H for which $T = U_d(H)$.*

Recall the following theorem.

Theorem 1.3 (Theorem 4.5 [7])**.** *Suppose $(G, u) \in \mathbf{W}$. Then $\mathrm{Max}_d(G)$ is a compact Hausdorff space.*

In [16], the authors showed that in the case of $\mathbf{W}$-objects, $\mathrm{Min}(G)^{-1}$ has a clopen π-base if and only if $\mathrm{Max}_d(G)$ has a clopen π-base. In what follows, our work is aimed at answering when $\mathrm{Max}_d(G)$ is zero-dimensional. (A space is said to have a *clopen π-base* if the space possesses a collection of nonempty clopen subsets such that every open subset contains one from the collection. When a space has a clopen base, then the space is said to be *zero-dimensional.*)

2. Wallman lattices

Throughout this section, X denotes a compact Hausdorff space. Let $\mathcal{R}(X)$ denote the collection of regular closed subsets of X. (Recall that $V \subseteq X$ is called *regular closed* if $V = \mathrm{cl}_X \mathrm{int}_X V$.) It is well known that $\mathcal{R}(X)$ is a (complete) boolean algebra when partially ordered by inclusion. The meet, join, and complement are given as follows: for $V_1, V_2 \in \mathcal{R}(X)$

(i) $V_1 \cup' V_2 = V_1 \cup V_2$;
(ii) $V_1 \cap' V_2 = \mathrm{cl}_X \mathrm{Int}_X (V_1 \cap V_2)$;
(iii) $V_1' = \mathrm{cl}_X (X \smallsetminus V_1)$.

We recall the definition of a Wallman lattice (see [7, Definition 2.1]).

Definition 2.1. Let $(L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice, and let $\mathbf{Ult}(L)$ denote the collection of L-ultrafilters. For $a \in L$, denote the set of ultrafilters containing a by $\mathcal{V}(a)$. For each $a, b \in L$, $\mathcal{V}(a) \cup \mathcal{V}(b) = \mathcal{V}(a \vee b)$ and $\mathcal{V}(a) \cap \mathcal{V}(b) = \mathcal{V}(a \wedge b)$. The collection $\{\mathcal{V}(a)\colon a \in L\}$ forms a base for a topology of closed sets on $\mathbf{Ult}(L)$. This is called the *Wallman topology* on $\mathbf{Ult}(L)$.

The lattice L is called a *Wallman lattice* if whenever $a < b$, there is some $c \in L$ such that $a \wedge c = 0 < b \wedge c$. The interest in Wallman lattices is that these are precisely the lattices for which the map $a \to \mathcal{V}(a)$ is injective. These lattices are also known as *disjunctive* lattices. For more information on Wallman lattices see [2] and [17].

Remark 2.2. We would like to point out that the definition of a Wallman lattice is equivalent to the property that starting with two distinct elements $a, b \in L$ one can find a $c \in L$ such that c is disjoint to exactly one of a or b. Wallman assumes that the lattice has a top element, while Banaschewski does not. Furthermore, these are equivalent to starting with an $a < 1$ and finding a $0 < c$ disjoint from a.

Theorem 2.3 (See [2]). *Let $(L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice.*

(i) *If L is a Wallman lattice, then $\mathbf{Ult}(L)$ is a compact T_1-space.*

(ii) *The space $\mathbf{Ult}(L)$ is a Hausdorff space if and only if for any $a, b \in L$ such that $a \wedge b = 0$, there exists $x, y \in L$ such that $x \vee y = 1$ and $a \wedge y = 0 = b \wedge x$. (When $\mathbf{Ult}(L)$ is a Hausdorff space, we shall say L is a normal lattice.)*

When we deal with a $\mathbf{W}$-object, say G, then we are interested in the following sublattices of $\mathcal{R}(YG)$. In our view, these are the appropriate generalizations of the objects studied in [15]; these were also studied in [7] and [16]:

(a) $Z^\sharp(G) = \{\mathrm{cl}\,\mathrm{int}\, Z(f)\colon\ f \in G\}$.

