# THE MAXIMAL RING OF QUOTIENTS OF $A_dL$

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Abstract. We start with a zero-dimensional frame $L$ and an arbitrary integral domain $A$. We equip $A$ with the discrete topology and consider the ring of $A$-valued continuous functions on $L$, which we denote by $A_dL$. In this article, we classify both the classical ring of quotients and maximal ring of quotients of $A_dL$, paying special attention to the case of $\mathfrak{Z}L$ the integer-valued continuous functions on $L$.

## 1. Introduction and preliminaries

When we began this project we were interested in classifying the classical ring of quotients on $\mathfrak{Z}L$, the ring of integer-valued continuous functions on a zero-dimensional frame. Our hope was to have a theorem in the same vein as the classical results of Fine, Gillman, and Lambek [4] and [5], which characterized both the classical and maximal ring of quotients of $C(X)$, the ring of real-valued functions on a Tychonoff space $X$. A recent paper by Abedi [1] generalized these classical results on $C(X)$ to $\mathcal{R}L$, the ring of continuous functions on a completely regular frame. After starting this project we found a recent paper by Olfati [12], which characterized the classical ring of quotients of $C(X,\mathbb{Z})$. We found Olfati's article to be encouraging in that it confirmed to us that we are on the right track.

Soon after, we realized that our results generalize to a much wider situation. This then led us to revisit the multi-layered paper by Vechtomov [15] which deals with continuous functions on a topological division ring. Our results can also be construed as a generalization of some of Vechtomov's result for the case when the field is discrete (e.g. [15, Theorem 14.1]).

Throughout, we assume that our rings are reduced commutative rings with identity. For such a ring $R$, we use $U(R)$ to denote the collection of units in $R$ and $\text{reg}(R)$ the collection of regular elements, that is, non zero-divisors. We let $\mathfrak{J}(R)$ denote the Jacobson radical of $R$. We let $q(R)$ denote the classical ring of quotients of $R$ and $Q(R)$ the maximal ring of quotients of $R$. (We shall recall the construction of $Q(R)$ in Section 2.)

Recall that a frame is a complete lattice which satisfies the frame law, that is, for all $a \in L$ and $S \subseteq L$,

$$a \wedge \bigvee S = \bigvee_{s\in S} (a \wedge s).$$

(Historically, these types of lattices have also been referred to as both complete Brouwerian lattices or locales, depending on the situation.)

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We let $1_L$ denote the top element of $L$ and $0_L$ the bottom element of $L$, dropping the subscripts when it is clear from context. (We shall assume that $0_L \neq 1_L$.) For $x \in L$, we denote the pseudo-complement of $x$ by $x^\perp$, the largest element of $L$ disjoint from $x$. Any element which satisfies $1_L = x \vee x^\perp$ is called a *complemented* element and $BL$ denotes the collection of all complemented elements of $L$, and is called the *boolean part of* $L$; $BL$ is a boolean algebra. The frame $L$ is said to be *zero-dimensional* if every element is the join of complemented elements. We shall assume that every frame discussed in the article is zero-dimensional. A zero-dimensional frame is completely regular.

A *frame homomorphism* is a function $h : L \to M$ between two frames such that $h$ preserves arbitrary joins and finite meets. We note that a frame homomorphism automatically sends $1_L$ to $1_M$ and similarly for the bottom elements. Any surjective frame homomorphism is known by the moniker *quotient map*.

The standard example of a frame is the topology of open subsets of a given topological space $X$, denoted $\Omega(X)$. Any frame which is frame isomorphic to $\Omega(X)$ is called a *spatial frame*.

Any element $x \in L$ for which $x^\perp = 0_L$ is called *dense* and the collection of dense elements is denoted by $\mathcal{D}(L)$. The set $\mathcal{D}(L)$ is a proper filter on $L$; $1_L \in \mathcal{D}$.

Recall that for any $x \in L$, the down set $\downarrow x = \{z \in L : z \leq x\}$ is also a frame, and the map $o_x : L \to \downarrow x$ defined by $o_x(a) = a \wedge x$ is called the *open quotient map*; it is a frame homomorphism. If $x \in \mathcal{D}(L)$, then the open quotient map $o_x$ has the property that $o_x(a) = 0$ if and only if $a = 0$. Such a frame homomorphism is known by the term *dense map*.

The frame $L$ is said to be *extremally disconnected* if $a^{\perp\perp} \vee a^\perp = 1$ for all $a \in L$. The space $X$ is called extremally disconnected if $\Omega(X)$ is an extremally disconnected frame.

Formally, for a zero-dimensional frame $L$, the ring of integer-valued continuous functions on $L$, denoted by $\mathfrak{Z}L$, is the collection of frame homomorphisms $h : \Omega(\mathbb{Z}) \to L$. In the next section, we shall see that there is a nicer way to view the continuous maps. The ring $\mathfrak{Z}L$ is a subring of the ring $\mathcal{R}L$ consisting of real-valued continuous functions on $L$.

Our main references for the theory of frames is [9] and [13]. We would be negligent if we did not mention Bernhard Banaschewski and the multitude of his work in frame theory.

## 2. The ring $A_dL$

We have generalized our work to the case for any integral domain $A$. We equip $A$ with the discrete topology and denote this space by $A_d$. Throughout, we let $K$ denote the field of fractions of $A$. We let $A_dL$ denote the collection of frame homomorphisms from the power set of $A$ into $L$.

The ring $A_dL$ has as its elements mappings $\alpha\colon A \to L$ such that

$$\alpha(k) \wedge \alpha(\ell) = 0_L \quad \text{for } k \neq \ell \qquad \text{and} \qquad \bigvee\{\alpha(m) \mid m \in A\} = 1_L,$$

with ring operations derived from those of $A$.

The ring operations of $A_dL$ are derived from those of $A$ as follows:

(1) for any $\diamond = +, \cdot,\quad (\alpha \diamond \beta)(m) = \bigvee\{\alpha(k) \wedge \alpha(l) \,|\, k \diamond l = m\}$;
(2) $(-\alpha)(m) = \alpha(-m)$;

We also have an embedding of $A$ into $A_dL$ given by associating to each $k \in A$, the element $\boldsymbol{k} \in A_dL$ defined by $\boldsymbol{k}(m) = 1_L$, if $m = k$ and $\boldsymbol{k}(m) = 0_L$ if $m \neq k$.

Associated with the ring $A_dL$ is the *cozero map*

$$\operatorname{coz}\colon A_dL \to L \quad \text{defined by} \quad \operatorname{coz}\alpha = \bigvee\{\alpha(m) \mid 0 \neq m \in A\}.$$

(We should point out that $\operatorname{coz}(\alpha)$ need not be a cozero-element of $L$ in the sense of $\mathcal{R}L$. If $A$ is countable, then it is.)

