UNIT-FUSIBLE PROPERTY VIA REGULARITY

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Abstract. An element in a ring is *left unit-fusible* if it is the sum of a left zero-divisor and a unit, and a ring is *left unit-fusible* if each of its nonzero elements is left unit-fusible. We explore the relation between the unit-fusible property and (unit-) regularity. Among others, we show that every (von Neumann) regular ring is left unit-fusible, and prove a characterization theorem of the rings whose left unit-fusible elements are (von Neumman) regular.

1. Introduction

In 2017, motivated by the well-studied notion of complemented rings, Ghashghaei and McGovern [5] initiated a study of left fusible rings. Here a ring is *left fusible* if every nonzero element is the sum of a left zero-divisor and a non-left zero-divisor, while a ring R is *left complemented* if for each element $a \in R$ there exists $b \in R$ such that $ba = 0$ and $a + b$ is a non zero-divisor. In this study, left unit-fusible rings arose as an interesting special class. A ring is called *left* (*resp. right*) *unit-fusible* if every nonzero element is left (resp. right) unit-fusible, i.e. it is the sum of a left (resp. right) zero-divisor and a unit. A ring which is both left and right unit-fusible is called a unit-fusible ring. Motivated by the result in [5] that every unit-regular ring is unit-fusible and the question in [5] whether every (von Neumann) regular ring is unit-fusible, we explore the relation between (unit-) regularity and the unit-fusible property. In section 2, first we observe that a nonzero element a in a ring is unit-regular if and only if $a = z + u$ where z is a left zero-divisor (resp. two-sided zero-divisor) and u is a unit such that $aR \cap zR = 0$. We show that a ring R is unit-regular if and only if for any $0 \neq a \in R$ and $b \in R$ with $Ra + Rb = R$, there exists a two-sided zero-divisor z

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such that $a + zb$ is a unit and $aR \cap zR = 0$. As one of our main results, we show that every regular ring is unit-fusible, answering affirmatively a question in [5]. In section 3, we address the question of when every left unit-fusible element of a ring is regular. A characterization theorem of these rings is proved (see Theorem 3.5). As revealed in the theorem, the structure of the rings whose left unit-fusible elements are regular is reduced to that of the rings whose left unit-fusible elements are units. Thus, we carry out a study of these rings in section 4, where many examples are presented.

Throughout, rings are associative with identity unless otherwise specified. For a ring R, let $\mathrm{zd}_l(R) = \{a \in R : \mathbf{r}(a) \neq 0\}$ be the set of left zero-divisors and $\mathrm{zd}_l^*(R) = \{a \in R : \mathbf{r}(a) = 0\}$ the set of non-left zero-divisors, where $\mathbf{r}(a)$ (resp. $\mathbf{l}(a)$) denotes the right (resp. left) annihilator of a in R. Similarly, $\mathrm{zd}_r(R)$ and $\mathrm{zd}_r^*(R)$ can be defined. Elements in $\mathrm{zd}(R) := \mathrm{zd}_l(R) \cap \mathrm{zd}_r(R)$ are called two-sided zero-divisors, and elements in $\mathrm{zd}^*(R) = \mathrm{zd}_l^*(R) \cap \mathrm{zd}_r^*(R)$ are called two-sided non-zero-divisors.

For a ring R, we write $J(R)$ for the Jacobson radical of R, $\mathrm{U}(R)$ for the group of units of R, $\mathrm{nil}(R)$ for the set of all nilpotent elements of R and $\mathrm{idem}(R)$ for the set of all idempotents of R. A ring R is called abelian if every idempotent of R is central, and reduced if $\mathrm{nil}(R) = \{0\}$. The $n \times n$ matrix ring over a ring R is denoted by $\mathbb{M}_n(R)$. For a ring R, $R[t]$ (resp. $R[t, t^{-1}]$) denotes the ring of polynomials (resp. Laurent polynomials) over R. For an endomorphism σ of a ring R, $R[[t; \sigma]]$ is the ring of left skew power series over R with multiplication, subject to the relation $ta = \sigma(a)t$, for all $a \in R$.

2. Unit-regularity and unit-fusible property

An element a in a ring R is (von Neumann) regular if $a \in aRa$, unit-regular if $a \in a\mathrm{U}(R)a$, and strongly regular if $a = eu = ue$ where $e^2 = e$ and $u \in \mathrm{U}(R)$. The ring is (von Neumann) regular (resp., unit-regular or strongly regular) if every element in the ring is regular (resp., unit-regular or strongly regular). It is a result in [5] that every unit-regular ring is unit-fusible, and a question left open in [5] asks whether every regular ring is unit-fusible. In this section, we explore the relation between (unit-) regularity and the unit-fusible property.

We first characterize nonzero unit-regular elements as some special kinds of unit-fusible elements. There are various characterizations of a unit-regular element of a ring (for example, see [9, Theorem 4.3]), most of which are associated with multiplication.