(b) $\mathrm{cl}\,\mathrm{Coz}(G) = \{\mathrm{cl}\, C\colon\ C \in \mathrm{Coz}(G)\}$.

(c) $\mathfrak{B}(G) = \{C \subseteq YG\colon\ C \text{ is clopen}\}$.

In [14] and [15], the authors consider a compact space X and the collection $Z^\sharp(X) := Z^\sharp(C(X))$. They show that this collection is a normal Wallman sublattice of $\mathcal{R}(X)$. We extend this result to $Z^\sharp(G)$; this was left to the reader in [7] so we include it here for the sake of completeness.

Lemma 2.4. *The collection $Z^\sharp(G)$ is a normal Wallman sublattice of $\mathcal{R}(YG)$.*

P r o o f. Let $e, f \in G^+$ such that $\mathrm{cl}\,\mathrm{int}\, Z(e) \subset \mathrm{cl}\,\mathrm{int}\, Z(f)$. Without loss of generality, we assume that $\emptyset \neq \mathrm{cl}\,\mathrm{int}\, Z(e)$. There is some $p \in \mathrm{int}\, Z(f) \smallsetminus \mathrm{cl}\,\mathrm{int}\, Z(e)$. Then, by the separation property, there is some $h \in G^+$ such that $0 \leqslant h \leqslant \mathbf{1}$ and $\mathrm{cl}\,\mathrm{int}\, Z(e) \subseteq \mathrm{int}\, Z(h)$ and $p \notin Z(h)$. Then

$$\mathrm{cl}\,\mathrm{int}\, Z(e+h) = \mathrm{cl}\,\mathrm{int}\,(Z(e) \cap Z(h)) = \mathrm{cl}\,\mathrm{int}\, Z(e) = \mathrm{cl}\,\mathrm{int}\, Z(e) \cap \mathrm{cl}\,\mathrm{int}\, Z(h).$$

Thus, by replacing e with $e' = e + h$, we know that $\text{cl int}\, Z(e) = \text{cl int}\, Z(e')$ and $p \notin Z(e')$. Next, we use the separation property again and choose a $g \in G^+$ such that $p \in \text{int}\, Z(g)$ and $Z(g) \cap Z(e') = \emptyset$. Then $\text{cl int}\, Z(g)$ is disjoint from $\text{cl int}\, Z(e')$ while $p \in \text{cl int}\, Z(g) \cap' \text{cl int}\, Z(f)$. Consequently, $Z^\sharp(G)$ is a Wallman sublattice of $\mathcal{R}(YG)$.

A proof that $Z^\sharp(G)$ is a normal lattice follows from [14, Theorem 2.4], *mutatis mutandis.* $\square$

In [7, Theorem 4.8], the authors showed that the space of ultrafilters of $Z^\sharp(G)$ is homeomorphic to $\text{Max}_d(G)$. In particular, they showed that for any d-subgroup H, the collection $Z^\sharp[H] = \{\text{cl int}\, Z(h)\colon\ h \in H\}$ is a $Z^\sharp(G)$-filter. Furthermore, for each $M \in \text{Max}_d(G)$,

$$M = \{g \in G\colon\ \text{cl int}\, Z(g) \in Z^\sharp[M]\}.$$

Since $\text{Spec}(G)$ is a root system and each $M \in \text{Max}_d(G)$ is a prime subgroup not containing u, M can be extended to a unique value of u, some $V_p \in YG$. This defines a map

$$\lambda^d\colon\ \text{Max}_d(G) \to YG.$$

We can use this for a description of $\lambda^d(M)$. (For more information on the map λ^d see [7, Proposition 5.2].)

Lemma 2.5. *Let $M \in \text{Max}_d(G)$. Then $\lambda^d(M) = V_p$, where $\{p\} = \bigcap\{\text{cl int}\, Z(g)\colon g \in M\}$.*

Proof. Let $p \in YG$ such that $\lambda^d(M) = V_p$. Then for each $g \in M$, $g \in V_p$ and thus $p \in Z(g)$. If for some $m \in M$, $p \notin \text{cl int}\, Z(m)$, then, by the separation property, there is some $h \in G^+$ such that $h(p) = 1$ and $\text{cl int}\, Z(m) \subseteq \text{int}\, Z(h)$, whence $\text{cl int}\, Z(m) \subseteq \text{cl int}\, Z(h)$, and so $h \in M$. But by design, $h \notin V_p$, the desired contradiction.