As in the real-valued case, every frame homomorphism $h\colon L \to M$ between zero-dimensional frames induces a ring homomorphism $A_d h\colon A_dL \to A_dM$ such that for any $\alpha \in A_dL$, $A_dh(\alpha) = h \circ \alpha$.

For any $c \in BL$, the idempotent element of $A_dL$ associated to $c$ is given by

$$\chi_c(x) = \begin{cases} c, & \text{if } x = 1 \\ c^{\perp}, & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 2.1.** *Suppose $\alpha, \beta \in A_dL$. If $\beta \cdot \alpha = 0$, then $\operatorname{coz}(\beta) \leq \alpha(0)$.*

*Proof.* Let $0 \neq m \in A$. Then for any $0 \neq k \in A$, $\beta(m) \wedge \alpha(k) = 0$ since $mk \neq 0$. It follows that $\beta(m) \wedge \bigvee_{k\neq 0} \alpha(k) = 0$, whence $\beta(m) \leq (\bigvee_{k\neq 0} \alpha(k))^{\perp} = \alpha(0)$. Since $m$ was arbitrarily chosen, $\operatorname{coz}(\beta) \leq \alpha(0)$. □

**Proposition 2.2.** *For any integral domain $A$ and any zero-dimensional frame $L$, the ring $A_dL$ is a weak Baer ring. In particular, for any $\alpha \in A_dL$*

$$\operatorname{Ann}(\alpha) = \langle \chi_{\alpha(0)} \rangle.$$

*Proof.* That $A_dL$ is weak Baer certainly follows from the displayed condition.

We first show that $\alpha\chi_{\alpha(0)} = 0$. To that end recall that for any $x \in A$

$$(\alpha \cdot \chi_{\alpha(0)})(x) = \bigvee_{mn=x} (\alpha(m) \wedge \chi_{\alpha(0)}(n)).$$

We first consider the case that $x \neq 0$. Then when $mn = x$, both $m, n \neq 0$. If $n \neq 1$, then $\chi_{\alpha(0)}(n) = 0$. So assume that $n = 1$ which means that $x = m$ and so

$$\begin{aligned} \alpha(m) \wedge \chi_{\alpha(0)}(n) &= \alpha(x) \wedge \chi_{\alpha(0)}(1) \\ &= \alpha(x) \wedge \alpha(0) \\ &= 0 \end{aligned}$$

where the last equality follows from the disjointness property of $\alpha \in A_dL$. It follows that $(\alpha \cdot \chi_{\alpha(0)})(x) = 0$ whenever $x \neq 0$.

Next, consider the case that $x = 0$. Then

$$\begin{aligned} (\alpha \cdot \chi_{\alpha(0)})(0) &= \bigvee_{mn=0} (\alpha(m) \wedge \chi_{\alpha(0)}(n)) \\ &= (\bigvee_{m=0,n\neq 0} (\alpha(0) \wedge \chi_{\alpha(0)}(n))) \vee (\bigvee_{m\neq 0,n=0} (\alpha(m) \wedge \chi_{\alpha(0)}(n))) \vee (\alpha(0) \wedge \chi_{\alpha(0)}(0)) \\ &= ((\alpha(0) \wedge \chi_{\alpha(0)}(1))) \vee (\bigvee_{m\neq 0,n=0} (\alpha(m) \wedge \alpha(0)^{\perp})) \vee (\alpha(0) \wedge \alpha(0)) \\ &= ((\alpha(0) \wedge \alpha(0))) \vee (\bigvee_{m\neq 0,n=0} (\alpha(m) \wedge \alpha(0)^{\perp})) \vee \alpha(0) \\ &= \alpha(0) \vee \bigvee_{m\neq 0} \alpha(m) \\ &= 1. \end{aligned}$$

We point out that in the second to last line of the string of equalities, this holds since by the disjointness property of $\alpha$, whenever $j \neq k$, then $\alpha(j) \leq \alpha(k)^{\perp}$. Therefore, $\alpha \cdot \chi_{\alpha(0)} = 0$. Thus, $\langle \chi_{\alpha(0)} \rangle \leq \mathrm{Ann}(\alpha)$.

Next, let $\beta \in \mathrm{Ann}(\alpha)$, i.e. $\beta \cdot \alpha = 0$. We aim to show that $\beta\chi_{\alpha(0)} = \beta$.
Let $x \in A$. First we consider $x \neq 0$. Then

$$\begin{aligned}(\beta \cdot \chi_{\alpha(0)})(x) &= \bigvee_{jk=x} (\beta(j) \wedge \chi_{\alpha(0)}(k)) \\ &= \beta(x) \wedge \chi_{\alpha(0)}(1) \\ &= \beta(x) \wedge \alpha(0) \\ &= \beta(x)\end{aligned}$$

where the last line follows from the previous lemma.
Next, let $x = 0$. Then

$$\begin{aligned}(\beta \cdot \chi_{\alpha(0)})(0) &= \bigvee_{jk=0} (\beta(j) \wedge \chi_{\alpha(0)}(k)) \\ &= \bigvee_{k\neq 0} (\beta(0) \wedge \chi_{\alpha(0)}(k)) \vee \bigvee_{j\neq 0} (\beta(j) \wedge \chi_{\alpha(0)}(0)) \vee (\beta(0) \wedge \chi_{\alpha(0)}(0)) \\ &= (\beta(0) \wedge \chi_{\alpha(0)}(1)) \vee (\bigvee_{j\neq 0} (\beta(j) \wedge \alpha(0)^{\perp}) \vee (\beta(0) \wedge \alpha(0)^{\perp}) \\ &= (\beta(0) \wedge \alpha(0)) \vee 0 \vee (\beta(0) \wedge \alpha(0)^{\perp} \\ &= \beta(0) \wedge (\alpha(0) \vee \alpha(0)^{\perp} \\ &= \beta(0).\end{aligned}$$

Thus, we have shown that for all $x \in A$, $(\beta \cdot \chi_{\alpha(0)})(x) = \beta(x)$. Consequently, $\beta \cdot \chi_{\alpha(0)} = \beta$, whence $\beta \in \langle \chi_{\alpha(0)} \rangle$. □

**Corollary 2.3.** *Let $\alpha \in A_dL$. The following are equivalent.*
1. *$\alpha$ is regular in $A_dL$.*
2. $\mathrm{Ann}(\alpha) = 0$.
3. $\chi_{\alpha(0)} = \mathbf{0}$.
4. $\alpha(0) = 0_L$.
5. $\mathrm{coz}(\alpha) = 1_L$.

