αCC-BAER RINGS

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Abstract. Let α denote an infinite cardinal or ∞ which is used to signify no cardinal constraint. This work introduces the concept of an αcc-Baer ring and demonstrates that a commutative semiprime ring A with identity is αcc-Baer if and only if Spec(A) is αcc-disconnected. Moreover, we prove that for each commutative semprime ring A with identity there exists a minimum αcc-Baer ring of quotients, which we call the αcc-Baer hull of A. In addition, we investigate a variety of classical α-Baer ring results within the contexts of αcc-Baer rings and apply our results to produce alternative proofs of some classical results such as A is α-Baer if and only if Spec(A) is α-disconnected. Lastly, we apply our results within the contexts of archimedean f-rings.

1. Preliminaries

In [C12] the author introduces the concept of an αcc-projectable ℓ-group and establishes the existence of an αcc-projectable hull . We develop the ring theoretic analogs of the aforementioned concepts. Throughout this article we shall assume that all rings are commutative and possess an identity. We call a ring semiprime if it has no non-zero nilpotent elements. In this section we provide the preliminary ring theoretic material -which the knowledgeable reader may elect to skip- and present an overview of some classical results for α-Baer rings.

Definitions and Remarks 1.1. (a) Let $S \subseteq A$. The *annihilator* of S, denoted $\mathrm{Ann}(S)$, is the set $\{t \in A | st = 0 \text{ for all } s \in S\}$. If $s \in A$, the annihilator of s is $\mathrm{Ann}(\{s\})$ and will be denoted by $\mathrm{Ann}(s)$. It is known that the following properties hold.

(i) $S \subseteq \mathrm{Ann}(\mathrm{Ann}(S))$
(ii) if $S \subseteq T$, then $\mathrm{Ann}(T) \subseteq \mathrm{Ann}(S)$, and

(iii) $\mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(S))) = \mathrm{Ann}(S)$.

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Throughout we write AnnAnn(S) for Ann(Ann(S)).
(b) For a ring A, Spec(A) denotes the set of all prime ideals of A endowed with the *hull-kernel* topology. For $S \subseteq A$ define $U(S) = \{P \in \text{Spec}(A) | S \nsubseteq P\}$. Recall that the basic open sets in Spec(A) are of the form $U(a)$ for $a \in A$. For future reference, we recall some of the properties of the basic open sets:

(i) $U(a) \cap U(b) = U(ab)$, for all $a, b \in A$;

(ii) $U(a) = \emptyset$ if and only if a is nilpotent; and when A is semiprime $U(a) = \emptyset$ if and only if $a = 0$.

(iii) $U(a) = \text{Spec}(A)$ if and only if a is a unit.

We let the reader verify that every open set in Spec(A) is of the form $U(I)$ for some ideal I and that $U(I) = \bigcup_{i \in I} U(i)$. Throughout we let $V(S)$ denote the complement of $U(S)$ in Spec(A). We now collect some technical lemmas which will be of particular interest to us in §3. The results are known but we include proofs for completeness sake.

Lemma 1.2. *If $I, J \leq A$ are ideals of the semiprime ring A such that $U(I), U(J)$ are disjoint sets in Spec(A) and Spec$(A) = U(I) \cup U(J)$, then $A = I \oplus J$. Moreover, I and J are each generated by a single idempotent.*

Proof. Suppose that I, J are ideals of A for which $U(I), U(J)$ are disjoint in Spec(A) and Spec$(A) = U(I) \cup U(J)$. We assert that $I \cap J = \{0\}$: otherwise, there exists an $0 \neq r \in I \cap J$, and since A is semiprime, it follows that $r^n \neq 0$ for any $n \in \mathbb{N}$. Define $S = \{r^n | n \in \mathbb{N}\}$ since S is a multiplicative set, there exists a prime ideal P which is maximal with respect to $P \cap S = \emptyset$. But then $P \in U(I)$ and $P \in U(J)$ implies that $P \in U(I) \cap U(J)$ which is a contradiction. Now suppose for the sake of contradiction that $A \neq I + J$. Then there exists a prime ideal P such that $I + J \subseteq P$. But $P \in \text{Spec}(A)$ implies that $P \in U(I)$ or $P \in U(J)$ which is a contradiction. ☐

Remark 1.3. Observe that if P is a prime ideal and AnnAnn$(I) \nsubseteq P$, then Ann$(I) \subseteq P$; for if $r \in \text{AnnAnn}(I) \setminus P$ and $a \in \text{Ann}(I)$, then $ra = 0 \in P$, and since P is prime, this implies that $a \in P$.

Lemma 1.4. *Let A be a semiprime ring and $I \subseteq A$ be an ideal, then $U(I)$ is dense in $U(AnnAnn(I))$.*

Proof. Let A be a semiprime ring and I be an ideal of A. It is clear that $U(I) \subseteq U(\text{AnnAnn}(I))$. Now let $\emptyset \neq U(r) \subseteq U(\text{AnnAnn}(I))$. For the sake of contradiction, suppose that $U(r) \cap U(I) = \emptyset$ then

$$\begin{aligned}\emptyset &= U(r) \cap U(I) \\ &= U(r) \cap \big(\bigcup_{i \in I} U(i)\big) \\ &= \bigcup_{i \in I} \big(U(r) \cap U(i)\big) \\ &= \bigcup_{i \in I} U(ri)\end{aligned}$$

But this implies that ri is nilpotent for every $i \in I$; since A is semiprime, we conclude that $ri = 0$ for every $i \in I$, and hence, $r \in \mathrm{Ann}(I)$ which is a contradiction. We have just shown that every basic open set which is contained in $U(\mathrm{AnnAnn}(I))$ intersects $U(I)$ nontrivially. Thus, $U(I)$ is dense in $U(\mathrm{AnnAnn}(I))$. □