Proposition 2.1. *The following are equivalent for a nonzero element a in a ring R:*

(1) *a is unit-regular.*

(2) *$a = z + u$ where $z \in R$, $u \in \mathrm{U}(R)$ and $aR \cap zR = 0$.*

(3) *$a = z + u$ where $z \in \mathrm{zd}_l(R)$, $u \in \mathrm{U}(R)$ and $aR \cap zR = 0$.*

(4) *$a = z + u$ where $z \in \mathrm{zd}(R)$, $u \in \mathrm{U}(R)$ and $aR \cap zR = 0$.*

Proof. (1) $\Rightarrow$ (4). Write $a = eu$, where $0 \neq e^2 = e$ and $u \in \mathrm{U}(R)$. Thus $a = (1-e)u + (-1+2e)u$, where $(1-e)u \in \mathrm{zd}(R)$ and $(-1+2e)u \in \mathrm{U}(R)$. Moreover, $aR \cap (1-e)uR = euR \cap (1-e)uR = eR \cap (1-e)R = 0$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2). The implications are obvious.

(2) $\Rightarrow$ (1). Let z, u be as given in (2). Then $au^{-1}a - a = (z+u)u^{-1}a - a = zu^{-1}a \in aR \cap zR = 0$, so $a = au^{-1}a$ is unit-regular. □

Remark 2.2. The equivalence (1) $\Leftrightarrow$ (2) in Proposition 2.1 was proved, independently, by G. Calugareanu [2] and by Tsiu-Kwen Lee [10]. A result of Camillo and Khurana [3] states that a ring R is unit-regular if and only if for any $a \in R$, $a = e + u$ where $e^2 = e$, $u \in \mathrm{U}(R)$ and $aR \cap eR = 0$. Note that the element-wise version of Camillo-Khurana's result does not hold, because a unit-regular element in a ring need not be the sum of an idempotent and a unit (see [8])

Proposition 2.1 gives rise to some other conditions equivalent to a ring being unit-regular. For instance, a ring R is unit-regular if and only if for any $0 \neq a \in R$, $a = z+u$ where $z \in \mathrm{zd}(R)$ and $u \in \mathrm{U}(R)$ such that $aR \cap zR = 0$. This is strengthened below. A result, called the Super Jacobson's Lemma, is needed.

Lemma 2.3. [9, 11] (**The Super Jacobson's Lemma**) *Let a, b, x be any elements in a ring R. Then $a + b - axb \in \mathrm{U}(R)$ if and only if $a + b - bxa \in \mathrm{U}(R)$.*

The next result can be compared with a result in [16] that a ring R is unit-regular if and only if for any $a, b \in R$ with $Ra + Rb = R$, there exists $e \in \mathrm{idem}(R)$ such that $a + eb \in \mathrm{U}(R)$ and $aR \cap eR = 0$.

Theorem 2.4. *The following are equivalent for a ring R:*

(1) *R is unit-regular.*
(2) *For any $0 \neq a \in R$ and $b \in R$ with $Ra + Rb = R$, there exists $z \in R$ such that $a + zb \in \mathrm{U}(R)$ and $aR \cap zR = 0$.*
(3) *For any $0 \neq a \in R$ and $b \in R$ with $Ra + Rb = R$, there exists $z \in \mathrm{zd}(R)$ such that $a + zb \in \mathrm{U}(R)$ and $aR \cap zR = 0$.*

Proof. (1) $\Rightarrow$ (3). Let $Ra + Rb = R$ where $a \neq 0$. Write $a = ue$ where $u \in \mathrm{U}(R)$ and $e^2 = e$. Then $Re + Rb = R$. Since R has stable range one, there exists $x \in R$ such that $v := e + xb \in \mathrm{U}(R)$. Then $v(1-e) = xb(1-e)$, so $R = Re + R(1-e) = Re + Rxb(1-e) = Re + Rb(1-e)$. Again there exists $y \in R$ such that $e + yb(1-e) \in \mathrm{U}(R)$. By Lemma 2.3, $e + (1-e)yb \in \mathrm{U}(R)$. Since $e \neq 0$, $(1-e)y$ is not a unit, so $(1-e)y \in \mathrm{zd}(R)$ (since $(1-e)y$ is unit-regular). Hence $a + u(1-e)yb = u[e + (1-e)yb] \in \mathrm{U}(R)$, where $z := u(1-e)y \in \mathrm{zd}(R)$ and $aR \cap zR = u[eR \cap (1-e)yR] = 0$.

(3) $\Rightarrow$ (2). The implication is obvious.

(2) $\Rightarrow$ (1). Let $0 \neq a \in R$. Since $Ra + R(-1) = R$, there exists $z \in \mathrm{zd}(R)$ such that $u := a + z(-1) \in \mathrm{U}(R)$ and $aR \cap zR = 0$. Hence $a = z + u$. So a is unit-regular by Proposition 2.1. □

We next show that any regular ring is unit-fusible, answering affirmatively a question in [5].

