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ARCHIMEDEAN ℓ-GROUPS WITH STRONG UNIT: COZERO-SETS AND COINCIDENCE OF TYPES OF IDEALS

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ABSTRACT. $\mathbf{W}^*$ is the category of the title. For $G \in \mathbf{W}^*$, we have the canonical compact space YG, and Yosida representation $G \leq C(YG)$, thus, for $g \in G$, one has the cozero-set $\operatorname{coz}(g)$ in YG. The ideals at issue in G include the principal ideals and polars, $G(g)$ and $g^{\perp\perp}$, respectively, and the $\mathbf{W}^*$-kernels of $\mathbf{W}^*$-morphisms from G. The "coincidences of types" include these properties of G: (M) Each $G(g) = g^{\perp\perp}$; (Y) Each $G(g)$ is a $\mathbf{W}^*$-kernel; (CR) Each $g^{\perp\perp}$ is a $\mathbf{W}^*$-kernel (iff each $\operatorname{coz}(g)$ is regular open). For each of these, we give numerous "rephrasings" and examples, and note that (M) = (Y) $\cap$ (CR). This paper is a companion to a paper in preparation by the present authors, which includes the present thrust in contexts less restrictive and more algebraic. Here, the focus on $\mathbf{W}^*$ brings topology to bear, and sharpens the view.

1. Introduction

A lattice-ordered group, or just ℓ-group, is a group G with a partial order $\leq$, invariant under the group operation (written $+$), which is a lattice. The theory is exposed in [1,7,10], where undefined terms, and well-known facts, may be found.

All our ℓ-groups G will be archimedean, which means if $g, h \in G$ and $0 \leq ng \leq h$ for every $n \in \mathbb{N}$, then $g = 0$. Every archimedean ℓ-group is abelian.

In such a G, a strong unit is $0 \leq u \in G$ for which for every $g \in G$ there is $n \in \mathbb{N}$ with $|g| \leq nu$. The quintessential example of such (G, u) is $(C(K), 1)$; here K is a compact Hausdorff space, $C(K)$ is the set of continuous real-valued functions under pointwise $+$ and $\leq$, and 1 is the constant function with value "1".

The domain of this paper is sub-ℓ-groups of $C(K)$'s which contain 1, as explained in 1.2 below. We proceed to

Definitions, etc. 1.1. In an ℓ-group G, let us say only Abelian, for the non expert an *ideal* I is the kernel of an ℓ-group homomorphism out of G, which means I is a convex sub-ℓ-group of G. Here, for each $g \in G$, we have the principal ideal $G(g) = \{f \in G \mid \exists n \in \mathbb{N}, |f| \leq n|g|\}$ ($\mathbb{N}$ is the positive integers), and the principal polar $g^{\perp\perp}$. $g^{\perp} = \{f \in G \mid |f| \wedge |g| = 0\}$, thus $g^{\perp\perp} = (g^{\perp})^{\perp}$ (where, for $S \subseteq G$, $S^{\perp} = \bigcap\{s^{\perp} \mid s \in S\}$.

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(a) The class (M) consists of those ℓ-groups G such that $G(g) = g^{\perp\perp}$ for all $g \in G$. As far as we know, this property was first identified in [6], but see [4] for further discussion. Our analysis of property (M) is in Section 4 below.

(b) The class (Y) consists of those ℓ-groups G such that $G(g)$ is an intersection of maximal ideals for every $g \in G$. Note that an ideal I in G is maximal (in the usual sense) if and only if G/I embeds into the real numbers $\mathbb{R}$ (Hölder's Theorem). The lattice of ideals of an ℓ-group in (Y) is a type of frame called a "Yosida frame" in [16], where this concept seems to originate. The description of (Y) in the Abstract will develop. Our analysis of (Y) is in Section 2 below.

(The notations (M) and (Y) are explained in [4].)

(c) The class (CR) is defined in Definition 1.5. Our analysis of (CR) is in Section 3 below. The definition of (CR) requires the following.

Now we restrict to $\mathbf{W}^*$, which is the category of all (G, u) such that G is Archimedean ℓ-group and $u \in G$ is a non-negative strong unit ($G(u) = G$). A $\mathbf{W}^*$-morphism $(G, u) \xrightarrow{\varphi} (H, v)$ is an ℓ-group homomorphism with $\varphi(u) = v$. By definition, $Y(G, u)$ is $\{I \mid I \text{ is an ideal maximal for } u \notin I\}$. In fact, because u is a strong unit, $Y(G, u)$ coincides with $\operatorname{Max}(G)$, the set of all maximal ideals of G ([14: Proposition 3.5]).

