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RESEARCH

Algebraic Frames and Minimal Prime Elements

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Abstract
The minimal prime elements of algebraic frames with the finite intersection property (FIP) can be characterized as those prime elements which are the joins of the pseudo-complements of all compact elements not below the said prime. Our goal is to study the class of algebraic frames with this latter property, generalizing the FIP case.

Keywords Algebraic frame · Prime elements · Ultrafilters

1 Algebraic Frames

In the study of algebraic frames with the finite intersection property on compact elements, the Lemma on Ultrafilters is a powerful result. Our aim was to study when this theorem holds weight on its own. We share what we have discovered.

Definition 1.1 A **frame** is a complete lattice $(L, \vee, \wedge, 0, 1)$ which satisfies the frame law: for every subset $S \subseteq L$

$$x \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} (x \wedge s).$$

Obviously, a frame is a distributive lattice. We point out that frames are also known as any of the following: locales, complete Heyting algebras, and Brouwerian lattices.

Examples of frames include the frame of open subsets of a topological space, the frame of radical ideals of a commutative ring with identity, and the frame of convex ℓ-subgroups of a lattice-ordered group.

For each $x \in L$, we let $x^{\perp}$ denote the **polar** of x, which is the largest element of L that is disjoint from x. The polar is also known as the pseudocomplement.

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For those studying algebraic structures, there is a special subclass of the class of frames that plays a pivotal role in the structure theory: the finitely generated objects. These are the compact elements in the theory of frames.

Definition 1.2 Let L be a frame and $c \in L$. If c has the property that whenever $c \leq \bigvee_i x_i$, then $c \leq x_{i_1} \vee \ldots \vee x_{i_k}$ for some finite subcollection, then c is called a **compact** element of L.

The set of all compact elements of a frame L will be denoted by $\mathfrak{K}(L)$; this set is a join semilattice of L with 0. When $1 \in \mathfrak{K}(L)$, we say that L is a **compact frame**.

Definition 1.3 A frame L is said to be **algebraic** if every element of L is a join of elements from $\mathfrak{K}(L)$.

In many nice examples, the set $\mathfrak{K}(L)$ is a sub-lattice of L. Such frames are said to satisfy (or have) **the finite intersection property**, or **FIP** for short. (These are also known as multiplicative or arithmetic lattices; see [5].) The well-known class of coherent frames is precisely the compact algebraic frames with FIP. It is a consequence of Hochster's Theorem and Banaschewski's Theorem that coherent frames are precisely the frame of radical ideals of an appropriate commutative ring with identity (see [6] and [2]).

A consequence of the Axiom of Choice is that every algebraic frame is isomorphic to a frame of open subsets of a topological space. Such frames are known as **spatial**. To determine whether a frame is spatial one determines if there are enough points. Recall that a point of L is a frame homomorphism $\rho : L \to \mathbf{2}$. The set of all points of a frame is denoted by ΣL (we should point out that there are pointless frames).

Each $\rho \in \Sigma L$ produces a filter on L, namely $\mathcal{F}_\rho = \{x \in L : \rho(x) = 1\}$. This is an example of a completely prime filter on L; that is, a proper filter $\mathcal{F}$ on L with the added property that whenever $\bigvee x_i \in \mathcal{F}$, then $x_i \in \mathcal{F}$ for some i. Moreover, each proper completely prime filter $\mathcal{F}$ on L corresponds to the point $\rho_{\mathcal{F}} : L \to \mathbf{2}$ defined by

$$\rho_{\mathcal{F}}(x) = \begin{cases} 1, & \text{if } x \in \mathcal{F} \\ 0, & \text{otherwise.} \end{cases}$$

The map $\rho \mapsto \mathcal{F}_\rho$ is a bijective correspondence between the set of points of L and the collection of proper completely prime filters of L.

Remark 1.4 It might appear that we are digressing, but the concepts we are now describing will be used in what follows.

To each completely prime filter $\mathcal{F}$, the element $p_{\mathcal{F}} = \bigvee\{x \in L : x \notin \mathcal{F}\}$ is a **prime** element of L, that is, an element $p \in L$ ($p < 1$) satisfying $a \wedge b \leq p$ implies either $a \leq p$ or $b \leq p$ (prime elements are also known as meet-irreducible elements). The set of all prime elements of L is denoted by $\mathrm{Spec}(L)$ and is called the prime spectrum of L. Conversely, if $p \in \mathrm{Spec}(L)$, then the set $\mathcal{F}_p = \{x \in L : x \nleq p\}$ is a proper completely prime filter of L. The map $p \mapsto \mathcal{F}_p$ is a one-to-one correspondence between the set of prime elements of L and the collection of proper completely prime filters of L. Observe that for all $y \in L$, $y \in \mathcal{F}_p$ implies $y^\perp \notin \mathcal{F}_p$.

Lemma 1.5 *Suppose L is an algebraic frame. Let $p \in \mathrm{Spec}(L)$ and $x \in L$ with $x^\perp \in \mathcal{F}_p$. Then there exists some $k \in \mathfrak{K}(L)$ such that $k \wedge x = 0$ and $k \in \mathcal{F}_p$.*

Proof Straightforward, using the definitions of polars and completely prime filters in an algebraic frame. □

Interestingly, non-zero compact elements in a frame produce points via Zorn's Lemma: Let $0 \neq c \in \mathfrak{K}(L)$ and consider the set $J = \{x \in L : c \nleq x\}$. By compactness, the join in L of any chain from J belongs to J and hence Zorn's Lemma applies to produce maximal elements in J. These are called **values** of c. Values of a compact element are prime elements. It follows then that in an algebraic frame every element is the meet of prime elements. In other words, every algebraic frame is spatial.

Theorem 1.6 [12, Proposition II.5.3] *The frame L is spatial if and only if every element of L is the meet of prime elements.*

Typically, investigations on algebraic frames include the assumption that L has FIP. Our interest has been to determine which results can be generalized to the non-FIP case. Parts of this work are from the second author's dissertation [8].

We assume familiarity with the theory of frames and lattice-ordered groups. Our main references for frames are [12] and [7].

2 Minimal Primes

Throughout this section, we assume that L is an algebraic frame. We are interested in studying the minimal prime elements of L; we denote this set by $\mathrm{Min}(L)$. In an algebraic frame, minimal primes exist. In algebraic frames with FIP, minimal primes can be completely characterized in the following way. We let $\mathcal{F}_p^* = \mathcal{F}_p \cap \mathfrak{K}(L)$.

Theorem 2.1 (Lemma on Ultrafilters) [9, Corollary 2.5.1] *Suppose L is an algebraic frame with FIP and $p \in \mathrm{Spec}(L)$. Then $p \in \mathrm{Min}(L)$ if and only if*

$$p = \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}.$$

Conversely, given any ultrafilter U of $\mathfrak{K}(L)$, the element $\bigvee\{c^{\perp} : c \in U\}$ is a minimal prime element of L.

We observe that in any algebraic frame L and any $p \in \mathrm{Spec}(L)$, $\bigvee_{c \in \mathcal{F}_p^*} c^{\perp} \leq p$.

Remark 2.2 Our aim in the current read is to determine whether we may remove, or possibly weaken, the restriction of FIP in Theorem 2.1. In Section 4, we provide an example of an algebraic frame (without FIP) with a minimal prime element that does not satisfy the above theorem.