Lemma 1.5. *Let A be a semiprime ring. If $I \subseteq A$ is an ideal, then $\overline{U(I)} = V(Ann(I))$.*

Proof. Let A be a semiprime ring and I, P be ideals of A, where P is prime. We assert that if $I \nsubseteq P$, then $\mathrm{Ann}(I) \subseteq P$. To see this observe that if $I \nsubseteq P$, then one can pick $i_0 \in I \setminus P$. Since for every $a \in \mathrm{Ann}(I)$, $0 = ai_0 \in P$, and P is prime, it follows that $\mathrm{Ann}(I) \subseteq P$. We have just demonstrated that $U(I) \subseteq V(\mathrm{Ann}(I))$, and hence, $\overline{U(I)} \subseteq V(\mathrm{Ann}(I))$. Conversely, suppose that $\mathrm{Ann}(I) \subseteq P$. For the sake of contradiction, suppose that there exists an $r \in A \setminus P$, such that $U(r) \cap U(I) = \emptyset$. But then, as demonstrated in the proof of Lemma 1.4, this implies that ri is nilpotent for every $i \in I$; since A is semiprime, we conclude that $ri = 0$ for every $i \in I$, and hence, $r \in \mathrm{Ann}(I)$ which is a contradiction. We have just shown that if $\mathrm{Ann}(I) \subseteq P$, then $U(r) \cap U(I) \neq \emptyset$ for every $r \in A \setminus P$; consequently, $P \in \overline{U(I)}$. □

Definitions 1.6. (a) A is called a *Baer ring* if for every subset S of A, $\mathrm{AnnAnn}(S)$ is a summand of A, i.e. there exists an idempotent $e \in A$ such that $\mathrm{AnnAnn}(S) = eA$.

(b) A is called a *weak-Baer ring* if for each $s \in A$, $\mathrm{AnnAnn}(s)$ is a summand of A.

(c) We say A is an α-*Baer* ring if for every subset S of A with $|S| < \alpha$, $\mathrm{AnnAnn}(S)$ is a summand of A. Let ω denote the first infinite cardinal. Observe that an ω-Baer ring is what is known as a weak Baer ring.

(d) A semiprime ring A is said to satisfy the *annihilator condition*, henceforth, abbreviated *a.c.*, if for every a, $b \in A$ there exists a $z \in A$ such that $\mathrm{Ann}(z) = \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$.

Definitions and Remarks 1.7. (a) Let X be a topological space and let $\omega_1 \leq \alpha \leq \infty$. An open set U of X is said to be an *α-cozeroset* if U is a union of fewer than α cozerosets. We say that X is *α-disconnected* if the closure of every α-cozeroset is clopen. Note that ∞-disconnected (resp., ω_1-disconnected) is what is known as extremally disconnected (resp., basically disconnected.)

(b) For a topological space X, $C(X)$ denotes the collection of real-valued continuous functions on X. $C(X)$ under the pointwise operations is a commutative semiprime ring where the constant function $\mathbf{1}$ is the identity. For additional information on $C(X)$ we refer the reader to [GJ60]. Let $\omega \leq \alpha \leq \infty$. It is well known that $C(X)$ is an α-Baer ring if and only if X is an α-disconnected space (see [HM96].) Consequently, $C(X)$ is weak-Baer (resp., Baer) when X is basically (resp., extremally) disconnected.

We end this section by noting some interesting results involving weak Baer rings. We urge the interested reader to check [P80] for more information.

1) If A is weak-Baer and $A \leq B \leq q(A)$, where $q(A)$ denotes the classical ring of fractions of A, then B is weak-Baer.

2) If A is weak Baer, then $A[x]$, the polynomial ring of A, is a weak-Baer ring.

3) If A is weak-Baer, then A is semiprime.

4) A is weak-Baer if and only if every principal ideal is projective.

2. αcc-Baer Rings

In this section we introduce the concept of an αcc-Baer ring and compare it to the classical concept of an α-Baer ring. In addition, we characterize the αcc-Baer rings in terms of their prime spectra.

Definitions 2.1. (a) For a ring A, $\mathcal{L}(A)$ denotes the collection of ideals of A and

$$\mathcal{P}(A) = \{I \in \mathcal{L}(A) \,|\, \mathrm{AnnAnn}(I) = I\}.$$

Equivalently, $\mathcal{P}(A) = \{I \in \mathcal{L}(A) \,|\, \mathrm{Ann}(S) = I \text{ for some } S \subseteq A\}$.
Observe that when partially-ordered by inclusion $\mathcal{L}(A)$ is a lattice, while $\mathcal{P}(A)$ is a complete boolean algebra (see [L86].)
(b) An ideal I is an *α-annihilator* if there exists an $S \subseteq I$ with $|S| < \alpha$ and $I = \mathrm{AnnAnn}(S)$.

(c) We call a set S in a ring A *product trivial* if the product of any two distinct elements of S is 0.

(d) An ideal I is an *αcc-ideal* if whenever $S \subseteq I$ is product trivial, then $|S| < \alpha$.

(e) $\mathcal{P}_\alpha^{cc}(A) = \{I \in \mathcal{P}(A) \,|\, I \text{ is } \alpha\text{cc-ideal}\}$

(f) We say A is *αcc-Baer* if for all $I \in \mathcal{P}_\alpha^{cc}(A)$, I is a summand of A.

(g) Let A be a ring and let $U(I)$ be an open set in Spec(A). $U(I)$ is said to be an *αcc-set* if whenever T is a family of pairwise disjoint open sets such that $\bigcup T \subseteq U(I)$, then $|T| < \alpha$. Spec(A) is said to be *αcc-disconnected* if every αcc-set in Spec(A) has clopen closure.