Our basic tool will be the following representation theorem, which developed from a theorem of Yosida [17].

Theorem 1.2 (See [3] or [14])**.**

(a) *Given the hull-kernel topology, $Y(G, u)$ is compact Hausdorff, and there is the $\mathbf{W}^*$-embedding $(G, u) \hookrightarrow (C(Y(G, u)), 1)$ with the image of (G, u) 0-1 separating closed sets of $Y(G, u)$. The space $Y(G, u)$ is unique for that data. We simply write "$G \leq C(Y(G, u))$", and sometimes just YG for $Y(G, u)$. If G is a vector lattice, the embedding is a vector lattice embedding.*

(b) *If $(G, u) \xrightarrow{\varphi} (H, v)$ is a $\mathbf{W}^*$-morphism, there is unique continuous $YG \xleftarrow{\tau} YH$ for which $\varphi(g)$ in $C(YH)$ is $\varphi(g) = g \circ \tau$, for each g. φ is onto if and only if τ is one-to-one (thus a homeomorphic embedding, and we may say "$YG \supseteq YH$").*

We usually shall adopt the view $G \leq C(YG)$ (recall $YG = Y(G, u)$), as above. Then, for $g \in G$, $\operatorname{coz}(g) = \{x \in YG \mid g(x) \neq 0\}$, $Z(g) = YG \setminus \operatorname{coz}(g)$, and $\operatorname{coz}(G) = \{\operatorname{coz}(g) \mid g \in G\}$ (which is a base for the topology of YG). Note that Theorem 1.2 tells us explicitly that our G's are ℓ-groups of bounded, continuous real-valued functions.

If $(G, u) \xrightarrow{\varphi} (H, v)$ is a $\mathbf{W}^*$-morphism, $\ker(\varphi) = \{g \mid \varphi(g) = 0\}$ is called a $\mathbf{W}^*$*-kernel*, and $\mathbf{W}^*\mathrm{k}(G)$ consists of all such $\ker(\varphi)$. The following is a synopsis from [3: Section 2] (an easy consequence of Theorem 1.2).

If S is a closed subset of $Y(g, u)$, then a $\mathbf{W}^*$-map is produced by restriction: $g \mapsto g|_S$, and this has kernel $I(S) = \{g \in G \mid Z(g) \supseteq S\}$. Observe that $S_1 \subseteq S_2$ if and only if $I(S_1) \supseteq I(S_2)$.

Proposition 1.3. *$\mathbf{W}^*\mathrm{k}(G)$ and $\{S \mid S \text{ closed in } YG\}$ are complete lattices, and lattice anti-isomorphic: if $I \in \mathbf{W}^*\mathrm{k}(G)$, then $I = I(S)$ for $S = \bigcap\{Z(g) \mid g \in I\}$.*

Armed with the "Yosida apparatus" just described, we discuss various types of ideals.

Facts 1.4. *Suppose $(G, u) \in \mathbf{W}^*$, viewed as $G \leq C(Y(G, u))$.*

(a) *The proper maximal ideals are, for $p \in Y(G, u)$, the $M_p \equiv I(\{p\}) = \{g \in G \mid g(p) = 0\}$.*

(b) *The intersections of maximal ideals are exactly the $I(S)$ for S closed in $Y(G, u)$, i.e., exactly the $\mathbf{W}^*$-kernels (since $G \twoheadrightarrow G/I(S)$ is (isomorphic to) the map of restricting $G \twoheadrightarrow G|_S$).*

(c) *If $G(g) = I(S)$ (S closed), then $S = Z(g)$.*

(d) $g^{\perp\perp} = I(\mathrm{cl}\,\mathrm{int} Z(g))$ *(*$\mathrm{cl}\,\mathrm{int}$ *is the topological closure of interior in* YG*).*

(e) *(A "regular open" (RO) subset of a topological space is a set* $S = \mathrm{int}\,\mathrm{cl}\, S$*.)*
For $g \in G$*,* $\mathrm{coz}(g)$ *is RO if and only if* $g^{\perp\perp} = I(Z(g))$*.*

(f) *(A "weak unit" in* G *(not necessarily in* $\mathbf{W}^*$*) is an element* $v \geq 0$ *for which* $v^{\perp} = \{0\}$*; or, equivalently,* $v^{\perp\perp} = G$*.)*
The element v *is a weak (resp., strong) unit in* $(G, u) \in \mathbf{W}^*$ *if and only if* $\mathrm{coz}(v)$ *is dense in* $Y(G, u)$ *(resp.,* $\mathrm{coz}(v) = Y(G, u)$*).*

(g) *For all* $g \in G$*,* $G(g) \subseteq I(Z(g)) \subseteq I(\mathrm{cl}\,\mathrm{int} Z(g)) = g^{\perp\perp}$*.*

Proof. (a) Clear.