We note that for an algebraic frame L, $\bigwedge \mathrm{Spec}(L) = \bigwedge \mathrm{Min}(L) = 0$

Lemma 2.3 *Let L be an algebraic frame and $p \in \mathrm{Spec}(L)$ such that*

$$p = \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}.$$

Then for each compact $c \leq p$, $c^{\perp} \nleq p$. Conversely, if for each $c \in \mathfrak{K}(L)$, exactly one of $c \leq p$ or $c^{\perp} \leq p$ occurs, then equality in the above display holds.

Proof Suppose $p = \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}$ and let $c \in \mathfrak{K}(L)$. Since $c \wedge c^{\perp} = 0 \leq p$, we know that at least one of either $c \leq p$ or $c^{\perp} \leq p$. It suffices to show that if $c \leq p$, then $c^{\perp} \nleq p$. By hypothesis and compactness, there is a finite collection $c_1, c_2, \ldots, c_n \in \mathcal{F}_p$ such that

$$c \leq c_1^{\perp} \vee \ldots \vee c_n^{\perp} \leq (c_1 \wedge \ldots \wedge c_n)^{\perp}.$$

Since $\mathcal{F}_p$ is a filter on L, $c_1 \wedge \ldots \wedge c_n \in \mathcal{F}_p$. Since the filter is completely prime, there exists some $0 < d \in \mathfrak{K}(L)$ belonging to $\mathcal{F}_p$ and $d \leq c_1 \wedge \ldots \wedge c_n$. Thus, $c \wedge d = 0$, and so $d \leq c^{\perp}$ from which we gather that $c^{\perp} \in \mathcal{F}_p$.

Conversely, suppose that $p \in \mathrm{Spec}(L)$ has the property that for every $c \in \mathfrak{K}(L)$ either $c \nleq p$ or $c^{\perp} \nleq p$. Let $d \in \mathfrak{K}(L)$ satisfy $d \leq p$. By hypothesis, $d^{\perp} \in \mathcal{F}_p$. Using Lemma 1.5, there is some compact $c \in \mathcal{F}_p$ such that $c \leq d^{\perp}$. It follows that $d \leq c^{\perp} \leq \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}$. Since d is an arbitrary compact element below p and L is algebraic, it follows that $p = \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}$. $\square$

Definition 2.4 Let $p \in \mathrm{Spec}(L)$. We say that p satisfies the **compact absoluteness property** (for short, we say p satisfies CAB) if

$$p = \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}.$$

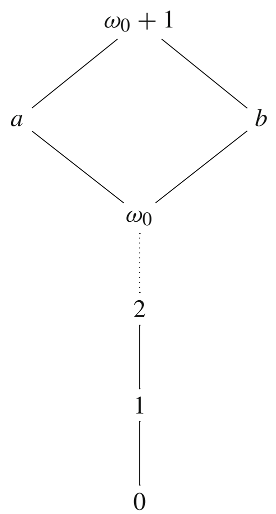
We let $\mathrm{Cab}(L) = \{p \in \mathrm{Spec}(L) : p \text{ satisfies CAB}\}$.

Lemma 2.5 *Suppose L is an algebraic frame. Then* $\mathrm{Cab}(L) \subseteq \mathrm{Min}(L)$.

Proof Let $p \in \mathrm{Cab}(L)$. To show that $p \in \mathrm{Min}(L)$ we need to prove that for any $q \in \mathrm{Spec}(L)$ if $q \leq p$, then $q = p$. To that end take such a q. We show $p \leq q$. Let $c \in \mathfrak{K}(L)$ satisfy $c \leq p$. So $c^{\perp} \nleq p$, since $p \in \mathrm{Cab}(L)$. It follows that $c^{\perp} \nleq q$. Since q is prime, $c \leq q$. Therefore $p \leq q$, since L is algebraic. $\square$

It should be apparent that if the algebraic frame L satisfies FIP, then $\mathrm{Cab}(L) = \mathrm{Min}(L)$. Here is an example of an algebraic frame that demonstrates that the converse is not true.

Example 2.6 Let $L = \mathbb{N} \cup \{\omega_0, a, b, \omega_0 + 1\}$.



Note that both a and b are compact elements but their meet ω_0 is not compact. Observe further that 0 is the unique minimal prime element and it does indeed satisfy CAB.

We would like to characterize the minimal primes belonging to $\mathrm{Cab}(L)$. In order to do this, we have to weaken our notion of compactness.

Definition 2.7 Let L be an algebraic frame. We call an element $s \in L$ **subcompact** if it is a finite meet of compact elements. We let $\mathfrak{s}(L)$ denote the set of subcompact elements of L.

Remark 2.8 Observe that in Example 2.6, $\mathfrak{K}(L) = L \setminus \{\omega_0\}$ and $\mathfrak{s}(L) = L$.

We leave it to the interested reader to show that $\mathfrak{s}(L)$ is a sublattice of L. Also, $\mathfrak{K}(L) \subseteq \mathfrak{s}(L)$. It should be obvious that $\mathfrak{K}(L) = \mathfrak{s}(L)$ if and only if L satisfies the FIP.

For algebraic frames satisfying the FIP, the Lemma on Ultrafilters says there is a correspondence between the ultrafilters of the lattice $\mathfrak{K}(L)$ and the minimal prime elements of L. There is always a one-one correspondence between filters on $\mathfrak{K}(L)$ and filters of L generated by compacts. We use the following definition of a filter on a join semi-lattice with 0.

Definition 2.9 Let S be a join semi-lattice with 0. A (non-empty) subset $F \subseteq S$ is called a **filter** if it is an up-set and for each $s_1, s_2 \in F$ there is some $s \in F$ such that $s \leq s_1$ and $s \leq s_2$. By an **ultrafilter** on S we mean a maximal proper filter on S.

Definition 2.10 By a **compact** filter on L, we mean a filter, say $\mathcal{F}$, that has a base of compact elements. This means that for each $x \in \mathcal{F}$ there is some compact element $c \in \mathcal{F}$ such that $c \leq x$. We define a **subcompact** filter analogously.

Clearly, a compact filter is a subcompact filter. Also, a completely prime filter in an algebraic frame is a compact filter. It should be apparent what we mean by a maximal compact filter, maximal subcompact filter, and maximal completely prime filter.

We use $\mathrm{Ult}_c(\cdot)$, $\mathrm{Ult}_{sc}(\cdot)$, and $\mathrm{Ult}_{cp}(\cdot)$ to denote the collection of these, respectively. Each of these is equipped with the Wallman and the inverse Wallman topologies.

Remark 2.11 Given a join semi-lattice, say K, and a collection of filters on K, say M, we equip M with the following two topologies. The **Wallman topology** on M is the topology generated by taking the collection $\{U_k : k \in K\}$, where $U_k = \{F \in M : k \notin F\}$, as a subbase for the open subsets. The **inverse Wallman topology** on M is the topology generated by taking the collection $\{V_k : k \in K\}$, where $V_k = \{F \in M : k \in F\}$, as a subbase for the open subsets. (We apply these to where M is one of $\mathrm{Ult}_c(L)$, $\mathrm{Ult}_{sc}(L)$, and $\mathrm{Ult}_{cp}(L)$.)