Clearly, p is the only element belonging to intersection since for any $q \neq p$, $M \nsubseteq V_q$, meaning that there is some $m \in M \setminus V_q$. It follows that $q \notin \text{cl int}\, Z(m)$. $\square$

Proposition 2.6. *Let $(G, u) \in \mathbf{W}$ and $g, h \in G^+$. The following are equivalent:*

(1) $U_d(g) = U_d(h)$,
(2) $V_d(g) = V_d(h)$,
(3) $\text{cl coz}(g) = \text{cl coz}(h)$,
(4) $\text{cl int}\, Z(g) = \text{cl int}\, Z(h)$,
(5) $g^{\perp\perp} = h^{\perp\perp}$.

P r o o f. That (1) and (2) are equivalent is purely set-theoretic. That (3) and (4) are equivalent is a straight forward topological argument. That (4) and (5) are equivalent follows from the characterization of $g^{\perp\perp}$ given in [7, Lemma 4.6 (b)], namely

$$g^{\perp\perp} = \{k \in G\colon \text{ cl int } Z(g) \subseteq \text{cl int } Z(k)\}.$$

That (5) implies (1) is obvious using the definition of d-subgroup.

So assume (1). If cl int $Z(g) \neq$ cl int $Z(h)$, then the disjunctive property of $Z^\sharp(G)$ produces an element $f \in G^+$ such that exactly one of cl int $Z(g) \cap'$ cl int $Z(f) = \emptyset$ or cl int $Z(h) \cap'$ cl int $Z(f) = \emptyset$ is satisfied. Since one of these is not empty, there must be a $Z^\sharp(G)$-ultrafilter containing exactly one of these, and thus a maximal d-subgroup containing exactly one g or h. This contradicts (1). Therefore, (1) implies (3), completing the proof. □

Definition 2.7. For any $g \in G^+$, we say that g is *complemented* if there is an $f \in G^+$ such that $g \wedge f = 0$ and $g \vee f$ is a weak order unit. We let $c(G)$ denote the set of all complemented elements of G. The cozeroset of such an element is a *complemented cozeroset* of YG, that is, there is another cozeroset of YG, say $D \in \operatorname{Coz}(G)$, for which $C \cap D = \emptyset$ and $C \cup D$ is dense in X. The set $c(G)$ is a sublattice of G^+. (G is called *complemented* when $G^+ = c(G)$. This is a well-known class of ℓ-groups, see [9].)

This leads us to recall the definition of the following sublattice of $Z^\sharp(G)$, see [15]:

$$\mathscr{G}(G) = \{\operatorname{cl} \operatorname{coz}(e)\colon\ e \in c(G)\}.$$

One ought to notice that $\mathscr{G}(G) = Z^\sharp(G) \cap \text{cl coz}(G)$. In particular, since inside the boolean algebra $\mathcal{R}(YG)$ the complement of cl int $Z(g)$ is cl coz(g), it follows that the bounded sublattice $\mathscr{G}(G)$ is closed under complements, and therefore $\mathscr{G}(G)$ is a boolean subalgebra of $\mathcal{R}(YG)$. From Stone Duality, we know that the space of ultrafilters on $\mathscr{G}(G)$ is a compact zero-dimensional Hausdorff space. This algebra will play a role in the next section. We end this section relating the complemented elements of G to the clopen subsets of $\operatorname{Max}_d(G)$.

Lemma 2.8 ([16, Lemma 5.1])**.** *The set $K \subseteq \operatorname{Max}_d(G)$ is clopen if and only if $K = U_d(e)$ for some $e \in c(G)$ if and only if $K = V_d(e)$ for some $e \in c(G)$.*

Proposition 2.9. *Let $e \in c(G)$. Then $\lambda^d(V_d(e)) =$ cl int $Z(e)$.*

P r o o f. Let $e \in c(G)$ and choose any $f \in c(G)$ such that $e \wedge f = 0$ and $e \vee f$ is a weak order unit.