(b) From (a), $I(S) = I(\overline{S})$, and Proposition 1.3.

(c) Suppose $G(g) = I(S)$. Then $g \in I(S)$, so $Z(g) \supseteq S$. If $x \in Z(g)$, then $x \in Z(f)$ for every $f \in G(g) = I(S)$, and it follows that $x \in S$. Hence $Z(g) \subseteq S$.

(d) It is noted in [13] that $g^{\perp\perp} = \{f \in G \mid \mathrm{coz}(f) \subseteq \overline{\mathrm{coz}(g)}\}$. Then, a little topology (see [11]) gives (d).

(e) From (c) and (d).

(f) First: In general, $v^{\perp} = \{f \in G \mid \mathrm{coz}(f) \cap \mathrm{coz}(v) = \emptyset\}$ and the statment about weak units follows. Second: It's easy that $\mathrm{coz}(v) = Y(G, u)$ if and only if $G = G(v)$.

(g) Follows from the preceding. ☐

Definition 1.5. The class (CR) consists of those $(G, u) \in \mathbf{W}^*$ such that $\mathrm{coz}(g)$ is regular open for every $g \in G$.

The following result shows that this definition is equivalent to one given in the Abstract.

Corollary 1.6. *In* $\mathbf{W}^*$*,*

(a) $G \in$ (Y) *if and only if* $\forall g \in G$*,* $G(g) = I(Z(g))$*;*

(b) $G \in$ (CR) *if and only if* $\forall g \in G$*,* $I(Z(g)) = g^{\perp\perp}$*;*

(c) (M) = (Y) $\cap$ (CR).

Proof. (a) here follows from (b) and (c) of Facts 1.4; (b) here follows from (b) and (e) of Facts 1.4; and (c) here follows from (a) and (b) here and (g) of Facts 1.4. ☐

Remark 1.7. In light of Corollary 1.6, one may express the relationship between the properties (Y), (CR), and (M) in terms of lattices associated with G. Indeed, in any ℓ-group G (not necessarily in $\mathbf{W}^*$), the sets $\mathrm{PI}(G) = \{G(g) \mid g \in G\}$ and $\mathrm{PP}(G) = \{g^{\perp\perp} \mid g \in G\}$ are lattices with respect to set-theoretic inclusion ([10: Propositions 7.15, 13.11]) and one has the surjective lattice homomorphism $\alpha\colon \mathrm{PI}(G) \to \mathrm{PP}(G)$ given by $\alpha(G(g)) = g^{\perp\perp}$. For $G \in \mathbf{W}^*$, one has also the lattice $\mathrm{coz}(G)$ and the commutative diagram

$$\begin{tikzcd}
\mathrm{PI}(G) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \mathrm{PP}(G) \\
& \mathrm{coz}(G) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

where β and γ are the surjective lattice homomorphisms given by $\beta(G(g)) = \mathrm{coz}(g)$ and $\gamma(\mathrm{coz}(g)) = g^{\perp\perp}$, respectively. It is easy to see that G has (Y) (resp., (CR)) if and only if β (resp., γ) is an isomorphism, and G has (M) if and only if each of α, β, and γ is an isomorphism.

2. The class (Y) in $\mathbf{W}^*$

We remind the reader that (in general) $G \in (\mathrm{Y})$ means that each $G(g)$ is an intersection of maximal ideals.

Theorem 2.1. *For $(G, u) \in \mathbf{W}^*$, the following are equivalent:*

(1) $G \in (\mathrm{Y})$.
(2) $\forall g \in G,\ G(g) = I(Z(g))$.
(3) $\forall g \in G,\ G(g)$ *is a* $\mathbf{W}^*$*-kernel.*
(4) $\forall g \in G,\ G/G(g)$ *is Archimedean.*
(5) *The Yosida embedding* $G \leq C(Y(G,u))$ *has the property:* $\mathrm{coz}(f) \subseteq \mathrm{coz}(g) \Rightarrow f \in G(g)$.