Remark 2.12 Suppose U is a subcompact filter of L. Consider $V_U = \{c \in \mathfrak{s}(L) : c \in U\}$. If $c, d \in V_U$, then $c \wedge d \in U \cap \mathfrak{s}(L)$ and hence belongs to V_U. If $c \in V_U$ and $d \in \mathfrak{s}(L)$ with $c \leq d$, then $d \in U$ and hence in V_U. Therefore, V_U is a filter on $\mathfrak{s}(L)$.

On the other hand, consider a filter $F \subseteq \mathfrak{s}(L)$. Let $\overline{F} = \{x \in L : x \geq s, s \in F\}$. It is easy to show that $\overline{F} = \langle F \rangle$ is the filter of L generated by F.

Furthermore, $V_{\overline{F}} = F$ and $\overline{V_U} = U$.

The following is a well-known result of Wallman. The application follows from the fact that $\mathrm{Ult}_{sc}(L)$ can be identified with the collection of ultrafilters on the lattice $\mathfrak{s}(L)$ (use the preceding remark).

Lemma 2.13 *Let L be an algebraic frame. The Wallman topology on* $\mathrm{Ult}_{sc}(L)$ *is compact.*

We present our main theorem of this section. We start with a well-known result on maximal filters of a lattice:

Lemma 2.14 *Let $U \in \mathrm{Ult}_{sc}(L)$ and $c \in \mathfrak{s}(L)$. $c \notin U$ implies $c^{\perp} \in U$.*

Proof Consider the subcompact filter, generated by U and c, call it $\overline{U}$, then $U \subset \overline{U}$. Thus, $\overline{U} = \{x \in L : x \geq u \wedge c, u \in U\}$. Since U is maximal, there exists some $u \in U$ such that $u \wedge c = 0$. Since $u \leq c^{\perp}$, $c^{\perp} \in U$. □

Theorem 2.15 *Let L be an algebraic frame and $p \in \mathrm{Min}(L)$. The following are equivalent:*

(1) $\mathcal{F}_p \in \mathrm{Ult}_{sc}(L)$.
(2) $p \in \mathrm{Cab}(L)$.
(3) $p = \bigvee_{s \in \mathcal{F}_p \cap \mathfrak{s}L} s^{\perp}$.
(4) $p = \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}$.
(5) $\bigvee_{c \in \mathcal{F}_p^*} c^{\perp} \in \mathrm{Cab}(L)$.

Proof Let p be a fixed minimal prime element of L.

(1) ⇒ (2). Take $c \in \mathfrak{K}L$ with $c \leq p$. Consequently $c \notin \mathcal{F}_p$, and using Lemma 2.14 it follows that $c^{\perp} \in \mathcal{F}_p$. Hence $c^{\perp} \nleq p$ which implies that $p \in \mathrm{Cab}(L)$.

(2) ⇒ (3). For each $s \in \mathcal{F}_p \cap \mathfrak{s}L$, since $s \nleq p$, we have $s^{\perp} \leq p$. For the other inequality, since $p \in \mathrm{Min}(L)$, it suffices to show that $\bigvee_{s \in \mathcal{F}_p \cap \mathfrak{s}L} s^{\perp} \in \mathrm{Spec}(L)$. Take any $a, b \in \mathfrak{K}L$ such that $a \wedge b \leq \bigvee_{s \in \mathcal{F}_p \cap \mathfrak{s}L} s^{\perp}$. Since $\mathcal{F}_p$ is a completely prime filter, and for every $s \in \mathcal{F}_p \cap \mathfrak{s}L$ we have $s^{\perp} \notin \mathcal{F}_p$, it follows that $\bigvee_{s \in \mathcal{F}_p \cap \mathfrak{s}L} s^{\perp} \notin \mathcal{F}_p$ and therefore $a \wedge b \notin \mathcal{F}_p$. Without loss of generality, we assume that $a \notin \mathcal{F}_p$, which implies $a \leq p$. Since $p \in \mathrm{Cab}(L)$ and $a \in \mathfrak{K}L$, it follows that $a^{\perp} \nleq p$; hence $a^{\perp} \in \mathcal{F}_p$. Since L is an algebraic frame, there exists $k \in \mathcal{F}_p \cap \mathfrak{K}L \subseteq \mathcal{F}_p \cap \mathfrak{s}L$ with $k \leq a^{\perp}$. This implies,

$$a \leq a^{\perp\perp} \leq k^{\perp} \leq \bigvee_{s \in \mathcal{F}_p \cap \mathfrak{s}L} s^{\perp}.$$

(3) ⇒ (4). Since $\mathfrak{K}L \subseteq \mathfrak{s}L$,

$$\bigvee_{c \in \mathcal{F}_p^*} c^{\perp} \leq \bigvee_{s \in \mathcal{F}_p \cap \mathfrak{s}L} s^{\perp} = p.$$

To show the other inequality, let $t \in \mathfrak{K}L$ with $t \leq p$. By compactness of t, it follows that

$$t \leq s_1^{\perp} \vee \cdots \vee s_k^{\perp} \leq (s_1 \wedge \cdots \wedge s_k)^{\perp}$$

for some $s_1, \ldots, s_k \in \mathcal{F}_p \cap \mathfrak{s}L$. Since $\mathcal{F}_p$ is a completely prime filter, there exists some $d \in \mathcal{F}_p^*$ with $d \leq s_1 \wedge \cdots \wedge s_k$. Thus we have,

$$t \leq (s_1 \wedge \cdots \wedge s_k)^{\perp} \leq d^{\perp} \leq \bigvee_{c \in \mathcal{F}_p^*} c^{\perp}.$$

(4) $\Rightarrow$ (5). Since $p \in \mathrm{Min}(L)$ with $p = \bigvee_{c \in \mathcal{F}_p^*} c^\perp$, we just need to show that $\bigvee_{c \in \mathcal{F}_p^*} c^\perp$ has the compact absoluteness property. Take $d \in \mathfrak{K}L$ with $d \leq \bigvee_{c \in \mathcal{F}_p^*} c^\perp$. There exists a finite collection of $c_i \in \mathcal{F}_p^*$ such that $d \leq c_1^\perp \vee \cdots \vee c_n^\perp \leq (c_1 \wedge \cdots \wedge c_n)^\perp$. Since $c_1 \wedge \cdots \wedge c_n \leq (c_1 \wedge \cdots \wedge c_n)^{\perp\perp} \leq d^\perp$ and $c_1 \wedge \cdots \wedge c_n \in \mathcal{F}_p$, we have $d^\perp \in \mathcal{F}_p$. Hence $d^\perp \nleq p = \bigvee_{c \in \mathcal{F}_p^*} c^\perp$.

(5) $\Rightarrow$ (2). Note that $\bigvee_{c \in \mathcal{F}_p^*} c^\perp \leq p$ is always true for any prime p. Given that $p \in \mathrm{Min}(L)$ and $\bigvee_{c \in \mathcal{F}_p^*} c^\perp \in \mathrm{Cab}(L) \subseteq \mathrm{Min}(L)$, it follows that $p = \bigvee_{c \in \mathcal{F}_p^*} c^\perp \in \mathrm{Cab}(L)$.

