ON FUSIBLE RINGS AND RELATED NOTIONS

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Abstract. A ring is called *left fusible* if each of its nonzero elements is the sum of a left zero-divisor and a non-left zero-divisor. This paper aims to extend the existing theory of left fusible rings and related notions such as being *left unit-fusible*, being *regular left fusible*, and being *uniquely left fusible*. New examples of these rings are presented. A new notion of left generalized unit-fusible rings is introduced and discussed. The asymmetry of these notions is addressed.

1. Introduction

Throughout, rings are associative with identity unless otherwise specified. For a ring R, let $\mathrm{zd}_l(R) = \{a \in R : \mathbf{r}(a) \neq 0\}$ be the set of left zero-divisors and $\mathrm{zd}_l^*(R) = \{a \in R : \mathbf{r}(a) = 0\}$ the set of non-left zero-divisors, where $\mathbf{r}(a)$ (resp. $\mathbf{l}(a)$) denotes the right (resp. left) annihilator of a in R. A both non-left and non-right zero-divisor is called a regular element. Let us first recall the well-studied notion of a complemented ring: A ring R is called *right complemented* if for each a in R there exists $b \in R$ with $ab = 0$ and $a + b$ regular. It is known that, for a ring R with a right classical quotient ring Q, R is right complemented if and only if Q is strongly regular (see [8], [20]). Complemented rings are the motivation of several other notions such as weakly complemented rings, quasi-complemented rings and feebly Baer rings (see [13]). Because every nonzero element in a complemented ring is the sum of a zero-divisor and a regular element, the notion of left fusible rings, interesting by its own nature, was introduced in [8]. Following [8], an element a in a ring R is called *left fusible* if a is the sum of a left zero-divisor and a non-left zero-divisor, and the ring R is called *left fusible* if every nonzero element of R is left fusible.

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In the study of left fusible rings, some related notions occurred. An element a in a ring R is called *left unit-fusible* if a is the sum of a left zero-divisor and a unit, and the ring R is called *left unit-fusible* if every nonzero element of R is left unit-fusible (see [8]). An element a in a ring R is called *uniquely left fusible* if there exists a unique left zero-divisor z such that $a-z$ is a non-left zero-divisor, and the ring R is *uniquely left fusible* if each nonzero element is uniquely left fusible (see [8]). An element a in a ring R is called *regular left fusible* if ra is left fusible for some regular element r, and the ring R is called *regular left fusible* if every nonzero element is regular left fusible (see [15]). The right versions of these notions are defined similarly. Various interesting results on left fusible (resp. left unit-fusible, regular left fusible, uniquely left fusible) rings have been established in [7, 8, 9, 15].

This is a continuation of the study of left fusible (resp. left unit-fusible, regular left fusible) rings. In Section 2, new examples of left fusible rings are presented. For instance, it is shown that, for a matrix A over a ring R, a single entry of A being left fusible (resp. left unit-fusible) in R implies that the matrix is left fusible (resp. left unit-fusible) in the matrix ring over R. Resultingly, matrix rings (of finite or infinite size) over a left fusible (resp. left unit-fusible) ring are left fusible (resp. left unit-fusible). Another consequence is that matrix rings over regular left fusible rings are regular left fusible, answering affirmatively a question in [15]. More examples of left fusible rings are perceived as a class of subrings of Laurent series rings. Section 3 is about being uniquely left fusible. It is proved that, for a ring R, every element outside the Jacobson radical (resp. outside the right singular ideal) is uniquely left fusible if and only if R is a domain or a UR-ring. Consequently, a ring R is uniquely left fusible if and only if R is a domain or a UR-ring. This result was proved in [8] for any ring R with $\text{ch}(R) \neq 2$, and in [9] for any ring R with $\text{ch}(R) = 2$. Because elements in the Jacobson radical of a ring are never left unit-fusible, Section 4 concerns with a class of rings, called left generalized unit-fusible rings. Here a ring is *left generalized unit-fusible* if every element outside the Jacobson radical is left unit-fusible. Because left unit-fusible rings are exactly these left generalized unit-fusible rings that are semiprimitive, the unit-fusible property can now be addressed in a more general context. We prove that there is a rich supply of left generalized unit-fusible rings. In [8], right fusible elements that are not left fusible were given. In Section 5, we give examples of regular right fusible elements that are not regular left fusible, and of left unit-fusible elements that are not right unit-fusible. Though we are unable to answer whether a left fusible (resp. left unit-fusible, or regular left fusible) ring is always right fusible (resp. right unit-fusible, or regular right fusible), we show that left generalized fusible rings need not be right generalized fusible.

For a ring R, we write $J(R)$ for the Jacobson radical of R, $\mathrm{U}(R)$ for the group of units of R, $\mathrm{Z}(R)$ for the center of R, $\mathrm{nil}(R)$ for the set of all nilpotent elements of R, $\mathrm{idem}(R)$ for the set of all idempotents of R, and $\mathrm{Sing}_r(R)$ for the right singular ideal of R. We let $\mathrm{reg}(R)$ be the set of regular elements of R, i.e. $\mathrm{reg}(R) = \mathrm{zd}_l^*(R) \cap \mathrm{zd}_r^*(R)$. An element a in R is a zero-divisor if $a \in \mathrm{zd}_l(R) \cap \mathrm{zd}_r(R)$. A ring R is called abelian if every idempotent of R is central, and reduced if $\mathrm{nil}(R) = \{0\}$. The $n \times n$ matrix ring over a ring R is denoted by $\mathbb{M}_n(R)$. For an endomorphism σ of a ring R, $R[[t;\sigma]]$ (resp. $R[t;\sigma]$) is the ring of left skew power series (resp. left skew polynomials) over R with multiplication subject to the relation $ta = \sigma(a)t$ for all $a \in R$. For an automorphism σ of a ring R, $R[[t,t^{-1};\sigma]]$ (resp. $R[t,t^{-1};\sigma]$) denotes the ring of left skew Laurent series (resp. skew Laurent polynomials) over R with multiplication subject to the relation $ta = \sigma(a)t$ for all $a \in R$.

2. To be left fusible

Let R be a ring. For $a \in R$, if $a = z + r$ where $z \in \mathrm{zd}_l(R)$ and $r \in \mathrm{zd}_l^*(R)$, a is called *left fusible* and such a sum is called a *left fusible representation* in R. If $a = z + u$ where $z \in \mathrm{zd}_l(R)$ and $u \in \mathrm{U}(R)$, a is called *left unit-fusible* and such a sum is called a *left unit-fusible representation* in R. A ring is called *left* (*unit-*) *fusible* if every nonzero element in it is left (unit-) fusible. Right (unit-) fusible elements and rings are defined similarly.