Proof. (1) $\Leftrightarrow$ (2) is Corollary 1.6(a).

(2) $\Rightarrow$ (3) from Facts 1.4(c).

(3) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (3). If $G/G(g)$ is Archimedean, the quotient map $G \twoheadrightarrow G/G(g)$ is a $\mathbf{W}^*$-map (since any quotient preserves strong unit), so $G(g)$ is a $\mathbf{W}^*$-kernel.

(3) $\Rightarrow$ (1). Any $\mathbf{W}^*$-kernel is an intersection of maximal ideals (Facts 1.4(b)). (3) says $Z(f) \supseteq Z(g) \Rightarrow f \in G(g)$, which is what (2) says.

(5) $\Leftrightarrow$ (2). Follows easily from the fact that $\mathrm{coz}(f) \subseteq \mathrm{coz}(g)$ if and only if $Z(f) \supseteq Z(g)$. ☐

Remark 2.2. Assuming nothing about G, [16: Proposition 5.2] says $G \in (\mathrm{Y})$ if and only if there is an embedding $G \hookrightarrow \mathbb{R}^X$ (for some X) with the property (5) of Theorem 2.1 (which specifies the embedding).

3. The class (CR) in $\mathbf{W}^*$

We remind the reader that $G \in (\mathrm{CR})$ was defined in the Abstract as: Each $g^{\perp\perp}$ is a $\mathbf{W}^*$-kernel, and this was shown in Facts 1.4 to be equivalent to: Each $\mathrm{coz}(g)$ is RO – regular open (hence "CR", for "cozero-regular").

We shall create a compendium of conditions equivalent to "$G \in (\mathrm{CR})$", at least eight. To avoid the mind-numbing effect of a single list of these, we first make two sub-groups of these, of rather similar conditions, as the following two lemmas.

Lemma 3.1 (Regularity conditions). *For $G \in \mathbf{W}^*$, the following are equivalent:*

(R1) $G \in (\mathrm{CR})$.
(R2) $\forall f, g \in G$, *if* $\mathrm{coz}(f) \subseteq \overline{\mathrm{coz}(g)}$, *then* $\mathrm{coz}(f) \subseteq \mathrm{coz}(g)$.
(R3) $\forall g \in G,\ I(Z(g)) = I(\mathrm{cl}\,\mathrm{int} Z(g))$ *(i.e., $Z(g) = \mathrm{cl}\,\mathrm{int} Z(g)$).*
(R4) $\forall g \in G,\ I(Z(g)) = g^{\perp\perp}$.

Proof. (R2) says that $\mathrm{coz}(g)$ is RO, i.e. (R1), and (R3) says $Z(g)$ is regular closed, which is equivalent to $\mathrm{coz}(g) = YG \setminus Z(g)$ is RO. From Facts 1.4, $g^{\perp\perp} = I(\mathrm{cl}\,\mathrm{int} Z(g))$, so (R4) if and only if (R3). ☐

Now, in any G, an ideal I is called a d-ideal if $g \in I$ implies $g^{\perp\perp} \subseteq I$. (For further comment on this, see Remark 3.4 below.) Observe that (i) any $g^{\perp\perp}$ is a d-ideal, and (ii) any intersection of d-ideals is a d-ideal.

Lemma 3.2 (d-ideal conditions)**.** *For $G \in \mathbf{W}^*$, the following are equivalent:*

(d1) *Each $\mathbf{W}^*$-kernel of G is a d-ideal.*

(d2) *$\forall g \in G$, $I(Z(g))$ is a d-ideal.*

(d3) *$\forall p \in YG$, $M_p \equiv \{g \in G \mid g(p) = 0\}$ is a d-ideal.*
(In our general notation, $M_p = I(\{p\})$.)

Proof. Recall from Facts 1.4(b) that the $\mathbf{W}^*$-kernels are the $I(S)$, S closed in YG.

(d3) $\Rightarrow$ (d1) by (ii) above, and (d1) $\Rightarrow$ (d2) obviously.

Suppose (d2) and $g \in M_p$. Then $p \in Z(g)$, so $M_p \supseteq I(Z(g))$. Thus $g^{\perp\perp} \subseteq I(Z(g)) \subseteq M_p$ and (d3) holds. $\square$

(d1) of Lemma 3.2 is a very curious property.