Theorem 2.5. *Suppose that for any direct summands A and B of R_R with $A \cong B$, $A \cap B$ is a direct summand of R_R, and $A + B$ is a direct summand of R_R if additionally $A \cap B = 0$. Then every nonzero regular element of R is left unit-fusible.*

Proof. Let $0 \neq x \in R$ be regular. Fix an inner inverse $x' \in R$ for x. That is, $x = xx'x$ and $x' = x'xx'$. Since the left multiplication by x takes $x'xR$ isomorphically to $xx'R$, $x'xR \cap xx'R$ is a direct summand of R_R, so we can write

$$x'xR = (x'xR \cap xx'R) \oplus A$$

for some principal right ideal A.

While the left multiplication by x takes $x'xR$ isomorphically to $xx'R$, it takes A isomorphically to a summand A' of $xx'R$. We then see that $A \cap A' \subseteq A \cap x'xR \cap xx'R = \{0\}$. Thus, $R_R = A \oplus A' \oplus X$ for some right ideal X. Let $u \in R$ be the element that takes $A \to A'$ like x, takes $A' \to A$ as the inverse, and fixes X. Then u is a unit and $x - u$ is zero on A. So, in order to avoid $x - u$ being a left zero-divisor, we must have $A = 0$.

Similarly, $xx'R = (x'xR \cap xx'R) \oplus B$ for a principal right ideal B. The left multiplication by x' is an isomorphism from $xx'R$ onto $x'xR$ which takes B to a summand B' of $x'xR$ and whose inverse is the left multiplication by x. We see that $B \cap B' \subseteq B \cap xx'R \cap x'xR = 0$. Thus, $R_R = B \oplus B' \oplus Y$ for a right ideal Y. Let $v \in R$ be the element that takes B' to B like x, takes $B \mapsto B'$ like x', and fixes Y. Then $v \in \mathrm{U}(R)$, and $x - u$ is zero on B'. So, in order to avoid $x - u$ being a left zero-divisor, we must have $B' = 0$ and hence $B = 0$.

Thus $x'xR = xx'R$. Then the left multiplication by x is an isomorphism from $x'xR$ to itself. Let $w \in R$ be the element that takes $x'xR$ to $x'xR$ like x and acts as the identity on $(1 - x'x)R$. Then $w \in \mathrm{U}(R)$ and $w - x$ is zero on $x'xR$. This shows that x is left unit-fusible. □

Corollary 2.6. *Every regular ring is unit-fusible.*

Proof. By [7, Theorem 1.1 and Lemma 2.2] a regular ring satisfies all the assumptions in Theorem 2.5. □

A unit-fusible ring need not be regular: By [1, Example 4.7], there exists a unit-fusible ring R such that the center of R is not unit-fusible. This ring R cannot be regular, otherwise its center will be regular and hence unit-fusible. More examples include $C(X)$, the ring of real-valued continuous functions on a Tychonoff space. This (commutative) ring is always unit-fusible but only von Neumann regular in particular cases (i.e. X is a P-space); see [6].

3. Left unit-fusible elements being regular

In this section, we continue to explore the relation between regularity and the unit-fusible property by addressing the question of when every left unit-fusible element of a ring is regular.

Lemma 3.1. *Suppose that every left unit-fusible element of a ring R is regular. Let $e \in R$ be a non-trivial idempotent. Then eRe is regular, and any element $a \in R$ with $eae \neq 0$ is regular in R.*

Proof. Let $e' = 1 - e$. Identify R with $R = \begin{pmatrix} e'Re' & e'Re \\ eRe' & eRe \end{pmatrix}$.

For $r \in eRe$, $\begin{pmatrix} e' & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & r-e \end{pmatrix} + \begin{pmatrix} e' & 0 \\ 0 & e \end{pmatrix}$ is a left unit-fusible representation in R. So, $\begin{pmatrix} e' & 0 \\ 0 & r \end{pmatrix}$ is regular in R, and hence there exists $\alpha := \begin{pmatrix} a & x \\ y & b \end{pmatrix} \in R$ such that $\begin{pmatrix} e' & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} e' & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} e' & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} a & xr \\ ry & rbr \end{pmatrix}$. Thus $r = rbr$. So eRe is regular.

As seen above, eae is regular in eRe, so it is left unit-fusible in eRe by Corollary 2.6. Write $eae = z + u$ where $z \in \mathrm{zd}_l(eRe)$ and $u \in \mathrm{U}(eRe)$. Then $a = \begin{pmatrix} e'ae' & e'ae \\ eae' & eae \end{pmatrix} = \begin{pmatrix} e'ae' - e' & 0 \\ eae' & z \end{pmatrix} + \begin{pmatrix} e' & e'ae \\ 0 & u \end{pmatrix}$ is a left unit-fusible representation. So $a \in R$ is regular. □

Lemma 3.2. *The following are equivalent for a ring R:*

(1) *Every left unit-fusible element in R is a unit.*
(2) $\mathrm{zd}_l(R) \subseteq J(R)$.