Lemma 2.1. *Let $a \in R$, $d \in \mathrm{zd}_l^*(R)$ and $u, v \in \mathrm{U}(R)$.*

(1) *If a is left fusible, then so is dau. In particular, a is left fusible if and only if vau is left fusible.*

(2) *a is left unit-fusible if and only if vau is left unit-fusible.*

Proof. (1) If $a = x + y$ is a left fusible representation, then $dau = dxu + dyu$ is a left fusible representation. (2) It is easily seen. □

Proposition 2.2. *Let R be a ring. Then every element in $\mathrm{Sing}_r(R)$ is not left fusible, and every element in $J(R) + \mathrm{Sing}_r(R)$ is not left unit-fusible.*

Proof. Let $a \in \mathrm{Sing}_r(R)$. If $a = x + y$ where $x \in \mathrm{zd}_l(R)$ and $y \in \mathrm{zd}_l^*(R)$, then $\mathbf{r}(x) \neq 0$ and $\mathbf{r}(a)$ is an essential right ideal of R. So $I := \mathbf{r}(a) \cap \mathbf{r}(x) \neq 0$, and $yI = (a - x)I = 0$. Since $\mathbf{r}(y) = 0$, it follows that $I = 0$, a contradiction.

Let $a \in J(R) + \mathrm{Sing}_r(R)$. If $a = j + y = x + u$ where $j \in J(R)$, $y \in \mathrm{Sing}_r(R)$, $x \in \mathrm{zd}_l(R)$ and $u \in \mathrm{U}(R)$, then $y = x + (u - j)$, which is left fusible and singular. This is a contradiction as shown above. □

Corollary 2.3. [8] *Every left fusible ring is right non-singular, and every left unit-fusible ring is right non-singular and semiprimitive.*

For a ring R, let $\mathbb{RCFM}_\kappa(R)$ be the ring of row and column-finite matrices, $\mathbb{RFM}_\kappa(R)$ the ring of row-finite matrices, and $\mathbb{CFM}_\kappa(R)$ the ring of column-finite matrices, indexed by an infinite set κ with entries from R (see [17]). Let $\mathbb{M}(R)$ denote a subring of $\mathbb{CFM}_\kappa(R)$ or $\mathbb{RFM}_\kappa(R)$ that contains $\mathbb{RCFM}_\kappa(R)$.

Theorem 2.4. *Let $A \in \mathbb{M}_n(R)$ and $B \in \mathbb{M}(R)$.*

(1) *If some entry of A is left fusible in R, then A is left fusible in $\mathbb{M}_n(R)$.*

(2) *If some entry of A is left unit-fusible in R, then A is left unit-fusible in $\mathbb{M}_n(R)$.*

(3) *If some entry of B is left fusible in R, then B is left fusible in $\mathbb{M}(R)$.*

(4) *If some entry of B is left unit-fusible in R, then B is left unit-fusible in $\mathbb{M}(R)$.*

(5) *If R is commutative and some entry of A is fusible (resp. unit-fusible) in R, then A is the sum of a zero-divisor and a regular element (resp. a unit) in $\mathbb{M}_n(R)$.*

Proof. (1) Let $A=(a_{ij})$ with some $a_{ij}\neq 0$. Interchanging row 1 and row i of A we carry A to A_1 whose $(1,j)$-entry is a_{ij}, and then interchanging column 1 and column j of A_1 we carry A_1 to A_2 whose $(1,1)$-entry is a_{ij}. Thus, there exist invertible matrices U,V in $\mathbb{M}_n(R)$ such that the $(1,1)$-entry of UAV is a_{ij}. Hence, to show that A is left fusible, by Lemma 2.1 we can assume that a_{11} is left fusible in R. Write $a_{11}=x+y$ where $x\in \mathrm{zd}_l(R)$ and $y\in \mathrm{zd}_l^*(R)$. Thus,

$$A=\begin{pmatrix} x & a_{12} & a_{13} & \cdots & a_{1n}\\ 0 & a_{22}-1 & a_{23} & \cdots & a_{2n}\\ 0 & 0 & a_{33}-1 & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a_{nn}-1\end{pmatrix}+\begin{pmatrix} y & 0 & 0 & \cdots & 0\\ a_{21} & 1 & 0 & \cdots & 0\\ a_{31} & a_{32} & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & 1\end{pmatrix}$$

is a left fusible representation.

(2) Argue as proving (1), where "$y\in \mathrm{zd}_l^*(R)$" is replaced by "$y\in \mathrm{U}(R)$".

(3) and (4). Argue as in (1) and (2), respectively.

(5) By [1, Theorem 9.1], $\mathrm{zd}_l(\mathbb{M}_n(R))=\mathrm{zd}_r(\mathbb{M}_n(R))$. The claims now follow from this, (1) and (2). □

Theorem 2.4 has some quick consequences.

Corollary 2.5. *Let R be a ring and $n\geq 1$.*

(1) [8, Theorem 2.18] *If R is left fusible, then so is $\mathbb{M}_n(R)$.*
(2) [8, Corollary 5.9] *If R is left unit-fusible, then so is $\mathbb{M}_n(R)$.*
(3) *If R is left fusible, then so is $\mathbb{M}(R)$.*
(4) *If R is left unit-fusible, then so is $\mathbb{M}(R)$.*
(5) *If R is a commutative fusible (resp. unit-fusible) ring, then every nonzero matrix in $\mathbb{M}_n(R)$ is the sum of a zero-divisor and a regular element (resp. a unit).*

Following [15], a ring R is called *regular left fusible* if, for each $0\neq a\in R$, there exists $r\in \mathrm{reg}(R)$ such that ra is left fusible. It is proved in [15, Theorem 2.17] that if R is a regular left fusible ring having a classical left quotient ring, then $\mathbb{M}_n(R)$ is regular left fusible. A question left open in [15, Question 2.19] asks whether the matrix ring over a regular left fusible ring is again regular left fusible. This question has an affirmative answer.

Corollary 2.6. *If R is a regular left fusible ring and $n \geq 1$, then $\mathbb{M}_n(R)$ is a regular left fusible ring.*

Proof. Let $0 \neq A = (a_{ij}) \in T := \mathbb{M}_n(R)$. Then $a_{ij} \neq 0$ for some i and j. So ta_{ij} is left fusible for some $t \in \text{reg}(R)$. Hence, by Theorem 2.4(1), $(tI_n)A = tA$ is left fusible. Since $tI_n \in \text{reg}(T)$, A is regular left fusible. □

In [8], it is proved that, for an automorphism σ of a ring R, R is left fusible if and only if $R[t;\sigma]$ is left fusible if and only if $R[[t;\sigma]]$ is left fusible. This result fits in the more general setting explained below.