We next turn to characterize the units of $A_dL$, which in turn, depend on the units in $A$. For each $\alpha \in A_dL$ let $\mathfrak{u}(\alpha) = \bigvee_{a\in U(A)} \alpha(a)$. Observe that $\mathfrak{u}(\alpha) \leq \mathrm{coz}(\alpha)$.

**Proposition 2.4.** *Let $\alpha \in A_dL$. Then $\alpha \in U(A_dL)$ if and only if $\mathfrak{u}(\alpha) = 1_L$.*

*Proof.* Suppose $\alpha \in U(A_dL)$. For simplicity sake we let $\beta \in A_dL$ denote the inverse of $\alpha$. Let $x \in A \setminus U(A)$. Then

$$\begin{aligned}0_L &= \mathbf{1}(x) \\ &= (\alpha \cdot \beta)(x) \\ &= \bigvee_{mn=x} (\alpha(m) \wedge \beta(n)).\end{aligned}$$

It follows that for any $m, n \in A$ at least one of which is not a unit, then $\alpha(m) \wedge \beta(n) = 0$. Fixing $m \in A \smallsetminus U(A)$ and allowing $n$ to run through all elements of $A$ yields

$$\begin{aligned} \alpha(m) &= \alpha(m) \wedge 1_L \\ &= \alpha(m) \wedge \Big( \bigvee_{x \in A} \beta(x) \Big) \\ &= \bigvee_{x \in A} \Big( \alpha(m) \wedge \beta(x) \Big) \\ &= 0_L. \end{aligned}$$

It follows that $\mathfrak{u}(\alpha) = 1$.

Conversely, suppose $\mathfrak{u}(\alpha) = 1_L$. It follows that for each $a \in A \smallsetminus U(A)$, $\alpha(a) = 0$. We define $\beta : A \to L$ as follows. For $x \in A$

$$\beta(x) = \begin{cases} \alpha(x^{-1}), & \text{if } x \in U(A) \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\beta \in A_dL$. We show that $\alpha \cdot \beta = 1$. To that end let $x \in A$. In the definition of multiplication if $x \in A \smallsetminus U(A)$, then whenever $mn = x$ both $m, n \in A \smallsetminus U(A)$. It follows then $\beta(n) = 0$ and so $(\alpha \cdot \beta)(x) = 0$. Therefore, we assume that $x \in U(A)$.

$$\begin{aligned} (\alpha \cdot \beta)(x) &= \bigvee_{mn = x} \Big( \alpha(m) \wedge \beta(n) \Big) \\ &= \bigvee_{m \in U(A)} \Big( \alpha(m) \wedge \beta\Big(\frac{x}{m}\Big) \Big) \\ &= \bigvee_{m \in U(A)} \Big( \alpha(m) \wedge \alpha\Big(\frac{m}{x}\Big) \Big). \end{aligned}$$

At this point we consider two cases: $x = 1$ and $x \neq 1$. In the first case, $m \neq \frac{m}{x}$, whence by the disjointness property of $\alpha$, $(\alpha \cdot \beta)(x) = 0$. In the second case, we are left with

$$\begin{aligned} (\alpha \cdot \beta)(x) &= \bigvee_{m \in U(A)} \Big( \alpha(m) \wedge \alpha\Big(\frac{m}{x}\Big) \Big) \\ &= \bigvee_{m \in U(A)} \alpha(m) \\ &= \mathfrak{u}(\alpha) \\ &= 1. \end{aligned}$$

It follows that $\alpha \cdot \beta = \mathbf{1}$, whence $\alpha \in U(A_dL)$. □

Let $A$ be an integral domain and let $K$ be the field of fractions of $A$. For any $\alpha \in A_dL$ we define the map $\bar{\alpha} \colon K \to L$ by

$$\bar{\alpha}(x) = \begin{cases} \alpha(x), & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\bar{\alpha} \in K_dL$ and that the map $\alpha \mapsto \bar{\alpha}$ is a ring embedding of $A_dL$ into $K_dL$.

We are now in position to classify the classical ring of quotients of a ring $A_dL$. This generalizes [12, Theorem 3.3].

**Theorem 2.5.** *Let $A$ be an integral domain and $K$ be the field of fractions of $A$. Then $q(A_dL) = K_dL$.*

*Proof.* Let $g\colon A_dL \to K_dL$ be the inclusion map. If $\gamma$ is a unit in $\in A_dL$, then $\mathfrak{u}(\gamma) = 1_L$. This implies that the function $\gamma^{-1}$ belongs to $K_dL$. Thus, the unit of the ring $A_dL$ is a unit of the ring $K_dL$, and the map

$$h\colon q(A_dL) \to K_dL \quad \text{given by} \quad h\Big(\frac{\alpha}{\gamma}\Big) = g(\alpha)g(\gamma)^{-1} = \alpha\gamma^{-1}$$

is a ring homomorphism, see [14, Proposition 5.10]. It is clear that the map $h$ is onto and one-one. □

When $L$ is a spatial frame, say $L = \Omega(X)$, then it is known that $\mathfrak{Z}L = C(X,\mathbb{Z})$. This yields the following corollary.

**Corollary 2.6.** [12, Theorem 3.3] *For any zero-dimensional space $X$,*

$$q(C(X,\mathbb{Z})) = C(X,\mathbb{Q}_d).$$

Next, we would like to classify the maximal ring of quotients. We recall the construction for a given ring $R$. For an ideal $I$ of $R$, we let $\mathrm{Hom}_R(I,R)$ denote the group of $R$-module homomorphisms from $I$ into $R$. We let $\mathfrak{D}(R)$ denote the collection of all dense ideals of $R$. This collection is a filter and thus the collection $\{\mathrm{Hom}_R(I,R) : I \in \mathfrak{D}(R)\}$ is a directed system. In particular, for any $I, J \in \mathfrak{D}(R)$, if $I \subseteq J$, then there is an embedding of $\mathrm{Hom}_R(J,R)$ into $\mathrm{Hom}_R(I,R)$ given by restriction. The direct limit of these hom groups is the maximal ring of quotients of $R$. The direct limit is given by taking the union and equipping the union with an equivalence relation $f \sim g$ defined by given $f \in \mathrm{Hom}_R(I,R)$ and $g \in \mathrm{Hom}_R(J,R)$, then there is some $K \subseteq I \cap J$ and $h \in \mathrm{Hom}_R(K,R)$ such that the restriction of $f$ and $g$ to $K$ equal $h$. (For more information the interested reader is referred to [10].)