Proof. (1) ⇒ (2). Let $a \in \mathrm{zd}_l(R)$. For any $b \in R$, $ba \in \mathrm{zd}_l(R)$, so $ba + 1$ is left unit-fusible in R. Hence $ba + 1 \in \mathrm{U}(R)$, for any $b \in R$. So $a \in J(R)$.

(2) $\Rightarrow$ (1). If $a = b + u$ is a left unit-fusible representation in R, then $b \in J(R)$ and $u \in \mathrm{U}(R)$. So $a \in \mathrm{U}(R)$. □

Lemma 3.3. [12] *Let* $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a Morita context. Then* $J(R) = \begin{pmatrix} J(A) & M_0 \\ N_0 & J(B) \end{pmatrix}$, *where* $M_0 = \{x \in M : xN = 0\}$ *and* $N_0 = \{y \in N : yM = 0\}$.

Lemma 3.4. *Let* $T = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a Morita context. If* A, B *are division rings with* $MN = 0$ *and* $NM = 0$, *then every left unit-fusible element of* R *is strongly regular.*

Proof. Let $\alpha = \begin{pmatrix} a & x \\ y & b \end{pmatrix} \in T$ be left unit-fusible. Then $\alpha \notin J(T)$. By Lemma 3.3, $J(R) = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}$. So $a \neq 0$ or $b \neq 0$.

If $a \neq 0$ and $b \neq 0$, then α is a unit, so it is strongly regular. If $b = 0$ and $a \neq 0$, then
$$\alpha = \begin{pmatrix} a & x - a^{-1}x \\ y - ya^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}x \\ ya^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}x \\ ya^{-1} & 0 \end{pmatrix} \begin{pmatrix} a & x - a^{-1}x \\ y - ya^{-1} & 1 \end{pmatrix}$$
is strongly regular (as the product of a unit and an idempotent that commute). If $a = 0$ and $b \neq 0$, then
$$\alpha = \begin{pmatrix} 0 & xb^{-1} \\ b^{-1}y & 1 \end{pmatrix} \begin{pmatrix} 1 & x - xb^{-1} \\ y - b^{-1}y & b \end{pmatrix} = \begin{pmatrix} 1 & x - xb^{-1} \\ y - b^{-1}y & b \end{pmatrix} \begin{pmatrix} 0 & xb^{-1} \\ b^{-1}y & 1 \end{pmatrix}$$
is strongly regular (as the product of an idempotent and a unit that commute). □

We now are ready to prove the second main result.

Theorem 3.5. (**Characterization Theorem**). *Every left unit-fusible element of* R *is regular if and only if one of the following holds:*

(1) R *is regular.*
(2) $\mathrm{zd}_l(R) \subseteq J(R)$.
(3) R *is isomorphic to the Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, *where* A, B *are division rings,* $MN = 0$ *and* $NM = 0$.

Proof. ($\Leftarrow$). The implication follows from Lemma 3.2 and Lemma 3.4.

($\Rightarrow$). Assume that neither (1) nor (2) holds. We next show that R satisfies (3). Note $\mathrm{idem}(R) = \{0, 1\}$ would imply that every nonzero regular element of R is a unit, i.e. (2) would hold by Lemma 3.2. Thus R has a non-trivial idempotent e. Then $R =$

$\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where $A = eRe, M = eR(1-e), N = (1-e)Re$ and $B = (1-e)R(1-e)$. By Lemma 3.1, A, B are regular rings. By Lemma 3.3, $J(R) = \begin{pmatrix} J(A) & M_0 \\ N_0 & J(B) \end{pmatrix} = \begin{pmatrix} 0 & M_0 \\ N_0 & 0 \end{pmatrix}$, where $M_0 = \{x \in M : xN = 0\}$ and $N_0 = \{y \in N : yM = 0\}$.

If $MN \neq 0$, then $M \neq M_0$. For any $x \in M \setminus M_0$, there exists $y_0 \in N$ such that $xy_0 \neq 0$. For any $y \in N_0$, let $\alpha = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in R$ and $\beta = \begin{pmatrix} 1_A & 0 \\ y_0 & 1_B \end{pmatrix} \in \mathrm{U}(R)$. Then $\alpha\beta = \begin{pmatrix} xy_0 & x \\ y & 0 \end{pmatrix}$ is regular in R by Lemma 3.1. So α is regular. Therefore, there exists $\begin{pmatrix} u & x' \\ y' & v \end{pmatrix} \in R$ such that $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \begin{pmatrix} u & x' \\ y' & v \end{pmatrix} \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} xvy & xy'x \\ yx'y & yux \end{pmatrix} = \begin{pmatrix} xvy & xy'x \\ 0 & 0 \end{pmatrix}$. Thus, $y = 0$ and $x \in xNx$. We have proved that, if $M \neq M_0$, then $N_0 = 0$ and $x \in xNx$ for any $x \in M \setminus M_0$.