Let $M \in V_d(e)$ and let $\lambda^d(M) = V_p$. Then by Lemma 2.5, $p \in$ cl int $Z(e)$, whence $\lambda^d(V_d(e)) \subseteq$ cl int $Z(e)$.

Next, we show that $\operatorname{int} Z(e) \subseteq \lambda^d(V_d(e))$. To that end, let $p \in \operatorname{int} Z(e)$ and let $M \in \operatorname{Max}_d(G)$ such that $\lambda^d(M) = V_p$. We claim that $e \in M$, i.e., $M \in V_d(e)$. If not, then $f \in M$. Applying Lemma 2.5, we gather that $p \in \operatorname{cl} \operatorname{int} Z(f)$, which is a contradiction.

Putting both pieces together produces the following string of containments:

$$\operatorname{int} Z(e) \subseteq \lambda^d(V_d(e)) \subseteq \operatorname{cl} \operatorname{int} Z(e).$$

Since λ^d is a continuous map between two compact Hausdorff spaces, it follows that $\lambda^d(V_d(e))$ is closed and therefore $\lambda^d(V_d(e)) = \operatorname{cl} \operatorname{int} Z(e)$. □

3. Main theorem

We now come to the main theorem of the article; a generalization of [15, Theorem 3.15] to **W**-objects.

Theorem 3.1. *Let (G, u) be a **W**-object. The following statements are equivalent.*

(1) $\operatorname{Max}_d(G)$ *is a zero-dimensional space.*
(2) *For each pair of distinct maximal d-subgroups, say $M_1 \neq M_2 \in \operatorname{Max}_d(G)$, there is a complemented element $0 < e \in G^+$ such that $e \in M_1 \smallsetminus M_2$.*
(3) *Whenever $k \in G^+$ and $M \in U_d(k)$, there is a $0 < e \in c(G)$ such that $M \in U_d(e)$ and $0 < e \leqslant k$.*
(4) *The collection $\mathscr{G}(G)$ forms a base for the closed sets on YG.*
(5) *The collection $\{\operatorname{coz}(e) \colon e \in c(G)\}$ is a base for the open sets of YG.*

Proof. (1) is equivalent to (2). This is clear as $\operatorname{Max}_d(G)$ is a compact Hausdorff space (Theorem 1.3) and (2) is stating that $\operatorname{Max}_d(G)$ is totally disconnected, which is equivalent to being zero-dimensional for compact Hausdorff spaces.

(1) implies (3). Suppose $\operatorname{Max}_d(G)$ is zero-dimensional and let $0 < k \in G^+$ and an $M \in \operatorname{Max}_d(G)$ such that $k \notin M$. Without loss of generality, assume that k is not a weak order unit. By hypothesis, there is a complemented element, say $e' \in c(G)$, such that $M \in U_d(e') \subseteq U_d(k)$. Then $e = e' \wedge k$ is also a complemented element (as $U_d(e) = U_d(e')$) and $e' \notin M$.

(3) implies (1). Let $k \in G^+$ and $M \in U_d(k)$. Without loss of generality, assume that $0 < k$. Now, $k \notin M$ and so by (3) there is an $0 < e \leqslant k$ with $e \in c(G)$ and $e \notin M$. Now, $M \in U_d(e) \subseteq U_d(k)$, whence $\operatorname{Max}_d(G)$ is zero-dimensional.