(2) $\Rightarrow$ (1). Since we know that $\mathcal{F}_q$ is a subcompactly generated proper filter on L for any prime q, to prove maximality of $\mathcal{F}_p$, it suffices to show that for any $x \in \mathfrak{s}L$, $x \notin \mathcal{F}_p$ implies $x^\perp \in \mathcal{F}_p$. Let $x = k_1 \wedge k_2 \wedge \cdots \wedge k_n \in \mathfrak{s}L$ with $x \notin \mathcal{F}_p$, where $k_i \in \mathfrak{K}L$. Since $x \leq p$, by primeness we have $k_i \leq p$ for some $k_i \geq x$. Since $p \in \mathrm{Cab}(L)$, $k_i^\perp \nleq p$; thence $k_i^\perp \in \mathcal{F}_p$. Finally, $k_i^\perp \leq x^\perp$ implies that $x^\perp \in \mathcal{F}_p$. □

3 Topologies on Min(L)

A common theme in the study of algebraic frames with FIP is to study the topological properties on the space of minimal prime elements of L, $\mathrm{Min}(L)$. When we remove the FIP condition, it is not surprising that some well known results fail to hold. Observe that for any algebraic frame L, $\bigwedge \mathrm{Min}(L) = 0$.

Recall the following definitions.

Definition 3.1 To each $x \in L$, we let $U(x) = \{p \in \mathrm{Min}(L) : x \nleq p\}$ and $V(x) = \{p \in \mathrm{Min}(L) : x \leq p\}$.

The following is a standard result regarding these operators.

Lemma 3.2 *For an algebraic frame L, the operators $U(\cdot)$ and $V(\cdot)$ satisfy the following.*

(i) *For all $x, y \in L$, $U(x) \cap U(y) = U(x \wedge y)$.*
(ii) *For any $T \subseteq L$, $U(\bigvee T) = \bigcup_{x \in T} U(x)$.*
(iii) *For all $x, y \in L$, $V(x) \cup V(y) = V(x \wedge y)$.*
(iv) *For any $T \subseteq L$, $V(\bigvee T) = \bigcap_{x \in T} V(x)$.*
(v) *$U(x) = \emptyset$ precisely if $x = 0$.*

It follows that when L satisfies FIP, then the collection $\mathfrak{B} = \{U(c) : c \in \mathfrak{K}(L)\}$ is closed under both intersection and union, and thus forms a base. It turns out that even without FIP the collection forms a base for a topology on $\mathrm{Min}(L)$. For more results and information on the hull-kernel topology on $\mathrm{Min}(L)$ for L an algebraic frame with FIP we refer to [1].

Proposition 3.3 *Suppose L is an algebraic frame. Then $\mathfrak{B}$ forms a base for a topology on* $\mathrm{Min}(L)$.

Proof Let $c, d \in \mathfrak{K}(L)$ and consider $p \in U(c) \cap U(d)$. Since $c, d \nleq p$, it follows by primality that $c \wedge d \nleq p$. Thus, there is some $k \in \mathfrak{K}(L)$ such that $k \leq c \wedge d$ and $k \nleq p$. If $q \in \mathrm{Min}(L)$ with $k \nleq q$, then $c \wedge d \nleq q$. Consequently, $p \in U(k) \subseteq U(c) \cap U(d)$. □

The topology generated by $\mathfrak{B}$ is known as the **hull-kernel topology** (or Zariski topology) on $\mathrm{Min}(L)$. On the other hand, the collection $\mathfrak{B}^{-1} = \{V(c) : c \in \mathfrak{K}(L)\}$ is closed under finite intersections and thus generates a topology called the **inverse topology**; we use $\mathrm{Min}(L)^{-1}$ to denote the space of minimal prime elements equipped with the inverse topology.

Remark 3.4 Given any $x \in L$, $U(x^{\perp}) = \bigcup_{\substack{c \wedge x = 0 \\ c \in \mathfrak{K}L}} U(c)$ is an open set in $\mathrm{Min}(L)$. Also, for any $x, y \in L$, $x \leq y$ implies $U(x) \subseteq U(y)$.

One of the results for algebraic frames with FIP is that the inverse topology is finer than the hull-kernel topology. We share an interesting generalization.

Proposition 3.5 *Let L be an algebraic frame. The hull-kernel topology on* $\mathrm{Min}(L)$ *is finer than the inverse topology if and only if* $\mathrm{Cab}(L) = \mathrm{Min}(L)$.

Proof ($\Rightarrow$) Let $p \in \mathrm{Min}(L)$ and $c \in \mathfrak{K}(L)$ with $c \leq p$. So $p \in V(c)$, an open subset of the hull-kernel topology. Therefore, there is some $d \in \mathfrak{K}(L)$ such that $p \in U(d) \subseteq V(c)$. Since $\emptyset = U(d) \cap U(c) = U(d \wedge c)$, we have $c \wedge d = 0$. So $d \leq c^{\perp}$ and $d \nleq p$ implies $c^{\perp} \nleq p$. Hence $p \in \mathrm{Cab}(L)$.

($\Leftarrow$) Consider an inverse basic open set $V(c)$ with $c \in \mathfrak{K}(L)$ and $p \in V(c)$. Using the CAB property, $c^{\perp} \nleq p$ which produces a $d \in \mathfrak{K}(L)$ such that $d \leq c^{\perp}$ and $d \nleq p$. Thus, $p \in U(d) \subseteq U(c^{\perp}) \subseteq V(c)$. It follows that $V(c)$ is an open set of $\mathrm{Min}(L)$. □

It is a well-known fact that for an algebraic frame L with FIP, $\mathrm{Min}(L)$ is a zero-dimensional Hausdorff space. For general algebraic frames (not necessarily with FIP), this is a special consequence of the condition $\mathrm{Cab}(L) = \mathrm{Min}(L)$.

Proposition 3.6 *Let L be an algebraic frame. For any $x \in L$,* $\mathrm{cl}_{\mathrm{Min}(L)} U(x) = V(x^{\perp})$.

Proof Observe that for any $p \in \mathrm{Min}(L)$, if $x \nleq p$, then $x^{\perp} \leq p$. In other words $U(x) \subseteq V(x^{\perp})$. Since $V(x^{\perp})$ is a closed subset in the hull-kernel topology, it follows that

$$\mathrm{cl}_{\mathrm{Min}(L)} U(x) \subseteq V(x^{\perp}).$$

Next, let $p \in V(x^{\perp})$ and $p \in U(d)$ for some $d \in \mathfrak{K}(L)$. If $x \wedge d = 0$, we have $d \leq x^{\perp} \leq p$, contradicting that $d \nleq p$. Consequently, $p \in \mathrm{cl}_{\mathrm{Min}(L)} U(x)$. □

Corollary 3.7 *Suppose L is an algebraic frame. $U(c)$ is clopen in* $\mathrm{Min}(L)$ *for each $c \in \mathfrak{K}(L)$ if and only if* $\mathrm{Cab}(L) = \mathrm{Min}(L)$.

Proof ($\Rightarrow$) Suppose $U(c)$ is clopen for each $c \in \mathfrak{K}(L)$. Therefore for each $c \in \mathfrak{K}(L)$, $U(c) = V(c^{\perp})$. Now, if $p \in \mathrm{Min}(L)$ and $c \leq p$, then $p \notin U(c) = V(c^{\perp})$. This implies that $c^{\perp} \nleq p$. Thus, $p \in \mathrm{Cab}(L)$.