Let $0 \neq y \in N$. There exists $x_0 \in M$ such that $x_0y \neq 0$. Let $x \in M_0$ and let $\alpha = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in R$. With $\beta = \begin{pmatrix} 1_A & x_0 \\ 0 & 1_B \end{pmatrix} \in \mathrm{U}(R)$, $\beta\alpha = \begin{pmatrix} x_0y & x \\ y & 0 \end{pmatrix}$ is regular in R by Lemma 3.1. So α is regular. Therefore, there exists $\begin{pmatrix} u & x' \\ y' & v \end{pmatrix} \in R$ such that $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \begin{pmatrix} u & x' \\ y' & v \end{pmatrix} \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} xvy & xy'x \\ yx'y & yux \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ yx'y & yux \end{pmatrix}$. Thus, $x = 0$ and $y \in yMy$. We have proved that, if $M \neq M_0$, then $N_0 = 0$, $M_0 = 0$, $x \in xNx$ for all $x \in M$, and $y \in yMy$ for all $y \in N$. Since A, B are regular, it follows from [13, Theorem 2.8(2)] that R is regular. This is a contradiction. So $MN = 0$. Similarly, $NM = 0$. It remains to show that A, B are division rings.

Since (1) does not hold, either $M \neq 0$ or $N \neq 0$. Without loss of generality, we can assume $M \neq 0$. To show that B is a division ring, it suffices to show that $\mathrm{idem}(B) = \{0, 1_B\}$. Assume on the contrary that f is a non-trivial idempotent in B. For $x \in M$, $\begin{pmatrix} 0 & x(1_B - f) \\ 0 & f \end{pmatrix} = \begin{pmatrix} -1_A & 0 \\ 0 & f - 1_B \end{pmatrix} + \begin{pmatrix} 1_A & x(1_B - f) \\ 0 & 1_B \end{pmatrix}$ is a left unit-fusible representation in R. So there exists $\begin{pmatrix} u & y \\ z & v \end{pmatrix} \in R$ such that

$$\begin{pmatrix} 0 & x(1_B - f) \\ 0 & f \end{pmatrix} = \begin{pmatrix} 0 & x(1_B - f) \\ 0 & f \end{pmatrix} \begin{pmatrix} u & y \\ z & v \end{pmatrix} \begin{pmatrix} 0 & x(1_B - f) \\ 0 & f \end{pmatrix} = \begin{pmatrix} 0 & x(1_B - f)vf \\ 0 & fvf \end{pmatrix}.$$

Thus, $x(1_B - f) = x(1_B - f)vf$, and it follows that $x(1_B - f) = 0$. That is, $x = xf$. Since $1_B - f$ is also a non-trivial idempotent of B, arguing as above with $1_B - f$ replacing f gives $x = x(1_B - f)$. Therefore, $x \cdot 1_B = 0$. But since x has the form $er(1-e)$ and $1-e \in B$, it follows from $xB = 0$, that $x = x(1-e) = 0$. Thus, $xB = 0$ for all $x \in M$. This contradiction shows that $\mathrm{idem}(B) = \{0, 1_B\}$. So B is a division ring. It is similar to show that A is a division ring. $\square$

Corollary 3.6. *Every left unit-fusible element of a ring R is unit-regular (resp. strongly regular) if and only if one of the following holds:*

(1) *R is unit-regular (resp. strongly regular).*

(2) $\mathrm{zd}_l(R) \subseteq J(R)$.

(3) $R \cong \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, *where A, B are division rings, $MN = 0$ and $NM = 0$.*

Proof. ($\Leftarrow$). This follows from Lemma 3.2 and Lemma 3.4.

($\Rightarrow$). If neither (2) nor (3) holds, then R is regular by Theorem 3.5. Therefore, R is left unit-fusible by Corollary 2.6, and (1) follows. $\square$

4. Rings R with $\mathrm{zd}_l(R) \subseteq J(R)$

The class of rings R with $\mathrm{zd}_l(R) \subseteq J(R)$ is determining for understanding the larger class of the rings whose left unit-fusible elements are regular, as suggested by Theorem 3.5. In this section, we present examples of the rings R with $\mathrm{zd}_l(R) \subseteq J(R)$.

Proposition 4.1. *Let R be a ring with an ideal I, and $n \geq 0$.*

(1) *If R is a local ring or a domain, then $\mathrm{zd}_l(R) \subseteq J(R)$.*

(2) *If $\mathrm{zd}_l(R) \subseteq J(R)$ then $\mathrm{zd}_l(eRe) \subseteq J(eRe)$ for any $e^2 = e \in R$.*

(3) $\mathrm{zd}_l(R) \subseteq J(R) \Leftrightarrow \mathrm{zd}_l(T) \subseteq J(T)$, *where T is any of the following*

  (a) *$T = R \propto I$, the trivial extension of R by I;*

  (b) *$T = R[t]/(t^{n+1})$;*

  (c) *$T = R[[t]]$.*

Proof. (1) This is clear.