Definition 2.7. Let R be a ring with an endomorphism σ and I be an ideal of R such that $\sigma(I) \subseteq I$. Denote by $[R;I][t;\sigma]$ the following subring of $R[[t;\sigma]]$:

$[R;I][t;\sigma] = \{f = \sum_{i=0}^{\infty} a_i t^i \in R[[t;\sigma]] : \exists 0 \leq n = n(f) \in \mathbb{Z}$ such that $a_i \in I$ for all $i \geq n\}$.

If $\sigma = 1_R$, $[R;I][t;\sigma]$ is the ring $[R;I][t]$ discussed in [14]. If $I = 0$, we have $R[t;\sigma] = [R;I][t;\sigma]$, and if $I = R$ we have $R[[t;\sigma]] = [R;I][t;\sigma]$.

Definition 2.8. Let R be a ring with an automorphism σ and I an ideal of R such that $\sigma(I) \subseteq I$. Denote by $[R;I][t,t^{-1};\sigma]$ the following subring of $R[[t,t^{-1};\sigma]]$:

$[R;I][t,t^{-1};\sigma] = \{f = \sum_{i\geq -s}^{\infty} a_i t^i \in R[[t,t^{-1};\sigma]] : s \geq 0, \exists 0 \leq n = n(f) \in \mathbb{Z}$ such that $-s \leq n$ and $a_i \in I$ for all $i \geq n\}$.

If $\sigma = 1_R$, $[R;I][t,t^{-1};\sigma]$ is the ring $[R;I][t,t^{-1}]$ discussed in [14]. If $I = 0$, we have $R[t,t^{-1};\sigma] = [R;I][t,t^{-1};\sigma]$, and if $I = R$ we have $R[[t,t^{-1};\sigma]] = [R;I][t,t^{-1};\sigma]$.

Proposition 2.9. *Let σ be an automorphism of a left fusible ring R and I be an ideal of R such that $\sigma(I) \subseteq I$. Then $[R;I][t;\sigma]$ is left fusible.*

Proof. Let $f = \sum_{i\geq k} a_i t^i \in [R;I][t;\sigma]$ with $a_k \neq 0$. Then $f = t^k g$ where $g = \sum_{i=0}^{\infty} \sigma^{-k}(a_{k+i}) t^i \in [R;I][t;\sigma]$ with $\sigma^{-k}(a_k) \neq 0$ and with $g \in [R;I][t;\sigma]$. By Lemma

2.1, to show that f is left fusible, it suffices to show that g is left fusible. Hence, without loss of generality, we can assume that $k = 0$. So $a_0 \in R$ is left fusible. Let $a_0 = a + b$ be a left fusible representation in R. Then $f = a + (b + \sum_{i=1}^{\infty} a_i t^i)$ is a left fusible representation in $[R; I][t; \sigma]$. □

Corollary 2.10. *Let σ be an automorphism of a left fusible ring R and I be an ideal of R such that $\sigma(I) \subseteq I$. Then $[R; I][t, t^{-1}; \sigma]$ is left fusible.*

Proof. Let $0 \neq f \in [R; I][t, t^{-1}; \sigma]$, and write $f = t^k g$ where $k \in \mathbb{Z}$ and $g = \sum_{i \geq 0} a_i t^i \in [R; I][t; \sigma]$ with $a_0 \neq 0$. By Proposition 2.9, $[R; I][t; \sigma]$ is left fusible, so there exist there exists $h \in [R; I][t; \sigma]$ such that $g = h + (g - h)$ is a left fusible representation in $[R; I][t; \sigma]$. It follows that $f = t^k g = t^k h + t^k (g - h)$ is a left fusible representation in $[R; I][t, t^{-1}; \sigma]$. □

Corollary 2.11. *If σ is an automorphism of a left fusible ring R, then:*

(1) [8, Proposition 2.9] *$R[t; \sigma]$ and $R[[t; \sigma]]$ are left fusible rings.*
(2) *$R[t, t^{-1}; \sigma]$ and $R[[t, t^{-1}; \sigma]]$ are left fusible rings.*

Note that Corollary 2.11(1) does not hold if σ is only an endomorphism (see [8, Exampe 2.10]).

Remark 2.12. A ring is called *clean* if each of its elements is the sum of an idempotent and a unit. The statement in [15, Introduction] that clean rings are fusible rings is incorrect. For example, any local ring with nonzero nil Jacobson radical is a clean ring that is not fusible.

3. To be uniquely left fusible

To state our next result, we need the notion of a UR-ring, due to Henriksen [11]. Following [11], a non-unital ring I is called a *UR-ring* if I has exactly one element e such that $\mathbf{r}(e) \cap \mathbf{l}(e) = 0$. UR-rings are characterized in [11].

Lemma 3.1. [11, Theorem 2.4] *Let I be a non-unital ring with $e \in I$. The following are equivalent:*

(1) *I is a UR-ring with $\mathbf{r}(e) \cap \mathbf{l}(e) = 0$.*
(2) *e is the unique element of I such that $\mathbf{l}(e) = 0$.*
(3) *e is the unique element of I such that $\mathbf{r}(e) = 0$.*
(4) *e is the unique element of I such that $\mathbf{r}(e) \cup \mathbf{l}(e) = 0$.*
(5) *I is a UR-ring that has an idenity element.*

It was proved in [8, Theorem 2.25] and [9, Theorem 2.2] that a ring R is uniquely left fusible if and only if it is a domain or a UR-ring. This is a consequence of the next result.

Theorem 3.2. *The following are equivalent for a ring R:*

(1) *For each $a \in R \setminus (J(R) + \mathrm{Sing}_r(R))$, there exists a regular element r such that ra is uniquely left fusible.*
(2) *R is a domain or a UR-ring.*

Proof. (1) $\Leftarrow$ (2). The implication is obvious.

(1) $\Rightarrow$ (2). First we assume $J(R) \neq 0$ and we show that R is a domain.

Claim 1: $\mathbf{r}(a) = 0$ for any $0 \neq a \in J(R)$. Indeed, if $\mathbf{r}(a) \neq 0$ with $a \in J(R)$, then $x := a + 1 \in \mathrm{U}(R)$, so $x \notin J(R) + \mathrm{Sing}_r(R)$. By (1), there exists a regular element r such that rx is uniquely left fusible. But $rx = ra + r = 0 + r(a+1)$ are two left fusible representations, and so $ra = 0$. That is, $a = 0$.