($\Leftarrow$) Conversely, if $\mathrm{Cab}(L) = \mathrm{Min}(L)$, then for any $c \in \mathfrak{K}(L)$ and each $p \in V(c^{\perp})$, we know that $c \nleq p$ by the CAB property and so $p \in U(c)$. Thus, $U(c) = V(c^{\perp})$ is clopen for each $c \in \mathfrak{K}L$. □

Corollary 3.8 *Suppose L is an algebraic frame satisfying* $\mathrm{Cab}(L) = \mathrm{Min}(L)$. *Then* $\mathrm{Min}(L)$ *is a zero-dimensional Hausdorff space.*

As was mentioned prior to Lemma 1.5, there is a one-to-one correspondence between prime elements of L and completely prime filters of L, which restricts to a bijection between $\mathrm{Min}(L)$ and $\mathrm{Ult}_{cp}(L)$. The hull-kernel topology on $\mathrm{Min}(L)$ is homeomorphic to the inverse Wallman topology on $\mathrm{Ult}_{cp}(L)$. And analogously, for $\mathrm{Min}(L)^{-1}$ and the Wallman topology on $\mathrm{Ult}_{cp}(L)$.

Here is an interesting characterization of when an algebraic frame satisfies CAB. We leave its proof to the interested reader. (Hint: see Theorem 2.15.)

Proposition 3.9 *Let L be an algebraic frame.* $\mathrm{Cab}(L) = \mathrm{Min}(L)$ *if and only if* $\mathrm{Ult}_{cp}(L) \subseteq \mathrm{Ult}_{sc}(L)$.

At this point a natural question is when the two collections of ultrafilters are equal. To that end, consider the following result.

Proposition 3.10 *The following are equivalent for any algebraic frame L:*

(1) $\mathrm{Min}(L)^{-1}$ *is compact.*
(2) Every non-empty subcompactly generated filter can be extended to a completely prime filter.
(3) Every maximal subcompactly generated filter is completely prime.

Proof (1) $\Rightarrow$ (2). Suppose that $\mathrm{Min}(L)^{-1}$ is compact and let $\mathcal{F}$ be a nonempty subcompactly generated filter. It suffices to find such a $p \in \mathrm{Min}(L)$ such that $\mathcal{F} \subseteq \mathcal{F}_p$. Define $K = \mathcal{F} \cap \mathfrak{s}(L)$. Note that for any finite subset $S \subseteq K$, if $\bigcup_{c \in S} V(c) = \mathrm{Min}(L)$, then since $V(\bigwedge S) = \bigcup_{c \in S} V(c)$ it follows that $\bigwedge S = 0$. Since S is finite we would be forced to conclude that $0 \in \mathcal{F}$. Thus, for any finite subset S of K, $\bigcup_{c \in S} V(c) \subset \mathrm{Min}(L)^{-1}$. By compactness, it follows that $\bigcup_{c \in K} V(c) \subset \mathrm{Min}(L)^{-1}$. This means that there exists a minimal prime $p \in \mathrm{Min}(L)$ such $p \notin V(c)$ for all $c \in K$. Therefore, $K \subseteq \mathcal{F}_p$. Since K generated $\mathcal{F}$, we conclude that $\mathcal{F} \subseteq \mathcal{F}_p$.

(2) $\Rightarrow$ (3). Let $\mathcal{F} \in \mathrm{Ult}_{sc}(L)$. By (2), we can extend $\mathcal{F}$ to some completely prime filter, and then extend to some $\mathcal{F}_p$ for some $p \in \mathrm{Min}(L)$. But $\mathcal{F}_p$ is compactly generated and hence subcompactly generated. By maximality, $\mathcal{F} = \mathcal{F}_p$.

(3) $\Rightarrow$ (1). In order to show that $\mathrm{Min}(L)^{-1}$ is compact we start with a collection of basic open subsets of $\mathrm{Min}(L)^{-1}$, say $\{V(s) : s \in K\}$ for some subset $K \subseteq \mathfrak{s}(L)$. We assume that for any finite subset S of K, $\bigcup_{s \in S} V(s) \subset \mathrm{Min}(L)$. It follows that $\bigwedge S \neq 0$. Thus, the collection K is a filter base for a proper filter. In particular, this filter is subcompactly generated and so can be extended to a maximal subcompactly generated filter, say $\mathcal{F}$. By (3) together with the fact that completely prime filters are subcompactly generated it follows that $\mathcal{F} = \mathcal{F}_p$ for some p. Thus, $p \notin \bigcup_{s \in K} V(s)$. □

Theorem 3.11 *Let L be an algebraic frame.* $\mathrm{Min}(L)^{-1}$ *is compact if and only if* $\mathrm{Ult}_{cp}(L) = \mathrm{Ult}_{sc}(L)$.

Proof Since $\mathrm{Ult}_{sc}(L)$ is always a compact space (Lemma 2.13), and the Wallman topology on $\mathrm{Ult}_{cp}(L)$ is homeomorphic to $\mathrm{Min}(L)^{-1}$, the sufficiency follows.

As to the necessity, suppose that $\mathrm{Min}(L)^{-1}$ is a compact space. It is clear from Proposition 3.10 that given any $F \in \mathrm{Ult}_{sc}(L)$, there exists some $p \in \mathrm{Min}(L)$ with $F = \mathcal{F}_p \in \mathrm{Ult}_{cp}(L)$. Therefore, $\mathrm{Ult}_{sc}(L) \subseteq \mathrm{Ult}_{cp}(L)$. To show reverse inclusion, suppose by contradiction that $\mathcal{M}$ is a maximal subcompactly generated filter that is not of the form $\mathcal{F}_p$ for any $p \in \mathrm{Min}(L)$. Since each $\mathcal{F}_p$ is subcompactly generated, it follows that for each $p \in \mathrm{Min}(L)$ there is some subcompact $s_p \in \mathcal{M} \setminus \mathcal{F}_p$. Consider the set $\{V(s_p) : p \in \mathrm{Min}(L)\}$. This collection is an open cover of $\mathrm{Min}(L)^{-1}$. By hypothesis, there is a finite set $\{s_{p_1}, \ldots, s_{p_n}\}$ such that

$$\mathrm{Min}(L) = V(s_{p_1}) \cup \ldots \cup V(s_{p_n}) = V(s_{p_1} \wedge \ldots \wedge s_{p_n}).$$

Let $s = s_{p_1} \wedge \ldots \wedge s_{p_n}$, a subcompact element that belongs to $\mathcal{M}$. Since L is algebraic and s is below every minimal prime element, it follows that $0 = s \in \mathcal{M}$, contradicting that $\mathcal{M}$ is a proper filter. $\square$

Corollary 3.12 *Suppose L is an algebraic frame.*

(1) If L satisfies the FIP, then $\mathrm{Min}(L)^{-1}$ is compact.
(2) If $\mathrm{Min}(L)^{-1}$ is compact, then $\mathrm{Cab}(L) = \mathrm{Min}(L)$.

Summing up the relation between different conditions on the algebraic frame L, we have:

$$\begin{tikzcd}
\text{FIP} \arrow[d, Rightarrow] & & \\
\mathrm{Min}(L)^{-1} \text{ compact} \arrow[r, Leftrightarrow] & \mathrm{Ult}_{sc}(L) = \mathrm{Ult}_{cp}(L) \arrow[d, Rightarrow] & \\
& \mathrm{Ult}_{cp}(L) \subseteq \mathrm{Ult}_{sc}(L) \arrow[r, Leftrightarrow] & \mathrm{Cab}(L) = \mathrm{Min}(L) \arrow[d, Rightarrow] \\
& & \mathrm{Min}(L) \text{ zero-dim and } T_2
\end{tikzcd}$$

In the next section we will construct examples of algebraic frames that show the converses of some of the above implications are not true. We will also show that there are examples for which $\mathrm{Min}(L)$ is a zero-dimensional Hausdorff space, yet $\mathrm{Cab}(L) = \emptyset$.