Proposition 2.2. *In a semiprime ring A every $I \in \mathcal{P}_\alpha^{cc}(A)$ is an α-annihilator.*

Proof. Let $I \in \mathcal{P}_\alpha^{cc}(A)$. By Zorn's Lemma, the collection of all product trivial subsets of I, ordered by inclusion, has maximal elements. Let S be such a maximal product trivial set. Since I is αcc it follows that $|S| < \alpha$, so it suffices to show that $I = \text{AnnAnn}(S)$. If $u \in I$ and $u \notin \text{AnnAnn}(S)$, then there is a $t \in \text{Ann}(S)$ with $ut \neq 0$. But then $uts = 0$ for all $s \in S$. By maximality of S, $ut \in S$ so $(ut)(ut) = 0$. Since A is semiprime it follows that $ut = 0$. This is a contradiction.

The reverse inclusion follows from the hypothesis that $I = \text{AnnAnn}(I)$ and that $S \subseteq I$. □

Proposition 2.3. *For a semiprime ring A, α-Baer implies αcc-Baer.*

Proof. Let $I \in \mathcal{P}_\alpha^{cc}(A)$, then by Proposition 2.2 there is a $S \subseteq I$ with $|S| < \alpha$ and $I = \text{AnnAnn}(S)$. Since A is α-Baer, AnnAnn(S) is a summand of A. □

However, as the following example demonstrates the converse of Proposition 2.3 need not hold.

Examples 2.4. Let $\omega_0 \subset \omega_1 ... \subset \omega_n ... \subset \omega_\omega$ denote the first infinite cardinals, where ω_ω is the first singular cardinal. If R denotes a commutative integral domain with identity, let $R_\sigma = R$ for $\sigma \in \omega_\omega$ and define the subring of the cartesian product $\prod_{\omega_\omega} R_\sigma$

$$A = \{f \in \prod_{\omega_\omega} R_\sigma \,|\; f \text{ is constant outside a set of cardinality less than } \omega_\omega\}.$$

Then A is an $\omega_1$cc-Baer ring which is not ω_1-Baer.

A is $\omega_1$cc-Baer ring:
Let I be an $\omega_1$cc-Baer ideal of A and let $S = \{i \in \omega_\omega : f(i) \neq 0 \text{ for some } f \in I\}$, then $\mathrm{Ann}(I) = \{g \in A : g(S) = 0\}$ and $I = \mathrm{AnnAnn}(I) = \{f \in A : g(\omega_\omega \setminus S) = 0\}$. The collection of characteristic functions $\{\chi_i : i \in S\}$ is a product trivial subset of I, therefore countable, so S is countable. It is easily seen that I is generated by the idempotent χ_S, thus a summand of A.

A is not ω_1-Baer ring:
Let $S \cup T = \omega_\omega$, disjoint sets, each of cardinality ω_ω. Define $f_n = \chi_{S\cap\omega_i}$ and claim that $\mathrm{AnnAnn}(\{f_n\}_N)$ is not a summand. It is enough to show that $\mathrm{Ann}(\{f_n\}_N)$ is not principal. For if $f \in A$ is a generator, then $f(S) = 0$. Now, due to cardinality considerations, $f(T \setminus U) = 0$ where $U \subset T$ has cardinality less than ω_ω. If $i \in T \setminus U$, then $\chi_i \in \mathrm{Ann}(\{f_n\}_N)$ but $\chi_i \notin Af$, so it is not principal.

We now present a variety of results which are akin to the α-Baer case.

Lemma 2.5. *Let $A \leq B$. If I is an αcc-ideal in B, then $I \cap A$ is an αcc-ideal in A.*

Proof. Given that $A \leq B$ let I be an αcc-ideal in B. Suppose $M \subseteq I \cap A$ be product trivial. Since $M \subseteq I$ is product trivial, and I is an αcc-ideal, it follows that $|M| < \alpha$. □

Theorem 2.6. *If A is αcc-Baer and $A \leq B \leq q(A)$, then B is αcc-Baer.*

Proof. Given that A is αcc-Baer let $A \leq B \leq q(A)$, where $q(A)$ denotes the classical ring of fractions of A. Suppose that $I = \mathrm{Ann}_B(S)$ for some $S \subseteq B$ is an αcc-ideal, then by Lemma 2.5 we get that $I \cap A$ is an αcc-ideal in A. Since A is αcc-Baer, it follows that $I \cap A$ is generated by an idempotent say e. We assert that I is generated by the idempotent $\frac{e}{1}$ in B. To see this, observe that $e \in I \cap A$ implies that $\frac{e}{1} \in I$, and therefore, $\frac{e}{1}B \leq I$. On the other hand, if $\frac{a}{b} \in I$, then $a \in I \cap A$. Since $eA = I \cap A$, there exists an $m \in A$ such that $em = a$. Now observe that $\frac{a}{b} = \frac{em}{b} = \frac{e^2m}{b} = (\frac{e}{1})(\frac{em}{b}) \in \frac{e}{1}B$. □

Lemma 2.7. [A73] *Let A be semiprime and $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j$ be polynomials of $A[x]$. $fg = 0$ if and only if $a_i b_j = 0$ for every $0 \leq i \leq n$, $0 \leq j \leq m$.*

Proposition 2.8. *Let A be a semiprime ring. A is αcc-Baer if and only if $A[x]$ is αcc-Baer.*