Claim 2: $J(R)$ is an essential right ideal of R. Indeed, if I is a right ideal of R such that $J(R) \cap I = 0$, then $IJ(R) = 0$. Thus, $(J(R)I)J(R) = 0$. Since $J(R) \neq 0$ and $J(R)I \subseteq J(R)$, it follows from Claim 1 that $J(R)I = 0$, and hence $I = 0$.

To show that R is a domain, assume $\mathbf{r}(y) \neq 0$ where $y \in R$. Then $J(R) \cap \mathbf{r}(y) \neq 0$ by Claim 2. Take $0 \neq a \in J(R)$ such that $ya = 0$, so $(ay)a = 0$. Since $ay \in J(R)$ and $a \neq 0$, it follows from Claim 1 that $ay = 0$. So $y = 0$ again by Claim 1. Hence R is a domain.

Next we assume that R is not a domain, so $J(R) = 0$ as seen above. There exists $0 \neq a \in R$ such that $\mathbf{r}(a) \neq 0$. It remains to show that R is a UR-ring.

If $a \in \mathrm{Sing}_r(R)$, then $0 \neq a+1 \notin \mathrm{zd}_l(R)$ (since $\mathbf{r}(a)$ is essential in R_R), and $a+1 \notin J(R)+\mathrm{Sing}_r(R)$. Then, for any regular element r, $r(a+1) = 0+r(a+1) = ra+r$ are two distinct left fusible representations of $r(a+1)$, a contradiction to (1). Hence $a \notin \mathrm{Sing}_r(R) = J(R) + \mathrm{Sing}_r(R)$. By (1), there exists a regular element r_0 such that $r_0(-a)$ is uniquely left fusible.

Now, for any $x \in \mathrm{zd}_l^*(R)$, since $x \notin \mathrm{Sing}_r(R) = J(R) + \mathrm{Sing}_r(R)$, there exists a regular element r such that rx is uniquely left fusible. If $\mathbf{r}(a+x) = 0$, then $rx = 0 + rx = r(-a) + r(a+x)$ are two distinct left fusible representations of rx, a contradiction. So $\mathbf{r}(a+x) \neq 0$ and, particularly, $\mathbf{r}(a+1) \neq 0$. Consequently, $r_0(-a) = -r_0(a+x) + r_0x = -r_0(a+1) + r_0$ are two left fusible representations of $r_0(-a)$. Since $r_0(-a)$ is uniquely left fusible, it must be that $x = 1$. Hence R is a UR-ring. □

Corollary 3.3. *The following are equivalent for a ring R:*

1. *Every element in $R \setminus \mathrm{Sing}_r(R)$ is uniquely left fusible.*
2. *R is uniquely left fusible.*
3. *Every element in $R \setminus J(R)$ is uniquely left fusible.*
4. *Every element in $R \setminus (J(R) + \mathrm{Sing}_r(R))$ is uniquely left fusible.*
5. *R is a domain or a UR-ring.*
6. *Each of the above conditions is equivalent to its right version.*

Proof. Either of (1), (2), (3) and (4) implies condition (1) of Theorem 3.2, and (5) implies all of (1), (2), (3) and (4). So the equivalences (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from Theorem 3.2. Because (5) is left-right symmetric, (6) holds true. □

The notion of left fusible rings was somehow motivated by that of right complemented rings. In reverse, the notion of uniquely left fusible rings leads to condition (1) of the next proposition.

Proposition 3.4. *The following are equivalent for a ring R:*

1. *For each $0 \neq a \in R$, there exists a unique $b \in \mathbf{r}(a)$ such that $a + b \in \mathrm{reg}(R)$.*

(2) *R is a domain or a Boolean ring.*

Proof. (2) $\Rightarrow$ (1). The implication is clear.

(1) $\Rightarrow$ (2). First note that R is reduced. To see this, assume $a^2 = 0$ but $a \neq 0$. Then $a + b \in \mathrm{reg}(R)$ for some $b \in \mathbf{r}(a)$. So $a(a + b) = 0$, and hence $a = 0$, a contradiction. Thus, R is reduced, and hence, for $x, y \in R$, $xy = 0$ if and only if $yx = 0$. We can assume that R is not a domain, and show that R is Boolean.

Let $r \in \mathrm{zd}_l^*(R)$. For any $0 \neq a \in R$ with $\mathbf{r}(a) \neq 0$, there exists $0 \neq b \in \mathbf{r}(a)$ such that $a + b \in \mathrm{reg}(R)$. If $x \in \mathbf{r}(ra + b)$, then $0 = b(ra + b)x = b^2x$ (as $bra = 0$), so $bxb = 0$ and hence $bx = 0$ (as $(bx)^2 = 0$). It follows that $rax = 0$, so $ax = 0$. Hence $(a + b)x = 0$, showing that $x = 0$. So $ra + b \in \mathrm{reg}(R)$. Since $b \neq 0$ and $a, ra \in \mathbf{r}(b)$ and $b + a, b + ra \in \mathrm{reg}(R)$, we must have $ra = a$. We have verified that

$$ra = a \text{ for all } 0 \neq a \in R \text{ with } \mathbf{r}(a) \neq 0. \tag{3.1}$$

We claim $r = 1$. If $r \neq 1$, then $1-r \neq 0$ and $\mathbf{r}(1-r) \neq 0$. So, by (3.1), $r(1-r) = 1-r$, i.e., $(1-r)^2 = 0$. Since R is reduced, $r = 1$, a contradiction. Therefore, $\mathrm{zd}_l^*(R) = \{1\}$, i.e. R is a UR-ring. Consequently, for any $0 \neq a \in R$, there exists $b \in \mathbf{r}(a)$ such that $a + b = 1$. It follows that $a^2 = a$. So R is Boolean. □

4. Generalized unit-fusible rings

By Proposition 2.2, elements in the Jacobson radical of a ring are not left unit fusible. This leads to the next notion.

Definition 4.1. A ring R is called *left generalized unit-fusible* if every element in $R \setminus J(R)$ is left unit-fusible.

As seen in this section, left generalized unit-fusible rings are rich in supply. A ring is called a *fine ring* if each of its nonzero elements is the sum of a nilpotent and a unit (see [2]), and a ring R is called a *generalized fine ring* if every element in $R \setminus J(R)$ is the sum of a nilpotent and a unit (see [21]). For a ring R, let $\mathrm{ucn}(R)$ be the set of uniquely clean elements in R, i.e. $a \in \mathrm{ucn}(R)$ if and only if there exists a unique $e \in \mathrm{idem}(R)$ such that $a - e \in \mathrm{U}(R)$.