In the classic text [4], the authors show that $Q(C(X))$ can be viewed as a direct limit of rings of continuous functions. For their construction they use the collection of dense open subsets of $X$, denoted $\mathcal{V}(X)$. (We shall use $\mathcal{V}_0(X)$ to denote the collection of dense cozero-sets of $X$; this is used at the end of this section.)

**Theorem 2.7.** [5, Representation Theorem 2.6] *For a Tychonoff space $X$,*

$$Q(C(X)) = \lim_{U \in \mathcal{V}(X)} C(U).$$

Recently, Abedi [1] generalized this to $\mathcal{R}L$ for a completely regular frame.

**Theorem 2.8.** [1, Theorem 3.8] *Let $L$ be a completely regular frame. Then*

$$Q(\mathcal{R}L) = \lim_{a \in \mathcal{D}(L)} \mathcal{R}(\downarrow a).$$

Our main theorem below (Theorem 2.12) is in the same vein as Theorem 2.8. Our proof will consist of a few lemmas that we will combine to conclude the theorem. We take some time to set up the appropriate notations and concepts.

Let $R$ be a ring. Recall that a *ring of quotients* of $R$ is an extension ring $S$ with the property that for each $s \in S$, $s^{-1}R$ is dense in $S$. For reduced rings, this is equivalent to saying that for each nonzero $s \in S$, there is an $r \in R$ such that $0 \neq rs \in R$. (We recall that $s^{-1}R = \{a \in R : sa \in R\}$.)

For any $x \in L$ and $\alpha \in A_dL$, the composition $o_x \circ \alpha$ belongs to $A_d(\downarrow x)$, and the induced map $\alpha \mapsto o_x \circ \alpha$ is a ring homomorphism. We shall denote this map by $\Psi_x : A_dL \to A_d(\downarrow x)$.

Let $I$ be an ideal of $A_dL$. We define the cozero-set of $I$ as

$$\operatorname{coz}(I) = \bigvee_{\alpha \in I} \operatorname{coz}(\alpha).$$

**Lemma 2.9.** *For any $x \in \mathcal{D}(L)$, the ring homomorphism $\Psi_x$ is injective. Furthermore, there is an embedding of $A_dL$ into $K_d(\downarrow x)$.*

*Proof.* The second statement follows from the first since $A_d(\downarrow x)$ is embedded in $K_d(\downarrow x)$.

Suppose $\alpha \in A_dL$ satisfies $\Psi_x(\alpha) = (\mathbf{0})$. This means that for all $0 \neq a \in A$, $\Psi_x(\alpha)(a) = 0$. But then for each $0 \neq a \in A$,

$$0 = o_x(\alpha(a)) = x \wedge \alpha(a)$$

so that by density, $\alpha(a) = 0$, from which it follows that $\alpha = \mathbf{0}$. $\square$

**Proposition 2.10.** *Let $K$ be a field. For any dense ideal $I$ of $K_dL$, $\operatorname{Hom}_{K_dL}(I, K_dL) \subseteq K_d(\downarrow \operatorname{coz}(I))$.*

*Proof.* Let $0 \neq \varphi \in \operatorname{Hom}_{K_dL}(I, K_dL)$. For any non-zero $e \in I$, the element $o_{\operatorname{coz}(e)} \circ e$ is an invertible element of $K_d(\downarrow \operatorname{coz}(e))$ and therefore we can define

$$\varphi_e = (o_{\operatorname{coz}(e)} \circ \varphi(e)) \cdot (o_{\operatorname{coz}(e)} \circ e)^{-1}$$

and observe that $\varphi_e \in K_d(\downarrow \operatorname{coz}(e))$.

Let $t_e = o_{\operatorname{coz}(e)} \circ e$. Then $t_e^{-1}(x) = \begin{cases} e(\frac{1}{x}), & \text{if } \frac{1}{x} \in A \\ 0_L, & \text{otherwise.} \end{cases}$

In particular, for any $0 \neq x \in K$

$$\begin{aligned} \varphi_e(x) &= \bigvee_{kj=x} \big(t_e^{-1}(k) \wedge \varphi(e)(j) \wedge \operatorname{coz}(e)\big) \\ &= \bigvee_{0 \neq k \in K} \big(e(k) \wedge \varphi(e)(kx) \wedge \operatorname{coz}(e)\big) \\ &= \bigvee_{0 \neq k \in K} \big(e(k) \wedge \varphi(e)(kx)\big) \\ &= \bigvee_{0 \neq k \in K} \big(e(k) \wedge \varphi(e)(kx)\big). \end{aligned}$$

Also, $\varphi_e(0) = \varphi(e)(0) \wedge \operatorname{coz}(e)$.

Observe that since $\varphi$ is a $K_dL$-module homomorphism, for any $e, d \in I$, $e \cdot \varphi(d) = d \cdot \varphi(e)$.

Next, define $\overline{\varphi} : K \to (\downarrow \operatorname{coz}(I))$ by

$$\overline{\varphi}(x) = \bigvee_{0 \neq e \in I} \varphi_e(x) = \bigvee_{0 \neq e \in I} \bigvee_{0 \neq k \in K} \big(e(k) \wedge \varphi(e)(kx)\big).$$

We argue that $\overline{\varphi} \in K_d(\downarrow \operatorname{coz}(I))$. Suppose $x, y \in K$ are distinct and non-zero. Then

$$\overline{\varphi}(x) \wedge \overline{\varphi}(y) = \bigvee_{0 \neq e \in I} \varphi_e(x) \wedge \bigvee_{0 \neq d \in I} \varphi_d(y).$$

To show that this equals $0_L$ it suffices to show, by two applications of the frame law, that

$$\varphi_e(x) \wedge \varphi_d(y) = 0_L$$

for all $0 \neq e, d \in I$. Applying the frame law twice, it suffices to show that

$$\big(e(k) \wedge \varphi(e)(kx)\big) \wedge \big(d(j) \wedge \varphi(d)(jy)\big) = 0_L.$$

Now, $e(k) \wedge \varphi(d)(jy) \leq (e \cdot \varphi(d))(kjy)$, and $d(j) \wedge \varphi(e)(kx) \leq (d \cdot \varphi(e))(kjx)$. We use our observation now.

$$\begin{aligned}
\big(e(k) \wedge \varphi(e)(kx)\big) \wedge \big(d(j) \wedge \varphi(d)(jy)\big) &= \big(e(k) \wedge \varphi(d)(jy)\big) \wedge \big(d(j) \wedge \varphi(e)(kx)\big) \\
&\leq (e \cdot \varphi(d))(kjy) \wedge (d \cdot \varphi(e))(kjx) \\
&= (e \cdot \varphi(d))(kjy) \wedge (e \cdot \varphi(d))(kjx) \\
&= 0_L.
\end{aligned}$$

We leave the case where $x \neq 0$ and $y = 0$ to the interested reader.