Theorem 3.3. *For $G \in \mathbf{W}^*$, the following are equivalent:*

(1) *$G \in$ (CR) (any/all of the conditions in* Lemma 3.1 *hold).*

(2) *For each $g \in G$, $I(Z(g))$ is a d-ideal (any/all of the conditions in* Lemma 3.2 *hold).*

(3) *In G, each weak unit is strong (see* Facts 1.4(f)*).*

Proof. (1) $\Leftrightarrow$ (2), in the form Lemma 3.1(R4) $\Leftrightarrow$ Lemma 3.2(d2): ($\Rightarrow$) $g^{\perp\perp}$ is always a d-ideal; ($\Leftarrow$) If $I(Z(g))$ is a d-ideal, then $g^{\perp\perp} \subseteq I(Z(g))$, thus $g^{\perp\perp} = I(Z(g))$ from Facts 1.4.

(1) $\Rightarrow$ (3), in the form Lemma 3.1(R4) $\Rightarrow$ (3). Suppose v is a weak unit, so $v^{\perp\perp} = G$. By (R4), $v^{\perp\perp} = I(Z(v))$, so $Z(v) = \emptyset$. Thus, v is a strong unit (Facts 1.4(f)).

(3) $\Rightarrow$ (1), in the form (3) $\Rightarrow$ Lemma 3.1(R2). Suppose (R2) fails, with $f, g > 0$: $\mathrm{coz}(f) \subseteq \overline{\mathrm{coz}(g)}$ but $\mathrm{coz}(f) \nsubseteq \mathrm{coz}(g)$. Choose $p \in \mathrm{coz}(f) \setminus \mathrm{coz}(g)$. Then, there is $h > 0$ with $h(p) = 0$ and $Z(f) \subseteq \mathrm{coz}(h)$ (since G separates closed sets in YG). Let $k = h+g$. We have $k(p) = h(p)+g(p) = 0+0 = 0$, so $\mathrm{coz}(k) = \mathrm{coz}(h) \cup \mathrm{coz}(g) \neq YG$, and k is not a strong unit. But k is a weak unit, because $\mathrm{coz}(k)$ is dense in YG: if $\emptyset \neq U$ is open, and if $U \cap \mathrm{coz}(h) = \emptyset$, then $U \subseteq Z(h) \subseteq \mathrm{coz}(f)$, so $U \subseteq \overline{\mathrm{coz}(g)}$ and therefore $U \cap \mathrm{coz}(g) \neq \emptyset$. $\square$

Remark 3.4. For $(G, u) \in \mathbf{W}$ – meaning u is merely a weak unit – we again have the Yosida space $Y(G, u) = YG$, and the Yosida representation now as

$$G \hookrightarrow D(YG) = \{f \in C(YG, [-\infty, +\infty]) \mid f^{-1}(-\infty, +\infty) \text{ dense}\},$$

so again the family $\mathrm{coz}(G)$, and $(G, u) \in$ (CR) can be defined. (One sees that $(G, u) \in$ (CR) if and only if $(G(u), u) \in$ (CR) in $\mathbf{W}^*$.) This "(CR) in $\mathbf{W}$" is addressed in [5: Theorem 5.3], which focuses on d-ideals and includes the condition (d3) of Lemma 3.2 here. The present treatment of (CR) owes much to that result.

4. The class (M) in $\mathbf{W}^*$

We exhibit a few ways of saying $G \in$ (M) within $\mathbf{W}^*$. (Of which there are many: Corollary 1.6 shows (M) = (Y) $\cap$ (CR), and Section 2 gives five ways of saying $G \in$ (Y) and Section 3 gives eight ways of saying $G \in$ (CR).)

Corollary 4.1. *For $G \in \mathbf{W}^*$, the following are equivalent:*

(1) *$G \in$ (M).*

(2) *$\forall g \in G$, $G(g)$ is a d-ideal.*

(3) *$\forall f, g \in G$, if $\mathrm{coz}(f) \subseteq \overline{\mathrm{coz}(g)}$, then $f \in G(g)$.*

(4) *$\forall g \in G$, $G/G(g)$ is Archimedean, and in G every weak unit is strong.*

Proof. (2) says $g^{\perp\perp} \subseteq G(g)$, thus $g^{\perp\perp} = G(g)$ by Facts 1.4.

(3) is Theorem 2.1(5) plus Lemma 3.1(R2).