Proof. ($\Rightarrow$) Given A is semiprime and αcc-Baer. Let $J = \mathrm{Ann}(S)$ for some $S \subseteq A[x]$ be an αcc-ideal. Define C to be the collection of coefficients of the polynomials in S. Let $I = \mathrm{Ann}(C)$ then by Lemma 2.7, $J = I[x]$. Since any

product trivial subset of I is contained in J, it follows that I is an αcc-ideal in A. By hypothesis there exists an idempotent e such that $I = eA$. Since $J = I[x] = (eA)[x] = eA[x]$, it follows that $A[x]$ is αcc-Baer.
($\Leftarrow$) Conversely, suppose A is semiprime and $A[x]$ is αcc-Baer. Let $J = \mathrm{Ann}(C)$ for some $C \subseteq A$ be an αcc-ideal in A. It follows from Lemma 2.7 that $J[x] = \mathrm{Ann}(C[x])$. Since α is infinite, it is easy to verify that $J[x]$ is an αcc-ideal in $A[x]$; consequently, there exists an idempotent $e \in A[x]$ such that $J[x] = eA[x] = (eA)[x]$ (since idempotents of $A[x]$ are in A). As a result we conclude that $J = eA$. □

Proposition 2.9. *$A = \prod_{i\in I} A_i$ is an αcc-Baer ring if and only if A_i for $i \in I$ is an αcc-Baer ring.*

Proof. For $a \in A$, a_i denotes the projection of a on to A_i. Given $A = \prod_{i\in I} A_i$ is an αcc-Baer ring. Let $i \in I$ and suppose that $J = \mathrm{Ann}_{A_i}\mathrm{Ann}_{A_i}(S)$ for some $S \subseteq A_i$ is an αcc-ideal. Let $\bar{S} = \{a \in A | a_k = 1 \text{ if } k \neq i \text{ and } a_i = s \text{ where } s \in S\}$. Since J is an αcc-ideal in A_i, one can easily verify that $\bar{J} = \mathrm{Ann}_A\mathrm{Ann}_A(\bar{S})$ is an αcc-ideal in A. Consequently, there exists an idempotent $e \in A$ such that $Ae = \bar{J}$. But then one can easily check that $e_i \in A_i$ is an idempotent for which $A_i e_i = J$.

Conversely, suppose that A_i for $i \in I$ is an αcc-Baer ring. Let $A = \prod_{i\in I} A_i$ and $\bar{J} = \mathrm{Ann}_A\mathrm{Ann}_A(\bar{S})$ for some $\bar{S} \subseteq A$ be an αcc-ideal. For $i \in I$ define $J_i = \mathrm{Ann}_{A_i}\mathrm{Ann}_{A_i}(S_i)$, where $S_i = \{s_i | s \in \bar{S}\}$. Since $\bar{J}$ is an αcc-ideal in A, it follows that for every $i \in I$, J_i is an αcc-ideal in A_i. But then for every $i \in I$ there exists an idempotent $e_i \in A_i$ such that $A_i e_i = J_i$. Now let $\bar{e}$ be the element of A such that $\bar{e}_i = e_i$. Then $\bar{e}$ is idempotent and $A\bar{e} = \bar{J}$. □

Definitions 2.10. Recall that a ring A is said to be a *p.p. ring* if every principal ideal of A is projective. We say that A is an *ωcc-p.p. ring* if whenever I is a principal ideal and $\mathrm{Ann}(I)$ is an ωcc-ideal, then I is projective.

Proposition 2.11. *Let A satisfy the a.c. ring. A is ωcc-Baer if and only if A is an ωcc-p.p. ring.*

Proof. ($\Rightarrow$) Suppose A is ωcc-Baer. Let Ar be an ideal for which the $\mathrm{Ann}(r)$ is an ωcc-ideal. Since A is ωcc-Baer, the $\mathrm{Ann}(r)$ and $\mathrm{Ann}\mathrm{Ann}(r)$ are generated by idempotents and thus are projective modules. Since $A \cong \mathrm{Ann}(r) \oplus \mathrm{Ann}\mathrm{Ann}(r)$ and $Ar \cong \mathrm{Ann}\mathrm{Ann}(r)$ (as A-modules), it follows that Ra is projective.
($\Leftarrow$) Given A is an ωcc-p.p. ring which satisfies the a.c. condition. Suppose that $S \subseteq A$ and $\mathrm{Ann}(S)$ is an ωcc-ideal. Then there exists $F \subseteq S$ such that F is finite and $\mathrm{Ann}(S) = \mathrm{Ann}(F)$. Since A satisfies the a.c. condition, there exists an $r \in R$ such that $\mathrm{Ann}(r) = \mathrm{Ann}(S)$. Now consider the following sequence

$$(\dagger) \qquad 0 \longrightarrow \mathrm{Ann}(r) \longrightarrow A \longrightarrow Ar \longrightarrow 0.$$

Since Ar is projective, it follows that (†) splits and $A \cong \mathrm{Ann}(r) \oplus \mathrm{AnnAnn}(r)$ as A-modules. Since $1 \in A$, it is easy to verify that $\mathrm{Ann}(r)$ is generated by an idempotent, and hence A is ωcc-Baer. □

As we have seen there are various α-Baer ring results which have analogs to the αcc–Baer case. However, this need not always be the case. It is well known that a weak-Baer ring is semiprime. However, our next example demonstrates the same is not true for ωcc-Baer rings.

Example 2.12. Let $Z = \mathbb{Z} \propto \mathbb{Z}$, where addition is coordinate-wise and multiplication is defined as follows:

$$(a,\, b) \cdot (c,\, d) = (ac,\, ad + bc).$$

(Rings of this forms are known as trivial extensions or idealizations.) Observe that

(i) Z is a commutative ring in which $(1,\, 0)$ is the multiplicative identity;

(ii) If $(0,\, a), (0,\, b) \in Z$, then $(0,\, a) \cdot (0,\, b) = (0,\, 0)$; consequently, Z is not semiprime.