(1) implies (4). (For the purpose of clarity in this proof we write λ for the map λ^d.) Suppose that $\operatorname{Max}_d(G)$ is zero-dimensional and let $p \in YG$ and $V \subseteq YG$, a closed

subset not containing p. By the separation property, there are disjoint G-zerosets, say Z_1 and Z_2, such that $p \in \operatorname{int} Z_1$ and $V \subseteq \operatorname{int} Z_2$. Since Z_1, Z_2 are disjoint in YG, then $\lambda^{-1}(Z_1)$ and $\lambda^{-1}(Z_2)$ are disjoint closed subsets of the zero-dimensional compact Hausdorff space $\operatorname{Max}_d(G)$. Thus, there is a clopen subset of $\operatorname{Max}_d(G)$, say B, such that $\lambda^{-1}(Z_1) \subseteq B$ and $\lambda^{-1}(Z_2) \cap B = \emptyset$. Set $B' = \operatorname{Max}_d(G) \smallsetminus B$. By Lemma 2.8, $B = U_d(e)$ and $B' = U_d(f)$, where e and f are a complementary pair. Also, $B = V_d(f)$ and $B' = V_d(e)$. Applying Proposition 2.9 and Proposition 2.6, $\lambda(B) = \operatorname{cl} \operatorname{int} Z(f) = \operatorname{cl} \operatorname{coz}(e)$ and $\lambda(B') = \operatorname{cl} \operatorname{int} Z(e) = \operatorname{cl} \operatorname{coz}(f)$. Now, $p \in \operatorname{int} Z(f)$ while $V \subseteq \operatorname{int} Z(e)$. Since e, f are a complementary pair, we know that

$$\operatorname{int} Z(f) \cap \operatorname{int} Z(e) = \emptyset.$$

Consequently, $p \notin \operatorname{cl} \operatorname{int} Z(e)$, while $V \subseteq \operatorname{cl} \operatorname{int} Z(e)$.

(4) implies (2). Let $M, N \in \operatorname{Max}_d(G)$ be distinct maximal d-subgroups. Then there are $0 < g, h \in G^+$ such that $g \in M$, $h \in N$ and $\operatorname{cl} \operatorname{int} Z(g) \cap' \operatorname{cl} \operatorname{int} Z(h) = \emptyset$. Thus, $\operatorname{int} Z(g) \cap \operatorname{int} Z(h) = \emptyset$. Choose $p \in \operatorname{int} Z(g)$ such that $p \notin Z(h)$. By hypothesis, there is a complemented element $e \in c(G)$ such that $p \notin \operatorname{cl} \operatorname{int} Z(e)$ and $Z(h) \subseteq \operatorname{cl} \operatorname{int} Z(e)$. It follows that $e \in N \smallsetminus M$.

(4) implies (5). Let $p \in YG$ and V be a closed subset of YG not containing p. By (4) together with the separation property of YG, there is some $e \in c(G)$, such that $V \subseteq \operatorname{cl} \operatorname{int} Z(e)$. There is some $g \in G^+$ such that $p \notin Z(g)$ and $\operatorname{cl} \operatorname{int} Z(e) \subseteq Z(g)$. It is straightforward to check that $\operatorname{cl} \operatorname{int} Z(e + g) = \operatorname{cl} \operatorname{int} Z(e)$, whence we can assume that $p \notin Z(e)$. It follows that $p \in \operatorname{coz}(e)$ and $\operatorname{coz}(e) \cap V = \emptyset$.

(5) implies (4). Suppose the collection $\{\operatorname{coz}(e)\colon e \in c(G)\}$ forms a base for the open subsets of YG. Notice that this collection is closed under finite unions and finite intersections. Let V be a closed subset of YG and $p \notin V$. Since YG is compact Hausdorff, and thus V is compact, we can find disjoint members in our base, say $\operatorname{coz}(e)$ and $\operatorname{coz}(f)$, such that $p \in \operatorname{coz}(f)$ and $V \subseteq \operatorname{coz}(e)$. It follows that $p \notin \operatorname{cl} \operatorname{coz}(e)$. Since $e \in c(G)$, $\operatorname{cl} \operatorname{coz}(e) \in \mathscr{G}(G)$. □

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Authors' addresses: *Ricardo Carrera*, Department of Mathematics, Halmos College of Arts and Sciences, Nova Southeastern University, Fort Lauderdale, FL 33328, USA; *Ramiro H. Lafuente-Rodriguez*, University of South Dakota, Department of Mathematical Sciences, Vermillion, SD 57069, USA; *Warren Wm. McGovern* (corresponding author), Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458, USA, e-mail: `warren.mcgovern@fau.edu`.