Proposition 4.2. *The following hold for a ring R:*

1. *Fine rings are unit-fusible, and generalized fine rings are generalized unit-fusible.*
2. *Local rings are generalized unit-fusible.*
3. *R is left unit-fusible if and only if R is semiprimitive and left generalized unit-fusible.*
4. *Any clean ring R with $\operatorname{ucn}(R) \cap (1 + \mathrm{U}(R)) \subseteq J(R)$ is generalized unit-fusible. Especially, any clean ring R with $\operatorname{ucn}(R) \subseteq \operatorname{idem}(R) + J(R)$ is generalized unit-fusible.*

Proof. It is easily seen that $(1), (2)$ and (3) hold.

(4) Let $a \in R \setminus J(R)$, and write $a = e + u$ where $e^2 = e$ and $u \in \mathrm{U}(R)$. If $e \neq 1$, then a is unit-fusible. If $e = 1$, then $a = 1 + u \notin \operatorname{ucn}(R)$. So a has at least two clean representations, one of which is a unit-fusible representation.

The second statement follows, because if $\operatorname{ucn}(R) \subseteq \operatorname{idem}(R) + J(R)$, then we have $\operatorname{ucn}(R) \cap (1 + \mathrm{U}(R)) \subseteq (1 + \mathrm{U}(R)) \cap (\operatorname{idem}(R) + J(R)) \subseteq J(R)$. □

A module M is called *continuous* if every submodule is essential in a summand and every submodule isomorphic to a summand is itself a summand. Dually, a module M is called *discrete* if (i) for every submodule A there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2$ is small in M, and (ii) any submodule B such that M/B is isomorphic to a summand of M is itself a summand of M. A module M is said to be a *Harada module* if there is a decomposition $M = \bigoplus_{i \in I} M_i$ that complements summands of M such that $\operatorname{End}(M_i)$ is a local ring for each i. A ring is called *strongly nil-clean* if every element is the sum of an idempotent and a nilpotent that commute with each other (see [6]). A ring R is *unit-regular* if $a \in aU(R)a$ for each $a \in R$.

Corollary 4.3. *The following rings R are left generalized unit-fusible:*

1. *R is a (von Neumann) regular ring for which every corner ring is clean.*
2. *$R = \operatorname{End}(M_S)$, where M_S is a continuous module, or a discrete module or a Harada module.*

(3) *$R/J(R)$ is a unit-regular ring and idempotents lift modulo $J(R)$.*

(4) *R is a semiperfect ring.*

(5) *R is a strongly nil-clean ring.*

Proof. (1)-(3). All these rings R are clean rings by [4, Theorems 8 and 9] and [3, Theorem 3.9], and $\mathrm{ucn}(R) = (\mathrm{idem}(R) \cap \mathrm{Z}(R)) + J(R)$ by [12, Theorems 4.4, 4.5, 4.8]. So, all these rings are left generalized unit-fusible by Proposition 4.2(4).

(4) and (5). Every semiperfect ring satisfies (3), and so does every strongly nil-clean ring (see [16, Theorem 2.7]). So (4) and (5) hold by (3). □

Corollary 4.4. [8] *Unit-regular rings are unit-fusible.*

Example 4.5. *A semilocal ring need not be left generalized unit-fusible.*

Proof. The ring $R = \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ is a semilocal domain but not a local ring, so it is not left generalized unit-fusible. □

Proposition 4.6. *Let σ be an endomorphism of a ring R and $n \geq 1$.*

(1) *If $R = \prod_{i \in I} R_i$ is a direct product of rings, then R is left generalized unit-fusible if and only if so is R_i for every i.*

(2) *$R[[t;\sigma]]$ is left generalized unit-fusible if and only if so is R.*

(3) *$R[t;\sigma]/(t^{n+1})$ is left generalized unit-fusible if and only if so is R.*

Proof. (1) ($\Rightarrow$). Here $J(R) = \prod_{i \in I} J(R_i)$. If $a \in R_k \setminus J(R_k)$ where $k \in I$, let $\alpha = (a_i) \in R$ with $a_k = a$ and $a_j = 0$ if $j \neq k$. Then $\alpha \in R \setminus J(R)$, so there is a left unit-fusible representation $\alpha = (b_i) + (u_i)$ in R. It follows that $a = b_k + u_k$ is a left unit-fusible representation in R_k.

($\Leftarrow$). If $\alpha = (a_i) \in R \setminus J(R)$, then there exists $k \in I$ such that $a_k \in R_k \setminus J(R_k)$. So there is a left unit-fusible representation $a_k = b + u$ in R_k. Let $\beta = (b_i), \gamma = (c_i) \in R$ be such that $b_k = b, c_k = u$ and $b_j = a_j - 1_{R_j}$, $c_j = 1_{R_j}$ for all $j \neq k$. Then $\alpha = \beta + \gamma$ is a left unit-fusible representation in R.

(2) ($\Rightarrow$). Let $T = R[[t;\sigma]]$. Then $J(T) = J(R) + Tt$. If $a \in R \setminus J(R)$, then $a \in T \setminus J(T)$, so a has a left unit-fusible representation $a = \sum_{i\geq 0} r_i t^i + \sum_{i\geq 0} s_i t^i$. It follows that $a = r_0 + s_0$ is a left unit-fusible representation.

($\Leftarrow$). Let $\alpha := \sum_{i\geq 0} a_i x^i \in T \setminus J(T)$. Then $a_0 \in R \setminus J(R)$, so a_0 has a left unit-fusible representation $a_0 = r + s$ in R. It follows that $\alpha = r + (s + \sum_{i\geq 1} a_i x^i)$ is a left unit-fusible representation.

(3) ($\Rightarrow$). Let $T = R[t;\sigma]/(t^{n+1})$. Then $J(T) = J(R) + Tt$. If $a \in R \setminus J(R)$, then a is contained in $T \setminus J(T)$. So a has a left unit-fusible representation $a = (a_0 + a_1 t + \ldots + a_n t^n) + (u_0 + u_1 t + \ldots + u_n t^n)$. Then $\mathbf{r}_R(a_0) \neq 0$ and $u_0 \in \mathrm{U}(R)$, so $a = a_0 + u_0$ is a left unit-fusible representation.