We claim that Z is an ωcc-Baer ring:

Let $I \in \mathcal{P}_{\omega}^{cc}(Z)$ and $(a,\, b) \in I$, then

$$(0,\, a) = (0,\, 1) \cdot (a,\, b) \in I,$$

and hence, the product trivial subset

$$S = \{(0,\, na) \,|\, n \in \mathbb{N}\}$$

is contained in I; but this is a contradiction unless $a = 0$. Consequently, $(a,\, b) = (0,\, b)$. Applying the same argument yields that $b = 0$, and hence, we conclude that the only ωcc-ideal in Z is (0). Since $\mathcal{P}_{\omega}^{cc}(Z)$ only contains the zero ideal, it follows that Z is an ωcc-Baer ring. It is worth noting that Z is an example of an ωcc-Baer ring, which is not weak-Baer.

Example 2.13. As this example demonstrates the class of αcc-Baer rings is not closed under homomorphic images. Consider $\mathbb{Z}$ the ring of integers. Since $\mathbb{Z}$ is an integral domain, $\mathbb{Z}$ is αcc-Baer. However, $\mathbb{Z}/8\mathbb{Z}$ is not an αcc-Baer ring.

Our goal for the remainder of this section is to characterize when a semiprime ring is αcc-Baer. We begin with some preliminary results.

Proposition 2.14. *Let A be a semiprime ring. I is an αcc-ideal of A if and only if $U(I)$ is an αcc-set in Spec(A).*

Proof. (⇒) Given $U(I)$ is an αcc-set in Spec(A). Suppose that $S \subseteq I$ is product trivial. Then $T = \{U(s) | s \in S\}$ will be a family of pairwise disjoint open sets such that $\bigcup T \subseteq U(I)$ and $|T| = |S|$. Since $U(I)$ is an αcc-set, it follows that $|T| < \alpha$.
(⇐) Conversely, let I is an αcc-ideal of A. Suppose that a collection of pairwise disjoint basic elements, each of which is contained in $U(I)$, are indexed by the members of $S = \{s \mid U(s) \subseteq U(I)$, and $U(s) \neq U(s')$ if $s \neq s'\}$. Since every element of $U(I)$ contains the Ann(I), it follows that

$$C = \{si_s \mid s \in S,\ i_s \in I, \text{ and } si_s \neq 0 \text{ if } s \neq 0\}$$

exists. Then C is a product trivial subset, and since A is semiprime, it follows that each of the si_s are distinct, and therefore, $|S| = |C|$. Since $C \subseteq I$ and I is an αcc-ideal, $|C| < \alpha$. □

Theorem 2.15. *Let A be a semiprime ring. A is αcc-Baer if and only if Spec(A) is αcc-disconnected.*

Proof. (⇒) Given A is semiprime and αcc-Baer ring. Let $U(J)$ be an αcc-set in Spec(A). Let $I =$ AnnAnn(J) by Lemma 1.4, $U(J)$ is dense in $U(I)$. Since $U(J)$ is an αcc-set, which is dense in $U(I)$, it follows that $U(I)$ is an αcc-set. By Proposition 2.14, I is an αcc-ideal in A, and since A is αcc-Baer, $I = eA$ for some idempotent $e \in A$. Since $U(I) = U(e)$ and $U(e)$, $U(1-e)$ is a separation of Spec(A), it follows that $U(I)$ is clopen. Therefore, $\overline{U(J)} = \overline{U(I)}$ is clopen, and thus, Spec(A) is αcc-disconnected.
(⇐) Conversely, suppose A is a semiprime ring and Spec(A) is αcc-disconnected. Let $I =$ AnnAnn(S) for some $S \subseteq A$ be an αcc-ideal in A. By Proposition 2.14, $U(I)$ is an αcc-set in Spec(A). Since Spec(A) is αcc-disconnected, it follows that $\overline{U(I)} = V(\text{Ann}(I))$ is clopen. Consequently, $U(\text{Ann}(I))$ is clopen. By Lemma 1.2 we get that Ann(I) is a summand of A, and hence, AnnAnn(I) is a summand of A. □

Remark 2.16. Recall that a topological space X is said to be extremally disconnected if the closure of every open set is clopen. A moments thought should reveal that X is ∞cc-disconnected if and only if it is extremally disconnected. The following corollary, which is also proved in [M71] is an immediate consequence of Theorem 2.15 and the fact that ∞-Baer (i.e., Baer) is equivalent to ∞cc-Baer.

Corollary 2.17. *Let A be a commutative semiprime ring with identity. A is ∞-Baer if and only if Spec(A) is extremally disconnected.*

We conclude this section with an observation about the minimal primes of an αcc-Baer ring.

Proposition 2.18. *Let α denote an infinite cardinal or ∞. If A is αcc-Baer, then $Spec(A) \cap \mathcal{P}_\alpha^{cc}(A) \subseteq Min(A)$.*

Proof. Let A be αcc-Baer and $P \in \mathcal{P}_\alpha^{cc}(A)$ be prime. Then there exists an idempotent $e \in A$ such that $P = Ae$. Now suppose that Q is a prime ideal that is contained in P. Since $e(1-e) \in Q$ and Q is prime, it follows that e or $(1-e)$ belongs to Q. But $(1-e) \notin Q$ (or else $1 = e + (1-e) \in P$ which is a contradiction.) Consequently, $e \in Q$, and thus $P \subseteq Q$. □

3. αcc-Baer Hull

It is known that every semiprime ring has a Baer hull as well as a weak Baer hull. In this section we construct the αcc-Baer hull of a semiprime ring. We begin by reviewing the background material.

Definitions and Remarks 3.1. [L86]

(a) A subgroup D of A is said to be *dense* if whenever $aD = 0$, then $a = 0$.

(b) Suppose that $A \leq B$. B is said to be a (Utumi) *ring of quotients* of A if and only if for every $b \in B$,

$$b^{-1}A = \{a \in A \,|\, ba \in A\}$$

is dense in B. Observe that $b^{-1}A$ is dense in B if and only if for every $0 \neq t \in B$, $t(b^{-1}A) \neq 0$.