(2) Let $a = b+u$ be a left unit-fusible representation in $S := eRe$. Then $u+(1-e) \in \mathrm{U}(R)$. So $a+(1-e) = b+(u+1-e)$ is a left unit-fusible representation in R. Applying Lemma 3.2, yields that $a + (1 - e) \in \mathrm{U}(R)$, and so $a \in \mathrm{U}(S)$.

The sufficiency of $(3a), (3b)$ and $(3c)$ is easily seen.

$(3a)(\Rightarrow)$. Let $(a, x) = (b, y) + (u, z)$ be a left unit-fusible representation in T. Then $u \in \mathrm{U}(R)$ and there exists $0 \neq (c, w) \in T$ such that $0 = (b, y)(c, w) = (bc, bw + yc)$. If $c \neq 0$, $b \in \mathrm{zd}_l(R)$. If $c = 0$, then $w \neq 0$ and $bw = 0$, so $b \in \mathrm{zd}_l(R)$. Hence $a = b + u$ is a left unit-fusible representation in R, so $a \in \mathrm{U}(R)$ and hence $(a, x) \in \mathrm{U}(T)$.

$(3b)(\Rightarrow)$. Let $\sum_{i=0}^{n} a_i t^i = \sum_{i=0}^{n} b_i t^i + \sum_{i=0}^{n} c_i t^i$ be a left unit-fusible representation in T. Then $c_0 \in \mathrm{U}(R)$ and there exists $0 \neq (\sum_{i=0}^{n-k} d_i t^i)t^k \in T$ with $d_0 \neq 0$ (where $0 \leq k \leq n$) such that $0 = (\sum_{i=0}^{n} b_i t^i)(\sum_{i=0}^{n-k} d_i t^i)t^k$. It follows that $b_0 d_0 = 0$. So $b_0 \in \mathrm{zd}_l(R)$. Thus $a_0 = b_0 + c_0$ is a left unit-fusible representation in R, so $a_0 \in \mathrm{U}(R)$ and hence $\sum_{i=0}^{n} a_i t^i \in \mathrm{U}(T)$.

$(3c)(\Rightarrow)$. Let $\sum_{i\geq 0} a_i t^i = \sum_{i\geq 0} b_i t^i + \sum_{i\geq 0} c_i t^i$ be a left unit-fusible representation in T. Then $c_0 \in \mathrm{U}(R)$ and there exists $0 \neq (\sum_{i\geq 0} d_i t^i)t^k \in T$ with $d_0 \neq 0$ (where $0 \leq k$) such that $0 = (\sum_{i\geq 0} b_i t^i)(\sum_{i\geq 0} d_i t^i)t^k$. It follows that $b_0 d_0 = 0$. So $b_0 \in \mathrm{zd}_l(R)$. Thus $a_0 = b_0 + c_0$ is a left unit-fusible representation in R, so $a_0 \in \mathrm{U}(R)$ and hence $\sum_{i\geq 0} a_i t^i \in \mathrm{U}(T)$. □

Example 4.2. *There exists a ring R with $\mathrm{zd}_\mathrm{l}(R) \subseteq J(R)$ but $\mathrm{zd}_l(R[t]) \not\subseteq J(R[t])$.*

Proof. Let R be a commutative reduced local ring that is not a domain. For example, let R be the localization of the ring of continuous function on the reals $\mathbb{R}$ at $p = 0$ (or any point in a space that is not an F-point). Then R is isomorphic to $C(X)/O_p$, where O_p denotes the semi-prime ideal consisting of functions which vanish on a neighbourhood of p (see [14]). Then $\mathrm{zd}(R) \subseteq J(R)$. Let $0 \neq a \in \mathrm{zd}(R)$. Then $1+at$ is unit-fusible in $R[t]$. We next verify $1+at$ is not a unit in $R[t]$, showing that $\mathrm{zd}_l(R[t]) \not\subseteq J(R[t])$ (by Lemma 3.2). Assume that $1 + at$ has an inverse $1 + b_1 t + \ldots + b_n t^n$. Then

$$a + b_1 = 0,\ ab_1 + b_2 = 0,\ \ldots, ab_{n-1} + b_n = 0,\ ab_n = 0.$$

It follows that $a^2b_{n-1} = 0$, $a^3b_{n-2} = 0$, $\ldots$, $a^nb_1 = 0$ and $a^{n+1} = 0$. Since R is reduced, $a = 0$, a contradiction. $\square$

A ring R is called a left Ore ring if for all elements x and y with $x \in \mathrm{zd}^*(R)$, there exist elements u and v with $v \in \mathrm{zd}^*(R)$ such that $ux = vy$. It is well-known that a ring R has a left classical ring of quotients if and only if R is left Ore.