($\Leftarrow$). Let $\alpha := a_0 + a_1 t + \ldots + a_n t^n \in T \setminus J(T)$. Then $a_0 \in R \setminus J(R)$, so a_0 has a left unit-fusible representation $a_0 = a + u$ in R. Then $\alpha = a + (u + a_1 t + \ldots + a_n t^n)$ is a left unit-fusible representation. $\square$

Example 4.7. *The factor ring of a left unit-fusible ring need not be left generalized unit-fusible; the center of a left unit-fusible ring need not be left generalized unit-fusible.*

Proof. Let R be a ring and S be a (unital) subring of R with $1_S = 1_R$, and $[R;S]$ be the ring of of all countably infinite matrices of the form

$$\begin{pmatrix} A & & & 0 \\ & r & & \\ & & r & \\ 0 & & & \ddots \end{pmatrix}$$

where $A \in \mathbb{M}_n(R)$ for some $n \geq 1$ and $r \in S$. This matrix is denoted by (A, r, n) or (A, r). Then (A, r) is in the center of $[R;S]$ iff $(A, r) = rI$ with $r \in \mathrm{Z}(R)$. So $\mathrm{Z}([R;S]) = \{rI : r \in \mathrm{Z}(R)\}$. Moreover, the map $[R;S] \to S$, $(A, r) \mapsto r$ is a ring epimorphism, so S is a factor ring of $[R;S]$.

We next show that, for a left unit-fusible ring R, $[R;S]$ is left unit-fusible. Indeed, let $0 \neq (A, r, n) \in [R;S]$. If $A \neq 0$, then by Corollary 2.5 A has a left unit-fusible representation $A = B+U$ in $\mathbb{M}_n(R)$, so $(A, r, n) = (B, r-1, n)+(U, 1, n)$ is a left unit-fusible representation in $[R; S]$. If $A = 0$ then $r \neq 0$. Then $(A, r, n) = (A', r, n + 1)$ with $0 \neq A' \in \mathbb{M}_{n+1}(R)$. As argued earlier, $(A', r, n + 1)$ is left unit-fusible in $[R; S]$.

Now let $T = [\mathbb{Q}; \mathbb{Z}]$. Since $\mathbb{Q}$ is unit-fusible, T is unit-fusible. Clearly, $\mathbb{Z}$ is not generalized unit-fusible. However, $\mathbb{Z}$ is both a factor ring and the center of T. □

For an (A, B)-bimodule M, $a \in A$ and $b \in B$, let $\mathbf{r}_M(a) = \{x \in M : ax = 0\}$ and $\mathbf{l}_M(b) = \{x \in M : xb = 0\}$.

Proposition 4.8. *Let M be a bimodule over a ring R. Then the trivial extension $T := R \propto M$ is left generalized unit-fusible if and only if, for any $a \in R \setminus J(R)$, $a = a_1 + a_2$ where $\mathbf{r}_R(a_1) \neq 0$ or $\mathbf{r}_M(a_1) \neq 0$, and $a_2 \in \mathrm{U}(R)$.*

Proof. ($\Rightarrow$). Let $a \in R \setminus J(R)$. Then $(a, 0)$ is contained in $T \setminus J(T)$, so it has a left unit-fusible representation $(a, 0) = (a_1, m) + (a_2, -m)$. It follows that $a = a_1 + a_2$ with $a_2 \in \mathrm{U}(R)$. Moreover, there exists a nonzero element (a_0, m_0) in T such that $(a_1, m)(a_0, m_0) = 0$. So $a_1a_0 = 0$ and $a_1m_0 + ma_0 = 0$. If $\mathbf{r}_R(a_1) = 0$, then $a_0 = 0$. Hence $m_0 \neq 0$ and $a_1m_0 = 0$. So $\mathbf{r}_M(a_1) \neq 0$.

($\Leftarrow$). Let $(a, m) \in T \setminus J(T)$. Then $a \notin J(R)$, so $a = a_1 + a_2$ where $\mathbf{r}_R(a_1) \neq 0$ or $\mathbf{r}_M(a_1) \neq 0$ and $a_2 \in \mathrm{U}(R)$. If $a_1a'_1 = 0$ where $0 \neq a'_1 \in R$, then $(a_1, 0)(a'_1, 0) = 0$. If $a_1m' = 0$ where $0 \neq m' \in M$, then $(a_1, 0)(0, m') = 0$. So $(a_1, 0)$ is a left zero-divisor in T. Hence $(a, m) = (a_1, 0)+(a_2, m)$ is a left generalized unit-fusible representation. □

Corollary 4.9. *Let M be a bimodule over a ring R. If R is left generalized unit-fusible, then so is the trivial extension $R \propto M$.*

The converse of Corollary 4.9 does not hold.

Example 4.10. *Consider $R = \mathbb{Z}$ and $T = R \propto M$ where $M = \bigoplus_{n\geq 2} \mathbb{Z}/n\mathbb{Z}$. Then R is not left generalized unit-fusible. We next show that every (a, x) in $T \setminus J(T)$ is left unit-fusible. Since $a \neq 0$ and $(-1, 0)(a, x) = (-a, -x)$, we can assume that $a > 0$ by Lemma 2.1. Then $m := a + 1 \geq 2$. Let $y = 1 + m\mathbb{Z} \in \mathbb{Z}/m\mathbb{Z} \subseteq M$. Then*

$(a+1,0)(0,y)=0$, *so* $(a,x)=(a+1,0)+(-1,x)$ *is a left unit-fusible representation. Hence* T *is left generalized unit-fusible.*

Proposition 4.11. *Let* $T=\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a Morita context. If* A,B *are left generalized unit-fusible, then so is* T.

Proof. By [19], $J(T)=\begin{pmatrix} J(A) & M_0 \\ N_0 & J(B) \end{pmatrix}$, where $M_0=\{x\in M : xN\subseteq J(A)\}$ and $N_0=\{y\in N : My\subseteq J(A)\}$. Let $\alpha:=\begin{pmatrix} a & x \\ y & b \end{pmatrix}\in T\setminus J(T)$. Then either $a\notin J(A)$, or $b\notin J(B)$, or $xN\nsubseteq J(A)$, or $My\nsubseteq J(A)$.

Case 1: $a\notin J(A)$. Since A is left generalized unit-fusible, $a=a_1+a_2$ where $\mathbf{r}_A(a_1)\neq 0$ and $a_2\in \mathrm{U}(A)$. Let $0\neq a_1'\in \mathbf{r}_A(a_1)$. Then $\begin{pmatrix} a_1 & x \\ 0 & b-1 \end{pmatrix}\begin{pmatrix} a_1' & 0 \\ 0 & 0 \end{pmatrix}=0$, so $\begin{pmatrix} a & x \\ y & b \end{pmatrix}=\begin{pmatrix} a_1 & x \\ 0 & b-1 \end{pmatrix}+\begin{pmatrix} a_2 & 0 \\ y & 1 \end{pmatrix}$ is a left unit-fusible representation.