It is well known that for every commutative semiprime ring A there exists a maximum ring of quotients $Q(A)$. For those not familiar with the construction of $Q(A)$ as a direct limit of of partial endomorphisms over dense ideals should preuse [L86] who gives an excellent treatment. Another source is [La98] which constructs $Q(A)$ as the injective hull of A. Some interesting facts about $Q(A)$ are now listed.

(1) If $A \leq B$ and B is a ring of quotients of A, then $A \leq B \leq Q(A)$. (And conversely.)
(2) $Q(A)$ is a semiprime von Neumann regular Baer ring.
(3) For any $A \leq B \leq Q(A)$, the trace map from $\mathcal{P}(B)$ to $\mathcal{P}(A)$ is a boolean isomorphism. The inverse of this map is $\text{Ann}_B\text{Ann}_B(\cdot) : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$

For $I \in \mathcal{P}(A)$, we let $\hat{I} = \text{Ann}_{QA}\text{Ann}_{QA}(I)$.

Lemma 3.2. *Suppose that $A \leq B \leq Q(A)$. Then I is an αcc-ideal in A if and only if $Ann_BAnn_B(I)$ is an αcc-ideal of B.*

Proof. The sufficiency is clear. As to the necessity, let S be a product trivial subset of $\mathrm{Ann}\mathrm{Ann}_B(I)$. Since for s in B, $s^{-1}A$ is dense, it follows that for every $s \in S$ one can find $r_s \in A$ such that $sr_s \in A$ and $sr_s \neq 0$ when $s \neq 0$; moreover, (since B is semiprime) each of the sr_s are distinct. Let T denote a collection of such sr_s's then $|T| = |S|$, and $T \subseteq I \subseteq A$. Since I is an αcc-ideal, $|S| = |T| < \alpha$. □

The following proposition is a direct consequence of Lemma 3.2.

Proposition 3.3. *Let B be a ring of quotients of A. If A is αcc-Baer, then B is αcc-Baer.*

Proof. Let $A \leq B \leq Q(A)$, and let $I \in \mathcal{P}_\alpha^{cc}(B)$. By Lemma 3.2, $I \cap A \in \mathcal{P}_\alpha^{cc}(A)$. Since A is αcc-Baer, it follows that there exists an idempotent $e \in A$ such that $Ae = I \cap A$. But then $I = B\phi(e)$ in B. □

Theorem 3.4. *For every commutative semiprime ring A there exists a minimum αcc-Baer ring of quotients extension denoted by $Q_\alpha^{cc}A$.*

Proof. Let A be a commutative semiprime ring with identity. Define

$$S = \{B \mid A \leq B \leq QA \text{ and } B \text{ is an } \alpha\text{cc-Baer ring of } A\}.$$

Observe $QA \in S$, hence, $S \neq \emptyset$. Now let $Q_\alpha^{cc}A = \bigcap S$. Then Lemma 3.2 along with the fact that the projection onto an annihilator ideal of QA is unique yields that $Q_\alpha^{cc}A$ is an αcc-Baer ring. It is clear that whenever $A \leq C$ is an αcc-Baer ring of quotients for A, then $A \leq Q_\alpha^{cc}A \leq C$. □

Although Theorem 3.4 establishes the existence of an αcc-Baer hull, it does not provide a concrete description of the hull itself. The remainder of this section will be devoted to giving an internal construction of the αcc-Baer hull.

Definition 3.5. For $I \in \mathcal{P}(QA)$ we let e_I denote the idempotent of QA that generates I in QA and $\mathcal{E}_\alpha^{cc}(QA)$ denotes the subalgebra of $\mathcal{P}(QA)$ generated by the e_I's for $I \in \mathcal{P}_\alpha^{cc}(QA)$. Recall that if e_1 and e_2, are idempotents in QA and $\mathrm{Ann}_{QA}\mathrm{Ann}_{QA}(e_1) = \mathrm{Ann}_{QA}\mathrm{Ann}_{QA}(e_2)$, then $e_1 = e_2$. Consequently, the generating idempotent of a summand of QA is unique.

Proposition 3.6. *Let $A \leq B \leq QA$ and $I \in \mathcal{P}(A)$. $\mathrm{Ann}_B\mathrm{Ann}_B(I)$ is a summand of B if and only if $be_{\hat{I}} \in B$ for every $b \in B$.*

Proof. Sufficiency: suppose that for every $b \in B$, $be_{\hat{I}} \in B$. Then $e_{\hat{I}} \in B$ and $\mathrm{Ann}_B\mathrm{Ann}_B(I) = Be_{\hat{I}}$. As to the necessity, suppose that for $I \in \mathcal{P}(A)$, $\mathrm{Ann}_B\mathrm{Ann}_B(I) = Be$, where e is an idempotent in B. Since $\hat{I}$
$= \mathrm{Ann}_{Q(A)}\mathrm{Ann}_{Q(A)}(Be) = \mathrm{Ann}_{Q(A)}\mathrm{Ann}_{Q(A)}(e)$, it follows (from the uniqueness of idempotents) that $e_{\hat{I}} = e$. Hence, for $b \in B$, $be_{\hat{I}} = be \in B$. □

In [P80] (resp., [M71]) the author constructs the weak-Baer (resp., Baer) hull of a semiprime ring A. We now construct αcc-Baer hull of a semiprime ring A.

Theorem 3.7. *Suppose A is a semiprime ring. The elements of $Q^{cc}_{\alpha}A$ are of the form*

$$\sum_{i=1}^{n} a_i e_i,$$

where $a_i \in A$, $e_i \in \mathcal{E}^{cc}_{\alpha}(Q(A))$ for every i.