Next, we show that $\bigvee_{x \in K} \overline{\varphi}(x) = \operatorname{coz}(I)$:

$$\begin{aligned}
\bigvee_{x \in K} \overline{\varphi}(x) &= \bigvee_{x \in K} \bigvee_{0 \neq e \in I} \varphi_e(x) \\
&= \bigvee_{0 \neq e \in I} \bigvee_{x \in K} \varphi_e(x) \\
&= \bigvee_{0 \neq e \in I} \operatorname{coz}(e) \\
&= \operatorname{coz}(I).
\end{aligned}$$

□

**Proposition 2.11.** *For any $x \in \mathcal{D}(L)$, the ring $K_d(\downarrow x)$ is a ring of quotients of $A_dL$.*

*Proof.* Observe that $A_dL$ embeds into $K_dL$ and that $K_dL$ embeds into $K_d(\downarrow x)$ since $x$ is a dense element of $L$.

Let $0 \neq h \in K_d(\downarrow x)$. We want to find $f \in A_dL$ such that $0 \neq fh \in A_dL$.

Let $0 \neq y \in K$ such that $0_L < h(y)$. Since $y \in K$ we can write $y = \frac{a}{b}$ for $a, b \in A$. Choose a complemented element of $L$, say $c$, such that $0_L < c \leq h(y)$. We claim that $0 \neq h(\bar{b}\gamma_c) \in A_dL$. Let $x \in K \smallsetminus A$.

First, observe that

$$(\bar{b}h)(a) = h(\frac{a}{b}) = h(y).$$

Next,

$$\begin{aligned}
(h \cdot (\bar{b}\gamma_c))(x) &= ((\bar{b}h) \cdot \gamma_c)(x) \\
&= \bigvee_{jk=x} ((\bar{b}h)(j) \wedge \gamma_c(k)) \\
&= (\bar{b}h)(x) \wedge c \\
&\leq (\bar{b}h)(x) \wedge h(y) \\
&= (\bar{b}h)(x) \wedge (\bar{b}h)(a) \\
&= 0_L.
\end{aligned}$$

Thus, $0 \neq h \cdot (\bar{b}\gamma_c) \in A_dL$ and we conclude that $K_d(\downarrow x)$ is a ring of quotients of $A_dL$. □

**Theorem 2.12.** *Let $A$ be an integral domain. The maximal ring of quotients of $A_dL$ is given by the following:*

$$Q(A_dL) = \lim_{x\in\mathcal{D}} K_d(\downarrow x).$$

*Proof.* We begin by pointing out that $A_dL \leq K_dL \leq Q(A_dL)$ and therefore, $Q(A_dL) = Q(K_dL)$. So without loss of generality we may assume that $A = K$.

We need to first show that the collection $\{K_d(\downarrow x)\}$ is a directed system. But this follows from the fact that if $x, y \in \mathcal{D}(L)$, then $x \wedge y \in \mathcal{D}(L)$, and both $K_d(\downarrow x)$ and $K_d(\downarrow y)$ embed into $K_d(\downarrow(x \wedge y))$. Thus, the ring on the right hand side of the display is a ring extension of $K_dL$; call it $R$.

Proposition 2.11 says that each member of the directed system forming $R$ is a ring of quotients of $K_dL$ and hence $R$ is a ring of quotients of $K_dL$. Therefore, $R \leq Q(A_dL)$. On the other hand, by Proposition 2.10, each $\mathrm{Hom}(I, K_dL) \leq K_d(\downarrow \mathrm{coz}(I))$ and since $\mathrm{coz}(I) \in \mathcal{D}$ whenever $I$ is a dense ideal it follows that $Q(K_dL) \leq R$. □

We return to the spatial case.

**Corollary 2.13.** *For any zero-dimensional space $X$*

$$Q(C(X,\mathbb{Z})) = \lim_{U\in\mathcal{V}(X)} C(U, \mathbb{Q}_d).$$

We would like to say more. Here we refer the reader to [6] and recall that a lattice-ordered group is said to be *laterally complete* if every pairwise disjoint collection of positive elements has a least upper bound. Given an $\ell$-group $G$, a lateral completion of $G$ is a laterally complete $\ell$-group $H$ that contains a dense copy of $G$, and no other intermediate $\ell$-subgroup of $H$ is laterally complete. Every archimedean $\ell$-group $G$ has an essentially unique lateral completion, which we shall denote by $G^L$. It is known (see [11, Corollary 2.7.1]) that, for an archimedean $f$-ring $A$,

$$Q(A) = q(A^L) = (q(A))^L.$$

The following is known. Recall that for a compact Hausdorff space $Z$, $EZ$ denotes the absolute of $Z$ (aka the Gleason cover). For a space $X$, $D(X)$ denote the collection of functions, say $f : X \to \bar{\mathbb{R}}$ whose domain is $X$ and whose codomain is the two-point compactification of the real numbers, satisfying that $\mathrm{re}(f) = f^{-1}(\mathbb{R})$ is dense. $D(X)$ need not be a group under addition but it is always a lattice. The notation $D(X,\mathbb{Z})$ should be understood.

**Proposition 2.14.** [6, Proposition 6.9] *Suppose $X$ is a zero-dimensional space. Then*

$$C(X,\mathbb{Z})^L = D(E\beta_0X, \mathbb{Z}).$$

Putting these two displays together yields the following.

**Proposition 2.15.** *Suppose $X$ is a zero-dimensional space. Then*

$$Q(C(X,\mathbb{Z})) = q(D(E\beta_0X, \mathbb{Z})).$$

Now, let $E$ be a compact zero-dimensional quasi $F$-space so that $D(E,\mathbb{Z})$ is an $\ell$-group. We aim to classify $q(D(E,\mathbb{Z}))$. Let $f, g \in D(E,\mathbb{Z})^+$ with $g$ regular; $Z(g) = \emptyset$. It follows that $h = \frac{f}{g} \in D(E)$. Set $U = \mathrm{re}(f) \cap \mathrm{re}(g)$; $U$ is a dense cozero-set of $E$. Then $h \in C(U, \mathbb{Q}_d)$. The proof of the converse should be apparent; use that $X$ being a quasi $F$-space implies that any $f \in C(U,\mathbb{Z})$ has an extension in $D(X,\mathbb{Z})$ . Consequently, we can add to Proposition 2.15. We let the reader generalize this to quasi $F$-spaces.