4 Partially-ordered Sets and Space of Ultrafilters

Throughout, $(P, \leq)$ denotes a partially-ordered set. A subset of P, say D, is called a **down-set** if whenever $x \in D$ and $y \leq x$, then $y \in D$. The dual notion yields the concept of an **up-set**. We let $\downarrow x = \{y \in P : y \leq x\}$, the principal down-set generated by x.

The collection of all down-sets of P shall be denoted by $\mathfrak{D}(P)$, and it is well-known that $\mathfrak{D}(P)$ is an algebraic frame when ordered by inclusion. In particular, the join of a collection of down-sets is, in fact, the union of the sets, while the meet is the intersection. The compact elements of $\mathfrak{D}(P)$ are the ones generated by finite sets; that is,

$$\mathfrak{K}(\mathfrak{D}(P)) = \{\downarrow F : F \subseteq P \text{ is finite}\}.$$

For more information on down-set lattices see [3] and [4]. These are precisely (up to isomorphism) the completely distributive algebraic frames. The top element of $\mathfrak{D}(P)$ is P, while the bottom element is $\emptyset$.

Definition 4.1 For each pair $\alpha, \beta \in P$ we can look at the collection of lower bounds of the pair which we denote by $L(\alpha, \beta)$; this set might be empty. We can then look at $\mathrm{Max}(L(\alpha, \beta))$

as the set of those $x \in L(\alpha, \beta)$ such that if $y \in L(\alpha, \beta)$ and $x \leq y$, then $x = y$. We say that $\mathrm{Max}(L(\alpha, \beta))$ covers $L(\alpha, \beta)$ if for each $\delta \in L(\alpha, \beta)$ there is some $x \in \mathrm{Max}(L(\alpha, \beta))$ such that $\delta \leq x$.

We generalize the above definition to include finite sets, e.g. if $F = \{\alpha_1, \ldots, \alpha_n\}$, then $L(F)$ is the set of lower bounds of F, that is, $L(F) = \{x \in P : x \leq \alpha_i, 1 \leq i \leq n\}$

We leave the proof of the following to the interested reader.

Lemma 4.2 *Let $(P, \leq)$ be a poset and $\alpha, \beta \in P$.*

$$(\downarrow \alpha) \cap (\downarrow \beta) = L(\alpha, \beta).$$

Lemma 4.3 *Let $(P, \leq)$ be a poset. The frame $\mathfrak{D}(P)$ satisfies FIP if and only if for each finite subset $F \subseteq P$, $\mathrm{Max}(L(F))$ is finite and covers $L(F)$.*

It is straightforward to check that the set-theoretic complement of a down-set is an up-set, and vice-versa. What is interesting is that one can use the complement of a down-set to determine if it is a prime element in $\mathfrak{D}(P)$.

Definition 4.4 Let $(P, \leq)$ be a poset. Recall that $\mathcal{F} \subseteq P$ is a **filter** of P if it is an up-set and down-directed subset, that is, for any $\alpha, \beta \in \mathcal{F}$ there is some $\gamma \in L(\alpha, \beta)$ such that $\gamma \in \mathcal{F}$.

Observe, we do not assume that our filters are proper subsets; it is possible that the whole poset P is a filter. In fact, P is a filter if and only if P is a downward-directed set (see [13, Remark following Definition 5.3] though the reader should be warned that the authors use compatible to mean downward-directed). On the other hand, if P is a trivially-ordered set, then P is not a filter of itself. We also observe that each element $a \in P$ generates a principal filter that we shall denote $\mathcal{F}_a$; so, $\mathcal{F}_a = \uparrow a$.

Lemma 4.5 *Let $(P, \leq)$ be a poset and $D \in \mathfrak{D}(P)$. D is a prime element of the frame if and only if $P \smallsetminus D$ is a filter of P.*

Proof Suppose $D \in \mathrm{Spec}(\mathfrak{D}(P))$ and let $x, y \in P \smallsetminus D$. Take the down-sets $D \cup \downarrow x$ and $D \cup \downarrow y$. Since D is prime it follows that

$$D \cup (\downarrow x \cap \downarrow y) = (D \cup \downarrow x) \cap (D \cup \downarrow y) \nsubseteq D$$

Thus, there is some $l \in (\downarrow x \cap \downarrow y) \setminus D$. Consequently, $l \in L(x, y)$, concluding that $P \smallsetminus D$ is a filter.

The converse follows in reverse. $\square$

Proposition 4.6 *Let $(P, \leq)$ be a poset and $D \in \mathrm{Spec}(\mathfrak{D}(P))$. Then $D \in \mathrm{Cab}(\mathfrak{D}(P))$ if and only if for each $x \in D$ there is some $y \notin D$ such that $L(x, y) = \emptyset$.*

Proof Suppose $D \in \mathrm{Cab}(\mathfrak{D}(P))$ and $x \in D$ for some $x \in P$, then $\downarrow x \leq D$. Since $\downarrow x$ is a compact element of the frame, $(\downarrow x)^{\perp} \nleq D$. So, there exists some down-set $T \leq (\downarrow x)^{\perp}$ with $T \nleq D$. This means that there exist some $y \in T \setminus D$. Note that $\downarrow y \leq T \leq (\downarrow x)^{\perp}$ implies that $(\downarrow y) \cap (\downarrow x) = \emptyset$. Hence, $L(x, y) = \emptyset$.

Conversely, let $F \subseteq P$ be a finite set with $\downarrow F \leq D$. Let $F = \{\alpha_1, \cdots, \alpha_n\}$. For each $\alpha_i \in D$, there exists $y_i \notin D$ such that $L(\alpha_i, y_i) = \emptyset$. Since each $y_i \in P \setminus D$, which is a filter, it follows that there exists some $y \in L(y_1, \cdots, y_n)$ with $y \notin D$. Observe further that $L(y, \alpha_i) = \emptyset$ for each i, proving that $y \in (\downarrow F)^{\perp}$. Hence, $(\downarrow F)^{\perp} \nleq D$. $\square$

Using Zorn's Lemma, every filter is contained in a maximal filter, i.e. an **ultrafilter**. Again, we do not assume that our ultrafilters are proper. So if P is down-directed, it will be the unique ultrafilter; this is acceptable. We denote the collection of ultrafilters on P by $\mathrm{Ult}(P)$. Notice that $\mathcal{F}_a \in \mathrm{Ult}(P)$ if and only if $a \in P$ is an atom of the poset. For those familiar with ultrafilters on a 0-lattice, a well-known property of a given ultrafilter $\mathcal{U}$ is: for each $x \notin \mathcal{U}$ there is some $y \in \mathcal{U}$ such that $x \wedge y = 0$.

Definition 4.7 A poset P is said to be *lattice-like with respect to ultrafilters* if every $\mathcal{U} \in \mathrm{Ult}(P)$ has the property that for each $x \notin \mathcal{U}$ there is some $y \in \mathcal{U}$ such that $L(x, y) = \emptyset$.

Proposition 4.8 *Suppose that $(P, \leq)$ is a partially-ordered set. Then P is lattice-like with respect to ultrafilters if and only if* $\mathrm{Cab}(\mathfrak{D}(P)) = \mathrm{Min}(\mathfrak{D}(P))$.