Proof. Let A be a semiprime ring and let

$$Q = \Big\{ \sum_{i=1}^{n} a_i e_i \mid a_i \in A,\, e_i \in \mathcal{E}^{cc}_{\alpha}(QA) \Big\}.$$

We demonstrate that Q is subring of $Q(A)$ and then show that Q is an αcc-Baer ring. Observe that if $a \in A$ and $e \in \mathcal{E}^{cc}_{\alpha}(Q(A))$, then $-(ae) = (-a)e \in Q$. And if $ae_1, be_2 \in Q$, then $ae_1 \cdot be_2 = (ab)e_1e_2 \in Q$. Consequently, Q is closed under subtraction and products, and hence, is a subring. To see that Q is an αcc-Baer ring, let $J \in \mathcal{P}^{cc}_{\alpha}(Q)$. Observe that by Proposition 3.2, $\hat{J} \in \mathcal{P}^{cc}_{\alpha}(Q(A))$, and thus, $e_{\hat{J}} \in \mathcal{E}^{cc}_{\alpha}(Q(A))$. Now let $\sum\limits_{i=1}^{n} a_i e_i \in Q$ then

$$\Big(\sum_{i=1}^{n} a_i e_i\Big) e_{\hat{J}} = \sum_{i=1}^{n} a_i e_i e_{\hat{J}},$$

where $e_i e_{\hat{J}} \in \mathcal{E}^{cc}_{\alpha}(QA)$ for every i. Consequently, $\Big(\sum\limits_{i=1}^{n} a_i e_i\Big) e_{\hat{J}} \in Q$, and by Proposition 3.6, J is a summand of Q. Next, observe that $A \leq Q \leq Q(A)$ by construction, and therefore, $Q^{cc}_{\alpha}A \leq Q$. On the other hand, since $Q(A)$ is a ring of quotients of the αcc-Baer ring $Q^{cc}_{\alpha}A$, it follows that $\mathcal{E}^{cc}_{\alpha}(Q(A)) = \mathcal{E}^{cc}_{\alpha}(Q^{cc}_{\alpha}A)$. But then Proposition 3.6 ensures that $ae \in Q^{cc}_{\alpha}A$ for every $a \in A$ and $e \in \mathcal{E}^{cc}_{\alpha}(Q(A))$. Consequently, $Q \leq Q^{cc}_{\alpha}A$.

□

4. Archimedean f-rings

In this section we prove that the αcc-Baer condition for an archmidean f-ring with identity is equivalent to the αcc-projectablity condition for an archimedean lattice-ordered group with designated unit.

Definitions and Remarks 4.1. (a) A *lattice-ordered group* abbreviated *ℓ-group* is a group with an underlying lattice structure such that whenever $a \leq b$, then $a + c \leq b + c$ for every $c \in G$. An *ℓ-homomoprhism* between lattice-ordered groups is a group homomorphism that preserves the lattice structure. A *convex ℓ-subgroup* of G is a subgroup which is a sublattice and order convex.

For $X \subseteq G$ define $X^{\perp} = \{g \in G \,|\, |g| \wedge |x| = 0 \text{ for every } x \in X\}$. A *polar* is a convex ℓ-subgroup of the form $X^{\perp\perp}$ for some $X \subseteq G$.

A lattice-ordered group G is said to be *archimedean* if whenever $0 \leq a, b \in G$ and $na \leq b$ for every positive integer n, then $a = 0$. As demonstrated [BKW77] an archimedean lattice-ordered group is abelian. The element $u \in G$ is said to be a *weak-unit* if $u^{\perp} = \{0\}$. **W** is the category of archimedean ℓ-groups with designated weak-unit and ℓ-homomorphisms that preserve the weak-unit.

(b) Let G be an ℓ-group. $S \subseteq G$ is *pairwise disjoint* if the infima of any two distinct elements of S is 0. A polar P of G is said to be an *αcc-polar* if whenever $S \subseteq P$ is pairwise disjoint, then $|S| < \alpha$. An ℓ-group G is said to be *αcc-projectable* if whenever P is an αcc-polar of G, then $G = P \oplus P^{\perp}$. For additional information the reader is referred to [C12].

(c) An *f-ring* A is a ring which is a lattice-ordered group under addition, and whenever $a \wedge b = 0$, then $a \wedge bc = 0$ for every $0 < c \in A$. As demonstrated in [BKW77] an archimedean f-ring with identity is necessarily commutative and semiprime. An *ℓ-ideal* of an f-ring A is a convex ℓ-subgroup which is a ring ideal. **Arf** denotes the category of archimedean f-rings with identity and ring ℓ-homomorphisms that preserve the identity. Throughout we make use of the fact that **Arf** is a monoreflective (full) subcategory of **W** (see [HR77] and [HR79] for details).

(d) Let X be compact Hausdorff space. $D(X)$ denotes the collection of continuous functions $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ such that $f^{-1}(\mathbb{R})$ is dense in X. It is well-known that $D(X)$ in general is not a group nor a ring under pointwise addition and multiplication. However, it is a lattice under pointwise suprema and infima.

The Henriksen-Johnson representation [HJ61]: For an archimedean f-ring A let mA denote the space of maximal ℓ-ideal with the hull-kernel topology. Then mA is a compact Hausdorff space and there is a ring ℓ-homomorphism ϕ from A onto an f-ring $\phi(A)$ in $D(mA)$, such that $\phi(1)$ is the constant function 1 and $\phi(A)$ separates the points of mA. The fact that $\phi(A)$ separates the points of mA makes the space unique (up to homeomorphism).

The Yosida representation [HR77]: For $G \in \mathbf{W}$ there exists a compact Hausdorff space YG and an ℓ-homomorphism ϕ from G onto an **W**-object $\phi(G)$ in $D(YG)$, such that the weak-unit is mapped to the constant function 1 and $\phi(G)$ separates the points of YG. As before, the fact that $\phi(G)$ separates the points of YG makes the space unique (up to homeomorphism). YG is the *Yosida space* for G.