**Theorem 2.16.** *Suppose $X$ is a zero-dimensional space. Then*

$$Q(C(X,\mathbb{Z})) = \lim_{U\in\mathcal{V}_0(E\beta_0X)} C(U, \mathbb{Q}_d).$$

Here are some final thoughts for this section. When we restrict our attention to a subring $\mathbb{A}$ of $\mathbb{R}$, our ring $\mathbb{A}_d L$ is an $f$-ring and as such it is common to discuss the collection of bounded continuous functions, denoted by $\mathbb{A}_d^* L$.

**Proposition 2.17.** *Let $\mathbb{A}$ be a subring of $\mathbb{R}$. Then $\mathbb{A}_d L$ is a ring of quotients of $\mathbb{A}_d^* L$. In particular,*

$$Q(\mathbb{A}_d^* L) = Q(\mathbb{A}_d L).$$

*Proof.* Let $0 \neq h \in \mathbb{A}_d L$. Choose $r \in \mathbb{A}$ such that $0_L < h(r)$ and set $c = h(r)$. Then

$$(h \cdot \gamma_c)(x) = h(x) \wedge c,$$

and so by disjointness it follows that $h \cdot \gamma_c \in A_d^* L$. □

## 3. Equalities of $A_d L$ to its rings of quotients

The classical result is that $C(X) = Q(C(X))$ precisely when $X$ is an extremally disconnected $P$-space. In order to classify when $A_d L$ is its own maximal ring of quotients we need a stronger concept than extremally disconnected. In [8], the authors define the concept of an $F$-quotient map where $F$ is the 4-element boolean algebra. The quotient map $h : L \to M$ is an *$F$-quotient map* if for any frame homomorphism $f : F \to M$ there is a frame homomorphism $\hat{f} : F \to L$ such that $h \circ \hat{f} = f$. On the other hand, in [3], Banaschweski is interested in determining when a quotient map $h : L \to M$ induces a ring isomorphism $\mathfrak{Z}^* h : \mathfrak{Z}^* L \to \mathfrak{Z}^* M$. He calls such a quotient map a *$\mathfrak{Z}^*$-isomorphism.*

We leave it to the interested reader to check the following; see [3] if needed.

**Lemma 3.1.** *Suppose $h : L \to M$ is a quotient map. The following statements are equivalent.*
1. *The map $h$ is an $F$-quotient map.*
2. *The map $h$ is a $\mathfrak{Z}^*$-isomorphism.*
3. *For all $x \in BM$, there is $y \in BL$ such that $h(y) = x$.*

We would like to point out that the notion of an $F$-quotient is, in the spatial context, known as 2-embedded. The subspace $Y$ of $X$ is *2-embedded* if for each clopen subset $K$ of $Y$ there is a clopen subset $J$ of $X$ such that $J \cap Y = K$. Recall that a Tychonoff space is extremally disconnected if and only if each subspace is 2-embedded. We first generalize this last result to extremally disconnected frames. Looking ahead we actually need a more general concept and thus will change the name of an $F$-quotient map.

**Definition 3.2.** Let $\kappa$ be any cardinal (greater than or equal to 2) and $L$ a zero-dimensional frame. A *$\kappa$-separation* of $L$ is a collection $\{c_\sigma\}_{\sigma<\kappa}$ such that for each $\sigma \neq \tau$, $c_\sigma \wedge c_\tau = 0_L$ and $\bigvee_{\sigma<\kappa} c_\sigma = 1_L$.

Note that in a $\kappa$-separation each member is automatically a complemented element of $L$. We also note that we are allowing for $c_\sigma$ to be equal $0_L$. A consequence of this is that if $\kappa < \tau$, then a $\kappa$-separation can be viewed as a $\tau$-separation.

Continuing, a quotient map $h \colon L \to M$ is a *$\kappa$-quotient map* if for every $\kappa$-separation of $\{c_\sigma\}_{\sigma<\kappa}$ of $M$ there exists a $\kappa$-separation of $L$, say $\{x_\sigma\}_{\sigma<\kappa}$ such that $h(x_\kappa) = c_\kappa$.

**Remark 3.3.** Observe that a 2-separation is simply an enumeration $\{c, c^\perp\}$ for any $c \in BL$. Furthermore, a 2-quotient map is precisely an $F$-quotient map. Furthermore, if $h : L \to M$ is a $\kappa$-quotient map, then it is a $\sigma$-quotient map for all $\sigma \leq \kappa$.

**Proposition 3.4.** *The following are equivalent for a zero-dimensional frame $L$.*
1. *$L$ is extremally disconnected.*
2. *Every dense quotient of $L$ is 2-quotient.*

3. *Every open quotient of $L$ is 2-quotient.*
4. *Every dense open quotient of $L$ is 2-quotient.*

*Proof.* 1. $\Rightarrow$ 2. Suppose that $L$ is extremally disconnected and let $h : L \to M$ be a dense quotient map. Take a seperation $\{a, b\}$ of a frame $M$, that is, $a \vee b = 1$ and $a \wedge b = 0$. Since $h : L \to M$ is a quotient map, there exist $x, y \in L$ such that $h(x) = a$ and $h(y) = b$. Then

$$h(x \wedge y) = h(x) \wedge h(y) = a \wedge b = 0$$

which implies that $x \wedge y = 0$ since $h : L \to M$ is dense. This implies that $y \leq x^\perp$ so that $b = h(y) \leq h(x^\perp)$. Observe that $a = h(x) \leq h(x^{\perp\perp})$ in $M$. Then $h(x^{\perp\perp}) \wedge b = 0$, which implies that $h(x^{\perp\perp}) \leq b^\perp \leq a$ and hence $h(x^{\perp\perp}) = a$.

2. $\Rightarrow$ 4. This is trivial.

4. $\Rightarrow$ 3. Suppose that every dense quotient of $L$ is 2-quotient and let $h\colon L \to {\downarrow}x$ for $x \in L$. Take a seperation $\{c, d\}$ of ${\downarrow}x$, that is, $c \vee d = x$ and $c \wedge d = 0$.