Proposition 4.3. *Let R be a left Ore ring with a left classical ring of quotients Q. Then $\mathrm{zd}_l(Q) \subseteq J(Q)$ if and only if $\mathrm{zd}_l(R) + \mathrm{zd}^*(R) = \mathrm{zd}^*(R)$.*

Proof. ($\Rightarrow$). Let $a = b + r$ where $b \in \mathrm{zd}_l(R)$ and $r \in \mathrm{zd}^*(R)$. Then $a = b + r$ is a left unit-fusible representation in Q. So $a \in \mathrm{U}(Q)$, and hence $a \in \mathrm{zd}^*(R)$.

($\Leftarrow$). Let $x = y + z$ be a left unit-fusible representation in Q. Then $yy' = 0$ for some $0 \neq y' \in Q$. We can write $x = r^{-1}a$, $y = r^{-1}b$, $z = r^{-1}c$ and $y' = r^{-1}d$ where $a, b, d \in R$ and $c, r \in \mathrm{zd}^*(R)$. Write $ar^{-1} = s^{-1}a'$, $br^{-1} = s^{-1}b'$, $cr^{-1} = s^{-1}c'$ where $a', b' \in R$ and $s, c' \in \mathrm{zd}^*(R)$. Then $0 = yy' = r^{-1}br^{-1}d = r^{-1}s^{-1}b'd$, showing $b'd = 0$. Since $d \neq 0$, $b' \in \mathrm{zd}_l(R)$. Then $a' = b' + c' \in \mathrm{zd}_l(R) + \mathrm{zd}^*(R)$, so $a' \in \mathrm{zd}^*(R)$. Thus, $x = r^{-1}a = r^{-1}s^{-1}a'r \in \mathrm{U}(Q)$. $\square$

Let R be a commutative ring and $f(t) = \sum_{i=0}^{n} a_it^i \in R[t]$. The following are well-known: $f(t) \in \mathrm{U}(R[t])$ if and only if $a_0 \in \mathrm{U}(R)$ and $a_i \in \mathrm{nil}(R)$ for $i = 1, \ldots, n$; $f(t) \in \mathrm{nil}(R[t])$ if and only if $a_i \in \mathrm{nil}(R)$ for $i = 0, \ldots, n$; $f(t) \in \mathrm{zd}(R[t])$ if and only if $f(t)c = 0$ for some $0 \neq c \in R$.

Proposition 4.4. *Let R be a commutative ring. Then $\mathrm{zd}(R[t, t^{-1}]) \subseteq J(R[t, t^{-1}])$ if and only if $\mathrm{zd}(R) = \mathrm{nil}(R)$.*

Proof. ($\Rightarrow$). Let $a \in \mathrm{zd}(R)$. Then, by (1), $1 + at \in \mathrm{U}(R[t, t^{-1}])$ with inverse $t^k(b_0 + b_1t + \ldots + b_nt^n)$ where $b_0 \neq 0$, $n \geq 0$ and $k \in \mathbb{Z}$. Therefore, we have

$$1 = [b_0 + (b_1 + ab_0)t + (b_2 + ab_1)t^2 + \ldots + (b_n + ab_{n-1})t^n + ab_nt^{n+1}]t^k.$$

Since $b_0 \neq 0$, it follows that $k = 0$ and

$$b_0 = 1,\ b_1 + ab_0 = 0,\ b_2 + ab_1 = 0,\ \ldots,\ b_n + ab_{n-1} = 0,\ ab_n = 0.$$

It follows $a^{n+1} = 0$.

($\Leftarrow$). Let $f(t) = g(t) + h(t)$ be a unit-fusible representation in $T := R[t, t^{-1}]$. Then $g(t) = t^n g_1(t)$ where $n \in \mathbb{Z}$ and $g_1(t) \in \mathrm{zd}(R[t])$. Thus, there exists $0 \neq b \in R$ such that $bg_1(t) = 0$. The hypothesis shows that all coefficients of $g_1(t)$ are nilpotent. Since R is commutative, $g_1(t) \in \mathrm{nil}(R[t])$, and hence $g(t) \in \mathrm{nil}(T)$. Since $h(t) \in \mathrm{U}(T)$, it follows that $f(t) = g(t) + h(t) \in \mathrm{U}(T)$. □

Let M be a left R-module. An element $a \in R$ is called a *left M-unit-fusible element* if $a = b + u$ where $\mathbf{r}_M(b) \neq 0$ and $u \in \mathrm{U}(R)$.