Case 2: $b\notin J(B)$. Since B is left generalized unit-fusible, $b=b_1+b_2$ where $\mathbf{r}_B(b_1)\neq 0$ and $b_2\in \mathrm{U}(B)$. Let $0\neq b_1'\in \mathbf{r}_B(b_1)$. Then $\begin{pmatrix} a-1 & 0 \\ y & b_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b_1' \end{pmatrix}=0$, so $\begin{pmatrix} a & x \\ y & b \end{pmatrix}=\begin{pmatrix} a-1 & 0 \\ y & b_1 \end{pmatrix}+\begin{pmatrix} 1 & x \\ 0 & b_2 \end{pmatrix}$ is a left unit-fusible representation.

Case 3: $a\in J(A)$ and $xN\nsubseteq J(A)$. Then there exists $y'\in N$ such that $xy'\in A\setminus J(A)$, so $a+xy'\notin J(A)$. Thus $\begin{pmatrix} a & x \\ y & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y' & 1 \end{pmatrix}=\begin{pmatrix} a+xy' & x \\ y+by' & b \end{pmatrix}$, which is left unit-fusible by Case 1. Since $\begin{pmatrix} 1 & 0 \\ y' & 1 \end{pmatrix}$ is a unit in T, $\begin{pmatrix} a & x \\ y & b \end{pmatrix}$ is left unit-fusible by Lemma 2.1.

Case 4: $a\in J(A)$ and $My\nsubseteq J(A)$. Then there exists $x'\in M$ such that $x'y\in A\setminus J(A)$, so $a+x'y\notin J(A)$. Thus $\begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ y & b \end{pmatrix}=\begin{pmatrix} a+x'y & x+x'b \\ y & b \end{pmatrix}$, which is left unit-fusible by Case 1. Since $\begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix}$ is a unit in T, $\begin{pmatrix} a & x \\ y & b \end{pmatrix}$ is left unit-fusible by Lemma 2.1. □

Corollary 4.12. *Let* R,A,B *are rings, and* M *an* (A,B)*-bimodule and* $n\geq 1$.

(1) *If* A *and* B *are left generalized unit-fusible, then so is* $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.

(2) *If* R *is a left generalized unit-fusible ring, then so is* $\mathbb{M}_n(R)$.

(3) $\mathbb{T}_n(R)$ *is left generalized unit-fusible if and only if so is* R.

Proof. The claims follow from Proposition 4.11. □

A ring is called *semipotent* if every left (or right) ideal not contained in the Jacobson radical contains a nonzero idempotent. A ring R is called *strongly semipotent* if for any $a \in R \smallsetminus J(R)$, there exists $0 \neq e^2 = e \in R$ such that $e = ab = ba$ for some $b \in R$ (see [10]).

Proposition 4.13. *Every strongly semipotent ring* R *is generalized unit-fusible.*

Proof. Let $a \in R \smallsetminus J(R)$. Then there exists $0 \neq e = e^2 \in R$ such that $e = ab = ba$ where $b \in R$. So, $ea = aba = ae$ and $eb = bab = be$, and hence $e = eae \cdot ebe = ebe \cdot eae$. Thus $ea = eae$ is a unit of eRe. Hence $a = (a-1)(1-e) + [ae + (1-e)]$ is a unit-fusible representation. So a is unit-fusible. □

Corollary 4.14. [8, Theorem 3.9] *Every semiprimitive commutative clean ring is* (*unit-*) *fusible.*

5. Asymmetry

The notions of fusible, regular fusible, and unit-fusible elements are not left-right symmetric by the examples below.

Example 5.1. [8, Example 2.10] *A right fusible element need not be left fusible.*

Proof. Let R be a domain and σ be a ring endomorphism of R which is not injective. For $T := R[t;\sigma]$, as proved in [8], $\mathrm{zd}_l(T) = Tt$, $\mathrm{zd}_l^*(T) = T \smallsetminus Tt$, and t is a right, but not left, fusible element in T. □

Example 5.2. *A regular right fusible element need not be regular left fusible; a left unit-fusible element need not be right unit-fusible.*

Proof. Let $T := R[t;\sigma]$ be given as in the previous example. Since t is right fusible in T, it is regular right fusible in T. But t is not regular left fusible in T. Indeed, if for some $f \in \mathrm{reg}(T)$, ft is left fusible, say $ft = \alpha + \beta$ is a left fusible representation in T. Then $\alpha \in \mathrm{zd}_l(T)$, so $\alpha = gt$ for some $g \in T$. Hence $\beta = (f - g)t \in \mathrm{zd}_l(T)$, contradicting $\beta \in \mathrm{zd}_l^*(T)$. So t is not regular left fusible in T. □

Example 5.3. *A left unit-fusible element need not be right unit-fusible.*

Proof. Let $T := R[t; \sigma]$ be given as in Example 5.1. Since σ is not injective, $\sigma(a) = 0$ for some $0 \neq a \in R$. Then $ta = 0$, so $t+1$ is left unit-fusible in T. We next show that $t+1$ is not right unit-fusible. Assume on the contrary that $1+t = f(t)+g(t)$ is a right unit-fusible representation in T. Then there exists $0 \neq \alpha \in T$ such that $\alpha f(t) = 0$. So $\alpha \in \mathrm{zd}_l^*(T) = Tt$. Write $\alpha = h(t)t^m$ where $m \geq 1$ and $h(t) = a_0 + a_1 t + \ldots + a_k t^k$ with $a_0 \neq 0$. Then $h(t) \in \mathrm{zd}_l^*(T)$ and $h(t)t^m f(t) = 0$. It follows that $t^m f(t) = 0$. Thus, $t^m(1+t) = t^m g(t)$. Let $g(t)^{-1} = b_0 + b_1 t + \ldots + b_n t^n$, and we have

$$
\begin{aligned}
t^m &= t^m(1+t)(b_0 + b_1 t + \ldots + b_n t^n) \\
&= t^m[b_0 + (b_1 + \sigma(b_0))t + \ldots + (b_n + \sigma(b_{n-1}))t^n + \sigma(b_n)t^{n+1}] \\
&= \sigma^m(b_0)t^m + \sigma^m(b_1 + \sigma(b_0))t^{m+1} + \ldots + \sigma^m(b_n + \sigma(b_{n-1}))t^{m+n} + \sigma^{m+1}(b_n)t^{m+n+1}
\end{aligned}
$$

It follows that

$$\sigma^{m+1}(b_n) = 0,\ \sigma^m(b_n) + \sigma^{m+1}(b_{n-1}) = 0,\ \cdots,\ \sigma^m(b_1) + \sigma^{m+1}(b_0) = 0,\ \sigma^m(b_0) = 1. \tag{5.1}$$

Applying σ to the second equality of (5.1) gives $\sigma^{m+2}(b_{n-1}) = 0$, and then applying σ^2 to the third equality of (5.1) gives $\sigma^{m+3}(b_{n-2}) = 0$, ..., and applying σ^n to the second last equality of (5.1) gives $\sigma^{m+n+1}(b_0) = 0$, which contradicts the last equality of (5.1). □

We next present an example of a left generalized unit-fusible ring that is not right generalized unit-fusible. An improvement of Corollary 4.12(1) is needed.