Consider the dense open quotient map $h\colon L \to {\downarrow}(x \vee x^\perp)$. Take a seperation $\{c', d'\}$ of ${\downarrow}(x \vee x^\perp)$, that is, $c' \vee d' = x \vee x^\perp$ and $c' \wedge d' = 0$, such that $c' = c$ and $d' = d \vee x^\perp$. By hypothesis there exists $a \in BL$ such that $a \wedge (x \vee x^\perp) = c'$ and $a^\perp \wedge (x \vee x^\perp) = d'$. We show that $h(a) = c$ and $h(a^\perp) = d$, that is $a \wedge x = c$ and $a^\perp \wedge x = d$, respectively. Then

$$c = a \wedge (x \vee x^\perp) = (a \wedge x) \vee (a \wedge x^\perp).$$

Observe that $a \wedge x^\perp \leq c \leq x$, and also $a \wedge x^\perp \leq x^\perp$ so that $a \wedge x^\perp \leq x \wedge x^\perp = 0$. So $c = a \wedge x$.

Next we show that $a^\perp \wedge x = d$. We have

$$(a^\perp \wedge x) \wedge c = (a^\perp \wedge x) \wedge (a \wedge x) = 0.$$

We know that $a^\perp \wedge x$ is the complement of $c$ in ${\downarrow}x$, but the complement of $c$ is $d$. So $a^\perp \wedge x = d$. Therefore $h(a) = a \wedge x$ and $h(a^\perp) = a^\perp \wedge x$.

3. $\Rightarrow$ 4. This is trivial.

4. $\Rightarrow$ 1. Assume every dense open quotient of $L$ is 2-quotient. Consider $a \in L$ for the purpose of showing that $a^\perp \vee a^{\perp\perp} = 1$. The open quotient map $h\colon L \to {\downarrow}(a^\perp \vee a^{\perp\perp})$ is dense because $a^\perp \vee a^{\perp\perp}$ is a dense element. Since $a^\perp$ and $a^{\perp\perp}$ are complements in ${\downarrow}(a^\perp \vee a^{\perp\perp})$, there exist elements $b$ and $c$ in $BL$ such that $h(b) = a^\perp$, $h(c) = a^{\perp\perp}$, $b \wedge c = 0$ and $b \vee c = 1$. Simple calculations shows that

$$a^\perp = h(b) = b \wedge (a^\perp \vee a^{\perp\perp}) = (b \wedge a^\perp) \vee (b \wedge a^{\perp\perp})$$

and

$$a^{\perp\perp} = h(c) = c \wedge (a^\perp \vee a^{\perp\perp}) = (c \wedge a^\perp) \vee (c \wedge a^{\perp\perp}).$$

But $c \wedge a^\perp = c \wedge (b \wedge (a^\perp \vee a^{\perp\perp})) = 0$ which implies $c \leq a^{\perp\perp}$. Also $b \wedge a^{\perp\perp} = b \wedge (c \wedge (a^\perp \vee a^{\perp\perp})) = 0$ implies $b \leq a^{\perp\perp\perp} = a^\perp$. Therefore $1 = b \vee c \leq a^\perp \vee a^{\perp\perp}$. Thus, $a^\perp \vee a^{\perp\perp} = 1$. □

**Definition 3.5.** Let $2 \leq \kappa$ be a cardinal. We call a frame $L$ *strongly $\kappa$-disconnected* if every open quotient is a $\kappa$-quotient map. Observe that a strongly $\kappa$-disconnected frame is necessarily extremally disconnected.

**Remark 3.6.** If we drop the word strongly from the above definition then we run into issues as others have used the term $\kappa$-disconnected to mean a cardinal weakening of things between extremally disconnected and basically disconnected. (For example, see [7].)

**Proposition 3.7.** *The following are equivalent for a zero-dimensional frame $L$.*

1. *$L$ is strongly $\kappa$-disconnected, that is, every open quotient of $L$ is a $\kappa$-quotient map.*
2. *Every dense quotient of $L$ is a $\kappa$-quotient map.*
3. *Every dense open quotient of $L$ is a $\kappa$-quotient map.*

*Proof.* Before we begin we would like to point out that any frame satisfying any of the above conditions is automatically an extremally disconnected frame by the previous proposition.

1. $\Rightarrow$ 2. Suppose that $L$ is strongly $\kappa$-disconnected and let $h : L \to M$ be a dense quotient map. Take a $\kappa$-separation $\{c_\sigma\}_{\sigma<\kappa}$ of $M$. Since $h : L \to M$ is a quotient map, there exist a collection $\{x_\sigma\}_{\sigma<\kappa}$ of $L$ such that $h(x_\sigma) = c_\sigma$. Fix $\sigma$ and let $\sigma \neq \tau < \kappa$. Then

$$h(x_\sigma \wedge x_\tau) = h(x_\sigma) \wedge h(x_\tau) = c_\sigma \wedge c_\tau = 0$$

which implies that $x_\sigma \wedge x_\tau = 0$ since $h : L \to M$ is dense. That $x_\tau \leq x_\sigma^\perp$ implies two things: $x_\sigma^{\perp\perp} \wedge x_\tau = 0$ and $c_\tau = h(x_\tau) \leq h(x_\sigma^\perp)$. The first part forces $h(x_\sigma^{\perp\perp}) \wedge c_\tau = 0$ for all $\tau \neq \sigma$. Therefore, $h(x_\sigma^{\perp\perp}) \wedge \bigvee c_\tau$, which implies that

$$h(x_\sigma^{\perp\perp}) \leq \Big(\bigvee_{\tau\neq\sigma} c_\tau\Big)^\perp = c_\sigma.$$

Clearly, $c_\sigma = h(x_\sigma) \leq h(x_\sigma^{\perp\perp})$ in $M$. Hence $h(x_\sigma^{\perp\perp}) = c_\sigma$.

We leave it to the interested reader to check that $x_\sigma^{\perp\perp} \wedge x_\tau^{\perp\perp} = 0$ so that without loss of generality we can assume that $x_\sigma = x_\sigma^{\perp\perp}$. Since $L$ is extremally disconnected each of these elements is a complemented element. So, we are left with explaining why $\bigvee x_\sigma = 1_L$. Set $x = \bigvee x_\sigma$ and let $j : L \to {\downarrow}x$ be the dense open question map. Since $\{x_\sigma\}$ is a $\kappa$-separation of ${\downarrow}x$ it follows by hypothesis there is a $\kappa$-separation of $L$, say $\{y_\sigma\}$ such that $j(y_\sigma) = x_\sigma$. By a similar argument we have already made, we know that $y_\sigma \wedge x_\tau = 0$. Furthermore, $h(x_\sigma \vee y_\sigma) = c_\sigma$. This is enough for the proof however, we point out that $y_\sigma = x_\sigma$.

2. $\Rightarrow$ 3. This is trivial.

3. $\Rightarrow$ 1. This proof is similar to the one given in the previous proposition. □

Our next theorem shows the importance of the concept of a strongly $\kappa$-disconnected frame. We begin with a lemma that shows the usefulness of certain $\kappa$-quotient maps.