Proof This follows from Proposition 4.6. □

Example 4.9 Suppose $(P, \leq)$ is a poset with an atom $a \in P$. Then the ultrafilter $\mathcal{F}_a$ has the property that for each $x \notin \mathcal{F}$, $L(a, x) = \emptyset$. Thus, if every ultrafilter on P has the form $\mathcal{F}_a$ for some atom, then P is lattice-like with respect to ultrafilters.

We recall the generalization of the Wallman and inverse Wallman topologies on meet-semi-lattices to posets.

In [13, Definition 5.6], the authors define a topology on the space of ultrafilters of a poset and show it is T_1. The base for the topology of open sets consists of sets of the form

$$N_\alpha = \{U \in \mathrm{Ult}(P) : \alpha \in U\}.$$

Let $\mathfrak{N} = \{N_\alpha : \alpha \in P\}$. We find this choice of topology interesting since the well-known Wallman topology on the set of ultrafilters of a bounded lattice is formed by taking the collection $\mathfrak{N}$ as a base for the closed sets. Wallman, himself, in [14], pointed out that the collection $\mathfrak{N}$ could be used as a base for the open sets of some topology, but that the topology would be totally disconnected, and so this was not useful for his purposes; he was only interested in ultrafilters on lattices in [14].

Moving to the more general setting, for a partially-ordered set $(P, \leq)$, the collection $\mathfrak{N}$ does indeed form an open base for a topology since

$$N_\alpha \cap N_\beta = \bigcup_{s \in L(\alpha,\beta)} N_s.$$

On the other hand, if we let $O_\alpha = \{U \in \mathrm{Ult}(P) : \alpha \notin U\}$ the best we can say is that the collection of such things forms a subbase. The topology generated by this subbase is the **Wallman** topology on $\mathrm{Ult}(P)$. On the other hand, we shall call the topology with $\mathfrak{N}$ as an open base the **inverse Wallman** topology on $\mathrm{Ult}(P)$. We shall denote the inverse Wallman topology by $\mathrm{Ult}(P)^{-1}$.

Example 4.10 Let $P = \{a, b, c\}$ be trivially-ordered. Then $\mathrm{Ult}(P) = \{\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c\}$ and for each $x \in P$, $O_x = \{\mathcal{F}_y, \mathcal{F}_z : x \neq y, z\}$. It follows that $O_a \cap O_b = \{\mathcal{F}_c\}$ which is not a union of sets of the form O_x.

Not too long ago, in the articles [10] and [11], the authors studied the space of ultrafilters of a poset and showed some properties that all such spaces must satisfy. For example, such a space must be a Baire space.

As was pointed out above, there is a correspondence between the prime elements of $\mathfrak{D}(P)$ and the filters on P. The restriction to $\mathrm{Min}(\mathfrak{D}(P))$ is a bijection onto $\mathrm{Ult}(P)$.

Theorem 4.11 *Given a poset $(P, \leq)$, the space of ultrafilters of P equipped with the inverse Wallman topology is homeomorphic to the space* $\mathrm{Min}(\mathfrak{D}(P))$ *equipped with the hull-kernel topology. Furthermore, the space* $\mathrm{Ult}(P)$ *equipped with the Wallman topology is homeomorphic to the space* $\mathrm{Min}(\mathfrak{D}(P))^{-1}$.

We are now ready for some examples. First, we start with a poset where the space of minimal prime elements of the frame of down-sets is not compact. Second, we give an example of poset P which has a $D \in \mathrm{Min}(\mathfrak{D}(P))$ that does not satisfy CAB. This example was used to produce our first example of an algebraic frame L for which $\mathrm{Cab}(L) \subset \mathrm{Min}(L)$. At the end of the section, we will show that there is a different way of creating such algebraic frames.

Example 4.12 ($\mathrm{Min}(L)$ not compact) Take a countably infinite trivially ordered set P. The only ultrafilters are the principal ultrafilters generated by points in the poset. Then for each $\alpha \in P$, $N_\alpha = \{\mathcal{F}_\alpha\}$. It follows that $\mathrm{Ult}(P)^{-1}$, and hence also $\mathrm{Min}(\mathfrak{D}(P))$, is discrete and hence not compact.

Example 4.13 ($\mathrm{Cab}(L) \neq \mathrm{Min}(L)$) We construct a partially ordered set P which has an ultrafilter whose associated minimal prime element does not satisfy CAB.

$$P = T \cup E \cup \{a\}$$

where $T = \{x_1, x_2, \ldots\}$ is a countable trivially ordered set and $E = \{y_1, y_2, \ldots\}$ is a chain for which $y_i > y_{i+1}$ for each $i \in \mathbb{N}$. Furthermore, $a > x_i$ for all $i \in \mathbb{N}$ and $x_i < y_k$ for each $k \leq i$.

First, we claim that E is an ultrafilter; obviously it is a filter. Suppose F is a filter of P that contains E. Note that $L(x_i, y_{i+1}) = \emptyset$ so $x_i \notin F$ for each x_i. If $a \in F$ and F is a filter then there must be a lower bound of a and y_i in F, but every lower bound of these two elements is of the form x_j for some $j \geq i$, none of which is in F. Therefore, $a \notin F$, whence $F = E$. Consequently, E is an ultrafilter. Observe that $a \in P \setminus E$ has the property that for each $y_i \in E$, $L(a, y_i) \neq \emptyset$. Consequently, $P \smallsetminus E \in \mathrm{Min}(\mathfrak{D}(P)) \smallsetminus \mathrm{Cab}(\mathfrak{D}(P))$.

The remaining ultrafilters on P are of the form $\mathcal{F}_{x_n}$ and each of these is an isolated point of both $\mathrm{Ult}(P)$ and $\mathrm{Ult}(P)^{-1}$, since for each $n \in \mathbb{N}$,

$$\{\mathcal{F}_{x_n}\} = O_{y_n} \cap O_{x_1} \ldots \cap O_{x_{n-1}} = N_{x_n}.$$

Let $\gamma \in P$ satisfy $E \in N_\gamma$. It is easy to check that $\gamma = y_n$ for some n. Therefore, $N_\gamma = \{E, \mathcal{F}_{x_n}, \mathcal{F}_{x_{n+1}}, \ldots, \}$. It follows that $\mathrm{Ult}(P)^{-1}$ is homeomorphic to the one-point compactification of the naturals. On the other hand, since $O_a = \{E\}$ it follows that $\mathrm{Ult}(P)$ is a countable discrete space.

Next, we turn to classifying those posets P for which $\mathrm{Min}(\mathfrak{D}(P))^{-1}$ is compact.

Definition 4.14 Let $S \subseteq P$. We say that S has the finite lower bound property if for each finite subset of S, say $F \subseteq S$, $L(F) \neq \emptyset$. In particular, S has the finite lower bound property if and only if each finite subset is contained in a filter.