Proposition 4.5. *Let M be a bimodule over R and $T := R \propto M$. Then $\mathrm{zd}_l(T) \subseteq J(T)$ if and only if $\mathrm{zd}_l(R) \subseteq J(R)$ and every left M-unit-fusible element of R is a unit.*

Proof. ($\Rightarrow$). If $a \in \mathrm{zd}_l(R)$, then $(a, 0) \in \mathrm{zd}_l(T)$, so $(a, 0) \in J(T) = J(R) \propto M$. Hence $a \in J(R)$. If $a = b + u$ where $\mathbf{r}_M(b) \neq 0$ and $u \in \mathrm{U}(R)$, let $0 \neq x \in \mathbf{r}_M(b)$. Then $(b, 0)(0, x) = 0$, so $(a, 0) = (b, 0) + (u, 0)$ is a left unit-fusible representation in R. Hence $(a, 0) \in \mathrm{U}(T)$ and so $a \in \mathrm{U}(R)$.

($\Leftarrow$). Let $(a, x) = (b, y) + (c, z)$ be a left unit-fusible representation in T. Then $c \in \mathrm{U}(R)$ and for some $0 \neq (d, w) \in T$, $0 = (b, y)(d, w) = (bd, bw + yd)$. If $d \neq 0$, then $b \in \mathrm{zd}_l(R) \subseteq J(R)$, so $a = b + c \in \mathrm{U}(R)$ and hence $(a, x) \in \mathrm{U}(T)$. If $d = 0$, then $w \neq 0$ and $bw = 0$, so $\mathbf{r}_M(b) \neq 0$. Thus, $a = b + c$ is left M-unit-fusible, so it is a unit in R by our assumption. Hence $(a, x) \in \mathrm{U}(T)$. □

Corollary 4.6. *Let R be a ring and $T = R \propto R/J(R)$. Then $\mathrm{zd}_l(T) \subseteq J(T)$ if and only if $\mathrm{zd}_l(R) \subseteq J(R)$ and $R/J(R)$ is a domain.*

Proof. ($\Rightarrow$). By Proposition 4.5, it suffices to show that $R/J(R)$ is a domain. Assume on the contrary that $ab \in J(R)$ where $a \notin J(R)$ and $b \notin J(R)$. There exists $r \in R$ such that $1 + ra \notin \mathrm{U}(R)$. Since $(ra, 0)(0, \bar{b}) = 0$ in $T := R \propto R/I$. So $(ra + 1, 0) = (ra, 0) + (1, 0)$ is a left unit-fusible representation in T. It follows from the hypothesis that $(ra + 1, 0) \in \mathrm{U}(T)$, and hence $ra + 1 \in \mathrm{U}(R)$. This is a contradiction.

($\Leftarrow$). Let $M = R/J(R)$ and $T = R \propto M$. Let $a = b + u$ be left M-unit-fusible, where $\mathbf{r}_M(b) \neq 0$ and $u \in \mathrm{U}(R)$. Then there exists $0 \neq \overline{c} \in M$ such that $b\overline{c} = 0$, i.e. $bc \in J(R)$. Since $R/J(R)$ is a domain, it follows that $b \in J(R)$. Hence $a = b + u \in \mathrm{U}(R)$. So we verified that every left M-unit-fusible element in R is a unit. Since $\mathrm{zd}_l(R) \subseteq J(R)$, the implication holds by Proposition 4.5. □

We conclude the paper by giving an example of a ring R with $\mathrm{zd}_l(R) \subseteq J(R)$ but not every nonzero right unit-fusible element is regular.

Example 4.7. *There exists a ring T with $\mathrm{zd}_l(T) \subseteq J(T)$, but not every right unit-fusible element of T is regular.*

Proof. Let R be a domain with $J(R) = 0$ and σ be a ring endomorphism of R that is not injective (e.g. $R = D[x]$ where D is a commutative domain and $\sigma : R \to R$, $\sigma(f(x)) = f(0)$). Let $T := R[[t, \sigma]]$. First we note that $J(T) = Tt$. If $f(t) = \sum_{i \geq 0} a_i t^i \in \mathrm{zd}_l(T)$, then there exists $g(t) = b_k t^k + b_{k+1} t^{k+1} + \ldots \in T$ with $b_k \neq 0$ such that $f(t)g(t) = 0$. Hence $a_0 b_k = 0$, and so $a_0 = 0$. Thus $f(t) \in J(T)$. We have verified $\mathrm{zd}_l(T) \subseteq J(T)$. Take $0 \neq a \in R$ with $\sigma(a) = 0$. Then $ta = \sigma(a)t = 0$, so $a \in \mathrm{zd}_r(T)$ but $a \notin J(T)$. Hence, for some $h(t) = \sum_{i \geq 0} c_i t^i \in T$, $\alpha := 1 + ah(t) \notin \mathrm{U}(T)$, but α is right unit-fusible in T. Assume that $\alpha \in T$ is regular. Then $\alpha = \alpha\beta\alpha$ where $\beta \in T$. Since T has only the trivial idempotents, $\alpha\beta = \beta\alpha = 1$, so $\alpha \in \mathrm{U}(T)$, a contradiction. □

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