Theorem 4.2. *Let α be an infinite cardinal or ∞ and let $A \in \mathbf{Arf}$. The following are equivalent:*

(i) *A is αcc-Baer*
(ii) *Spec(A) is αcc-disconnected*
(iii) *A is an αcc-projectable $\mathbf{W}$-object.*

Proof. Let α be an infinite cardinal or ∞ and let $A \in \mathbf{Arf}$. Theorem 2.15 establishes the equivalence of (i) and (ii). As to the equivalence of (i) and (iii), we make some observations from which the result will follow:

(a) For $a, b \in A$, $ab = 0$ if and only if $|a| \wedge |b| = 0$. Therefore, for $S \subseteq A$, $\mathrm{Ann}(S) = S^{\perp}$.
(b) If T is a product trivial subset of A, then $\bar{T} = \{|t| \,|\, t \in T\}$ is a pairwise disjoint set of A. Moreover, since for any distinct $t_1, t_2 \in T$, $|t_1| = |t_2|$ implies that $t_1 = 0 = t_2$, it follows that $|T| = |\bar{T}|$. On the other hand, it is evident that any pairwise disjoint subset of A is product trivial.

Since items (a) and (b) establish that every $I \in \mathcal{P}^{cc}_{\alpha}(A)$ is an αcc-polar, and vice-versa, it follows that A is αcc-Baer if and only if A is αcc-projectable. □

Remark 4.3. Recall that an open set U of the space X is said to be an αcc-set if whenever T is a family of pairwise disjoint open sets such that $\bigcup T \subseteq U$, then $|T| < \alpha$. X is αcc-disconnected if the closure of every αcc-set is clopen. Theorem 4.2 demonstrates the αcc-Baer condition for an archimedean f-ring A is characterized by the αcc-disconnectedness of Spec(A). But what can be said about the relationship between A and mA when A is αcc-Baer? To answer this question we again use the fact that an archimedean f-ring is a $\mathbf{W}$-object. In [C12] the author characterizes the αcc-projectable $\mathbf{W}$-objects and demonstrates that the Yosida space of an αcc-projectable $\mathbf{W}$-object is αcc-disconnected. Since for any $A \in \mathbf{Arf}$, mA is homeomorphic to YA, and αcc-Baer is equivalent to αcc-projectabitlity, it follows that if A is αcc-Baer, then mA is αcc-disconnected. However, in general the αcc-disconnectedness of mA is not sufficient for A to be αcc-projectable (equivalently, αcc-Baer); in fact, there are variety of conditions which must be satisfied in order for $A \in \mathbf{Arf}$ to be αcc-projectable (αcc-Baer). We refer the reader to [C12] for additional information. That being said, there are some instances when A is αcc-Baer if and only if mA is αcc-disconnected.

Definition 4.4. For a completely regular topological space X, βX denotes the *Stone-Čech compactification* of X.

Theorem 4.5. *For a Tychonoff space X the following are equivalent:*

(i) *X is αcc-disconnected.*
(ii) *βX is αcc-disconnected.*
(iii) *$C(X)$ is αcc-projectable.*
(iv) *$C(\beta X)$ is αcc-projectable.*

(v) $C(X)$ *is* α*cc-Baer.*
(vi) $C(\beta X)$ *is* α*cc-Baer.*

Proof. First, observe that for any Tychonoff space X, $mC(X)$ is homeomorphic to βX. As to the equivalence of items (i) through (iv) we refer the reader to [C12]. Since Theorem 4.2 establishes the equivalence of items (iii) and (v), and (iv) and (vi), this completes the proof. □

Examples and Remarks 4.6. As demonstrated in [C12] every Tychonoff space X is ωcc-diconnected. In light of this and Theorem 4.5, we get that for any Tychonoff space X, $C(X)$ is an ωcc-Baer ring. But as the following group ring example demonstrates not all commutative semiprime rings with identity are ωcc-Baer: Let $C_2 = \{1, g\}$ be a multiplicative cyclic group or order two, and R an integral domain such that 2 is not invertible nor zero. Define $S = R[C_2]$. An arbitrary element of S is of the form $a+bg$ where $a, b \in R$. We first demonstrate that S is semiprime. Note the trivial group homomorphism from $\{1, g\}$ to $\{1\}$ induces a ring homomorphism from S to R such that $a+bg$ maps to $a+b$. Observe that if $(a+bg)^n = 0$ for $1 \leq n$, then $b = -a$ (since R is a semiprime.) Since $0 = (a+bg)^n = a^n(1-g)^n = (a^n)(2^{n-1})(1-g)$, it follows that $a^n 2^{n-1} = 0$ in R. But R is an integral domain thus we conclude that $a = 0$, and hence $b = 0$ and $a+bg = 0$. If $a+bg$ is idempotent, $(a+bg)^2 = a+bg$, then $a^2+b^2+2abg = a+bg$, which implies that $b = 0$ and $a = 0$ or 1. We claim that S is not ωcc-Baer: Since the only idempotents of S are 0 and 1, it follows that the only summands of S are the ideals 0 and S. Thus the principal ideal $(1-g)S$ is not a summand ($g \notin (1-g)S$). Moreover since $(a+bg)(1+g) = 0$ if and only if $b = -a$, an easy computation shows that $Ann(1+g) = (1-g)S = (1-g)R$, thus an ωcc-ideal. On the other hand, the largest product trivial subset of $(1-g)S$ has order 2, so we conclude that S is not ωcc-Baer.

Although the previous example demonstrates that R being αcc-Baer is not a sufficient condition for $R[G]$ to be αcc-Baer, the general question of when a group ring is αcc-Baer remains open.

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