Proposition 4.15 *Let $(P, \leq)$ be a poset. The Wallman topology on* $\mathrm{Ult}(P)$ *is compact if and only if every subset of P with the finite lower bound property can be extended to an ultrafilter.*

Proof Suppose $\mathrm{Ult}(P)$ is compact and $S \subseteq P$ has the finite lower bound property. Consider the collection $\mathcal{C} = \{N_s : s \in S\}$ of closed sets in the Wallman topology on $\mathrm{Ult}(P)$. We claim this collection of subsets has the finite intersection property. Let $s_1, \dots, s_n \in S$. By hypothesis, there is some $\mathcal{U} \in \mathrm{Ult}(P)$ such that $s_i \in \mathcal{U}$ for each $i = 1, \dots, n$. Thus, $\mathcal{U} \in N_{s_1} \cap \dots \cap N_{s_n}$ and so $\mathcal{C}$ has the finite intersection property. Since $\mathrm{Ult}(P)$ is compact, the intersection of $\mathcal{C}$ is non-empty, say $\mathcal{V} \in \cap\,\mathcal{C}$. Whence $S \subseteq \mathcal{V}$.

Conversely, let $\mathcal{O} = \{O_s : s \in S\}$ be a collection of basic open subsets of $\mathrm{Ult}(P)$. If $\mathcal{O}$ has no finite subcover, then S has the finite lower bound property. Thus, by hypothesis, there is some $\mathcal{U} \in \mathrm{Ult}(P)$ containing S, whence $\mathcal{O}$ is not an open cover of $\mathrm{Ult}(P)$. □

Example 4.16 ($\mathrm{Cab}(L) = \mathrm{Min}(L)$ but $\mathrm{Min}(L)^{-1}$ not compact) Let P be the disjoint union of two pairwise incomparable countable sets, say $\{x_n\}_{n\in\mathbb{N}}$ and $\{a_n\}_{n\in\mathbb{N}}$.

The partial order on P is further defined by $a_i \geq x_j$ whenever $i \leq j$. Notice that this is an atomic partially ordered set with atoms x_n. We claim that the only ultrafilters of P the ones of the form $\mathcal{F}_{x_n}$. First notice that for each $n \in \mathbb{N}$,

$$\mathcal{F}_{x_n} = \{x_n, a_1, \dots, a_n\}.$$

Let $\mathcal{F}$ be an arbitrary ultrafilter. If $\mathcal{F}$ contains x_n, then $\mathcal{F} = \mathcal{F}_{x_n}\}$. Now, $\mathcal{F}$ cannot contain an infinite number of elements of $S = \{a_n\}$. If it did, then taking two of the elements, there must be a lower bound of those two which must be of the form x_n. But choose an a_m with $m > n$, producing a contradiction. It follows that

$$\mathrm{Ult}(P) = \{\mathcal{F}_{x_n}\}_{n\in\mathbb{N}}.$$

By Example 4.9, we know that P is lattice-like and so $\mathrm{Cab}(\mathfrak{D}(P)) = \mathrm{Min}(\mathfrak{D}(P))$.

Furthermore, notice that the collection $S = \{a_n\}$ has the property that each finite set belongs to a filter, while S itself does not. Therefore, by Proposition 4.15, the Wallman topology on $\mathrm{Ult}(P)$ is not compact.

Now, for each natural $n > 1$,

$$O_{a_n} = \{\mathcal{F}_{x_1}, \dots, \mathcal{F}_{x_{n-1}}\}.$$

Therefore, the collection $\{O_{\alpha_n}\}$ is a countable cover of $\mathrm{Ult}(P)$ which has no finite subcover. So, $\mathrm{Ult}(P)$ is not countably compact. Consequently, $\mathrm{Min}(\mathfrak{D}(P))^{-1}$ is not compact but $\mathrm{Cab}(\mathfrak{D}(P)) = \mathrm{Min}(\mathfrak{D}(P))$.

Example 4.17 ($\mathrm{Min}(L)^{-1}$ compact but L does not satisfy FIP)

We modify the previous example by letting $P = \{x_n\}_{n\in\mathbb{N}} \cup \{a, b\}$, where $\{x_n\}_{n\in\mathbb{N}}$ is a pairwise incomparable countable set, $a \geq x_n$ for all $n \geq 1$, and $b \geq x_n$ for all $n > 1$. Each ultrafilter is of the form $\mathcal{F}_{x_n}$, where $\mathcal{F}_{x_1} = \{x_1, a\}$ and $\mathcal{F}_{x_n} = \{x_n, a, b\}$ for all $n > 1$. Observe that for each $n \in \mathbb{N}$, $O_{x_n} = \{\mathcal{F}_{x_k} : k \neq n\}$. Also, $O_a = \emptyset$ and $O_b = \{\mathcal{F}_{x_1}\}$.

Applying Proposition 4.15, we gather that $\mathrm{Ult}(P)$ is compact (and thus, so is the space $\mathrm{Min}(\mathfrak{D}(P))^{-1}$). Furthermore, for the finite subset $\{a, b\} \subseteq P$, $\mathrm{Max}(L(\{a, b\})) = \{x_k\}_{k\geq 2}$ is infinite. Therefore, by Lemma 4.3, $\mathfrak{D}(P)$ does not satisfy FIP.

In [11], the authors share two ways of constructing certain spaces as the space of ultrafilters of a poset. One of these constructions is pertinent to our discussion.

Start with X, a locally compact Hausdorff space. Recall that an open subset O is called pre-compact if its closure is compact. Then, by taking Γ_X to be the collection of nonempty pre-compact open subsets with the partial order $O_1 \leq O_2$ if either $O_1 = O_2$ or $\mathrm{cl}\, O_1 \subseteq O_2$, they show that $\mathrm{Ult}(\Gamma_X)$ is homeomorphic to X. The homeomorphism is the one that sends an element $x \in X$ to the ultrafilter consisting of those pre-compact open subsets that contain x, call this ultrafilter U_x. Interestingly, the minimal prime element corresponding to this ultrafilter satisfies CAB if and only if the point x is an isolated point of X as we now show. We observe that if x is not isolated, then there is a pre-compact open subset O such that $x \in \mathrm{cl}\, O \setminus O$ and for every $O' \in U_x$, $L(O, O') \neq \emptyset$.

Theorem 4.18 (Theorem 2.2 [11]) *For every locally compact Hausdorff space X there is a poset P such that X is homeomorphic to* $\mathrm{Ult}(P)^{-1}$.

Example 4.19 ($\mathrm{Min}(L)$ T_2+ zero-dim but $\mathrm{Cab}(L) \neq \mathrm{Min}(L)$) Take any locally compact Hausdorff space X that is dense-in-itself (i.e. X has no isolated point), e.g. the Cantor space. We then form Γ_X and construct $\mathrm{Ult}(\Gamma_X)$, an algebraic frame for which $\mathrm{Cab}(\mathfrak{D}(\Gamma_X)) = \emptyset$. One can further choose X to be zero-dimensional, showing that the converse to Corollary 3.8 is not true.

Remark 4.20 Observe that the poset of pre-compact subsets of the one-point compactification of the naturals, $\alpha\mathbb{N}$, is not the same as the poset constructed in Example 4.13. In fact, since $\alpha\mathbb{N}$ is compact it follows that every nonempty open subset of $\alpha\mathbb{N}$ is already pre-compact. But notice that the partial orders on these are not the same. For example, let O_1 be the set of even numbers and $O_2 = O_1 \cup \{1\}$. Then $O_1 \subseteq O_2$ but $\mathrm{cl}\, O_1 \nsubseteq O_2$. Interestingly, the poset of pre-compacts and Example 4.13 both produce homeomorphic copies of $\alpha\mathbb{N}$ as the space of ultrafilters of a poset under the inverse Wallman topology.

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