Theorem 5.4. *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. *Then*

(1) *R is left generalized unit-fusible if and only if B is left generalized unit-fusible and, for each $0 \neq a \in A$, $a = x + y$ where $y \in \mathrm{U}(A)$ and $\mathbf{r}_A(x) \neq 0$ or $\mathbf{r}_M(x) \neq 0$.*

(2) *R is right generalized unit-fusible if and only if A is right generalized unit-fusible and, for each $0 \neq b \in B$, $b = x + y$ where $y \in \mathrm{U}(B)$ and $\mathbf{l}_B(x) \neq 0$ or $\mathbf{l}_M(x) \neq 0$.*

Proof. (1)($\Leftarrow$). Let $\alpha = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in R \setminus J(R)$. Then either $a \notin J(A)$ or $b \notin J(B)$.

Suppose $a \notin J(A)$. Then $a = a_1 + a_2$ where $\mathbf{r}_A(a_1) \neq 0$ or $\mathbf{r}_M(a_1) \neq 0$ and $a_2 \in \mathrm{U}(A)$. If $a_1 a_1' = 0$ where $0 \neq a_1' \in A$, then $\begin{pmatrix} a_1 & 0 \\ 0 & b-1 \end{pmatrix}\begin{pmatrix} a_1' & 0 \\ 0 & 0 \end{pmatrix} = 0$. If $a_1 m' = 0$ where $0 \neq m' \in M$, then $\begin{pmatrix} a_1 & 0 \\ 0 & b-1 \end{pmatrix}\begin{pmatrix} 0 & m' \\ 0 & 0 \end{pmatrix} = 0$. So $\begin{pmatrix} a_1 & 0 \\ 0 & b-1 \end{pmatrix} \in \mathrm{zd}_l(R)$ and hence $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & b-1 \end{pmatrix} + \begin{pmatrix} a_2 & m \\ 0 & 1 \end{pmatrix}$ is a left generalized unit-fusible representation.

Suppose $b \notin J(B)$. Then $b = b_1 + b_2$ where $b_1 b_1' = 0$ for some $0 \neq b_1' \in B$ and $b_2 \in \mathrm{U}(B)$. Then $\begin{pmatrix} a-1 & 0 \\ 0 & b_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b_1' \end{pmatrix} = 0$. So $\begin{pmatrix} a-1 & 0 \\ 0 & b_1 \end{pmatrix} \in \mathrm{zd}_l(R)$ and hence $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} a-1 & 0 \\ 0 & b_1 \end{pmatrix} + \begin{pmatrix} 1 & m \\ 0 & b_2 \end{pmatrix}$ is a left unit-fusible representation.

(1)($\Rightarrow$). Let $a \in A \setminus J(A)$. Then $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is contained in $R \setminus J(R)$, so it has a left unit-fusible representation $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & m \\ 0 & y \end{pmatrix} + \begin{pmatrix} a-x & -m \\ 0 & -y \end{pmatrix}$. It follows that $a = x + (a-x)$ with $a - x \in \mathrm{U}(A)$. There exists a nonzero element $\begin{pmatrix} x_0 & m_0 \\ 0 & y_0 \end{pmatrix}$ in R such that $\begin{pmatrix} x & m \\ 0 & y \end{pmatrix}\begin{pmatrix} x_0 & m_0 \\ 0 & y_0 \end{pmatrix} = 0$. Since $y \in \mathrm{U}(B)$, $y_0 = 0$ and it follows that $xx_0 = 0$ and $xm_0 = 0$ with either $x_0 \neq 0$ or $m_0 \neq 0$.

Let $b \in B \setminus J(B)$. Then $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ is contained in $R \setminus J(R)$, so it has a left unit-fusible representation $\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} x & m \\ 0 & y \end{pmatrix} + \begin{pmatrix} -x & -m \\ 0 & b-y \end{pmatrix}$. It follows that $b = y + (b-y)$ with $b - y \in \mathrm{U}(B)$. There exists a nonzero element $\begin{pmatrix} x_0 & m_0 \\ 0 & y_0 \end{pmatrix}$ in R such that $\begin{pmatrix} x & m \\ 0 & y \end{pmatrix}\begin{pmatrix} x_0 & m_0 \\ 0 & y_0 \end{pmatrix} = 0$. Since $x \in \mathrm{U}(A)$, $x_0 = 0$ and it follows that $xm_0 + my_0 = 0$ and $yy_0 = 0$. So $m_0 = -x^{-1}my_0$, and we deduce that $y_0 \neq 0$. So $b = y + (b-y)$ is a left unit-fusible representation.

The proof of (2) is parallel to that of (1). □

Example 5.5. *A left generalized unit-fusible ring need not be right generalized unit-fusible.*

Proof. Let $A = \mathbb{Z}$, $B = \mathbb{Z} \propto (\bigoplus_{n\geq 2} \mathbb{Z}/n\mathbb{Z})$, and $M = 0 \propto (\bigoplus_{n\geq 2} \mathbb{Z}/n\mathbb{Z})$. Then M is an (A, B)-bimodule. Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Since A is not right generalized unit-fusible, R is not right generalized unit-fusible by Theorem 5.4(2).

By Example 4.10, B is left generalized unit-fusible. For any $a \in A \setminus J(A)$, $a \neq 0$. We show that $a = x+y$, where $y \in \mathrm{U}(A)$ and $\mathbf{r}_A(x) \neq 0$ or $\mathbf{r}_M(x) \neq 0$. We can assume that a is not a unit. Then either $a \geq 2$ or $a \leq -2$. If $a \geq 2$, then $a = (a+1) + (-1)$ where $\mathbf{r}_M(a+1) \neq 0$. If $a \leq -2$, then $a - 1 \leq -3$, so $a = (a-1) + 1$ where $\mathbf{r}_M(a-1) \neq 0$. Hence, by Theorem 5.4, R is left generalized unit-fusible. □

In conclusion, we raise the following question.

Question 5.6. *Are left fusible rings right fusible? Are regular left fusible rings regular right fusible? Are left unit-fusible rings right unit-fusible?*

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