**Lemma 3.8.** *Let $L$ be a zero-dimensional frame and $A$ an integral domain. For each $x \in L$, the map $\varphi : A_dL \to A_d({\downarrow}x)$ is a surjection if and only if $o_x$ is a $|A|$-quotient map.*

*Proof.* Set $\kappa = |A|$ and enumerate $A = \{a_\sigma\}_{\sigma<\kappa}$.

Recall that the map $\varphi$ is defined by $\varphi(f) = o_x \circ f$. First suppose that $\varphi$ is onto and let $\{c_\sigma\}_{\sigma<\kappa}$ be a $\kappa$-separation of ${\downarrow}x$. Define $h : A \to ({\downarrow}x)$ by $h(a_\sigma) = c_\sigma$. Clearly, $h \in A_d({\downarrow}x)$. By hypothesis, there is an $f \in A_dL$ such that $\varphi(f) = h$. The collection $\{f(a_\sigma)\}_{\sigma<\kappa}$ is a $\kappa$-separation of $L$, and

$$\begin{aligned} o_x(f(a_\sigma)) &= (o_x \circ f)(a_\sigma) \\ &= h(a_\sigma) \\ &= c_\sigma. \end{aligned}$$

Therefore, $o_x$ is a $\kappa$-quotient map.

Conversely, suppose that $o_x$ is a $\kappa$-quotient map. Let $h \in A_d({\downarrow}x)$. Then $\{h(a_\sigma)\}_{\sigma<\kappa}$ is a $\kappa$-separation of ${\downarrow}x$. By hypothesis, there is a $\kappa$-separation of $L$, say $\{b_\sigma\}_{\sigma<\kappa}$, such that $x \wedge b_\sigma = h(a_\sigma)$. Define $f : A \to L$ by $f(a_\sigma) = b_\sigma$; $f \in A_dL$. Moreover, $\varphi(f) = o_x \circ f = h$. □

**Theorem 3.9.** *The following are equivalent for a zero-dimensional frame $L$ and any integral domain $A$.*

1. *$Q(A_dL) = A_dL$.*
2. *$A_dL$ is self-injective.*

3. *$A$ is a field and $L$ is a strongly $|A|$-disconnected frame.*

*Proof.* 1. $\Leftrightarrow$ 2. This is patent.

1. $\Rightarrow$ 3. It is clear that $A$ must be a field. Our goal is to show that every dense open quotient is a $\kappa$-quotient map. Let $x \in L$ be a dense element. By Theorem 2.12, we know that the injective map $A_dL = A_d(\downarrow x)$ is also onto. Consequently, by Lemma 3.8, $o_x$ is a $\kappa$-quotient map.

3. $\Rightarrow$ 1. Let $x \in L$ be a dense element. By 3., the map $o_x$ is a $\kappa$-quotient map and so the embedding $\varphi : A_dL \to A_d(\downarrow x)$ is onto, and hence an isomorphism. It follows by Theorem 2.12 that $Q(A_dL) = K_dL = A_dL$. □

We can now focus on when $q(A_dL) = Q(A_dL)$ and in particular obtain when $A_dL$ is self-injective. Since any integral domain has the same cardinality as its field of fractions, we gather that:

**Corollary 3.10.** *Let $A$ be an integral domain. Then $q(A_dL) = Q(A_dL)$ if and only if $L$ is strongly $|A|$-disconnected.*

Observe then that to say $\mathbb{Q}_dL$ is self-injective is equivalent to saying that $L$ is strongly $\aleph_0$-disconnected. We can show that, for a zero-dimensional frame $L$, if $L$ is strongly $\aleph_0$-disconnected frame, then it is a $P$-frame. Interestingly, in the spatial case and barring measurable cardinals, an extremally disconnected $P$-space is necessarily discrete. Banaschewski studied the point-free situation in [2], calling any extremally disconnected $P$-frame an *almost boolean frame*. Our next result characterizes the case for $\aleph_0$, and surprisingly also for the continuum $\mathfrak{c}$.

**Proposition 3.11.** *Suppose $L$ is zero-dimensional. The following statements are equivalent.*

1. *The frame $L$ is strongly $\aleph_0$-disconnected.*
2. *The frame $L$ is an almost boolean frame.*
3. *The frame $L$ is strongly $\mathfrak{c}$-disconnected.*

*Proof.* 3. $\Rightarrow$ 1. Obvious.

1. $\Rightarrow$ 2. Suppose $L$ is a strongly $\aleph_0$-disconnected. In order to show that $L$ is a $P$-frame, we need to demonstrate that every cozero-element of $L$ is a complemented element. Since $L$ is zero-dimensional and extremally disconnected, it is strongly zero-dimensional which implies that every cozero-element is a join of complemented elements. Let $x$ be a cozero-element of $L$. We can write $x = \bigvee_{i \in \mathbb{N}} c_i$ where each $c_i$ is a complemented element of $L$. Moreover, we can assume that $c_i \wedge c_j = 0_L$ whenever $i \neq j$. Set $y = x \vee x^{\perp}$, a dense element of $L$. Since $L$ is extremally disconnected $z^{\perp} \in BL$ and so we notice that $\{c_i\} \cup \{x^*\}$ is an $\aleph_0$-separation of $y$. By hypothesis, there is an $\aleph_0$-separation of $L$, say $\{b_i\} \cup \{z\}$ such that $b_i \wedge y = c_i$ and $z \wedge y = x^{\perp}$.

Set $s_i = b_i \wedge c_i^{\perp}$; $s_i \in BL$. Observe that $c_i \vee s_i = b_i$. Consider the collection $\{c_i\} \cup \{s_i\} \cup \{z\}$. This a pairwise disjoint collection which joins up to $1_L$; the collection is an $\aleph_0$-separation. It follows that any join of elements from this collection is again a complemented element. In particular, $x = \bigvee c_i$ is a complemented element of $L$.

2. $\Rightarrow$ 3. Suppose $L$ is an almost boolean frame. Then $\mathcal{R}L = \mathcal{R}_dL$ since $L$ is a $P$-frame. Furthermore, $\mathcal{R}L$ is self-injective by [1, Theorem 4.7]. Thus, we can apply Theorem 3.9 and conclude that $L$ is strongly $\mathfrak{c}$-disconnected. □

**Corollary 3.12.** *Let $L$ be a zero-dimensional frame. Then $L$ is an almost boolean frame if and only if $Q(\mathfrak{Z}L) = \mathbb{Q}_dL$.*

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