(4) is Theorem 2.1(4) plus Theorem 3.3(3). ☐

Now we relate (M) to two well-studied properties (see [10], inter alia).

Definitions and Facts 4.2.

(a) G is hyperarchimedean (HA) if every quotient of G is Archimedean (see [9]).

(b) G is projectable (Pr) if each $g^{\perp\perp}$ is an ℓ-group direct summand.

(c) For $G \in \mathbf{W}^*$, as usual taking the "Yosida view" $G \le C(YG)$:
  - (i) $G \in$ (HA) if and only if each $\operatorname{coz}(g)$ is closed (see [12]). (One sees (HA) $\subseteq$ (M), e.g., by Corollary 4.1.)
  - (ii) $G \in$ (Pr) if and only if each $\overline{\operatorname{coz}(g)}$ is open (see [13]).

Bigard [6] notes that in general ℓ-groups (Pr) $\cap$ (M) $=$ (HA). This sharpens in $\mathbf{W}^*$.

Corollary 4.3. *In $\mathbf{W}^*$,* (Pr) $\cap$ (CR) $=$ (HA).

Proof. Evident using 4.2(b). ☐

5. Some examples

Within $\mathbf{W}^*$, we shall: (1) for compact K, indicate when $C(K)$ is in one class or another;

(2) give examples showing that (M) is neither (Y) nor (CR) (thus (M) $=$ (Y) $\cap$ (CR) is more or less sharp);

(3) give examples in (M) (especially not (HA)).

Theorem 5.1. *Suppose K is compact.*

(a) *The following are equivalent:*
  - (i) *$C(K) \in$ (HA);*
  - (ii) *$C(K) \in$ (Y);*
  - (iii) *$C(K) \in$ (M);*
  - (iv) *K is finite.*

(b) $C(K) \in$ (CR) *if and only if K is an almost-P-space ($\equiv$ every nonempty zero-set has nonempty interior, or, is regular-closed; see* [15]*).*

Proof. (a) If K is finite, then all subsets, and hence all cozero-sets, are closed, so $C(K) \in$ (HA) $\subseteq$ (M) $\subseteq$ (Y). If K is not finite, a non-closed $\operatorname{coz}(g)$ is easily found (so $C(K) \notin$ (HA)), and then (take $g > 0$), $G(g^2) \neq I(Z(g^2))$ (with $\sqrt{g^2} = g$), so Theorem 2.1(2) fails and $C(K) \notin$ (Y).

(b) Clear. ☐

Several examples to follow use the one-point compactification of the discrete natural numbers $\mathbb{N}$, denoted $\alpha\mathbb{N} = \mathbb{N} \cup \{\alpha\}$.

***Example* 5.2.** (Y) $\nsubseteq$ (CR).

We give three (types of) examples.

(1) Let $G \leq C(\alpha\mathbb{N})$ be generated by $F \equiv \{f \in C(\alpha\mathbb{N}) \mid f(\alpha\mathbb{N}) \text{ finite}\}$ and

$$v(x) = \begin{cases} \dfrac{1}{x}, & x \in \mathbb{N}, \\ 0, & x = \alpha. \end{cases}$$

Note that $g \in G$ can be put $(*)$ $g \doteq r + sv$ for some $r, s \in \mathbb{R}$, with $\doteq$ denoting "eventually equal on $\mathbb{N}$". Here, $\operatorname{coz}(v) = \mathbb{N}$, so v is a weak unit, not strong. Thus $G \notin$ (CR). That $G \in$ (Y) can be checked using any of Theorem 2.1.

(2) (A simplification of [16: Example 5.5]). Let $G \leq C([0, +\infty])$ consist of the g which are "finitely piecewise linear" ($[0, +\infty] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n]$, with $x_0 = 0$, $x_n = +\infty$, and g linear on each $[x_i, x_{i+1}]$). Take $g \in G$ such that

$$g(x) = \begin{cases} x, & x \in [0, 1], \\ 1, & x \in [1, +\infty]. \end{cases}$$

Then g is a weak unit that is not strong, which shows that $G \notin$ (CR). That $G \in$ (Y) can be checked using Theorem 2.1(2) and some arithmetic.

(3) Let n be a positive integer, and let $\mathrm{FVL}(n)$ be the free vector lattice on n generators. It is well-known (see, e.g., [2]) that $\mathrm{FVL}(n)$ may be represented as the sub-vector lattice of $C(\mathbb{R}^n)$ generated by the coordinate projections, i.e., by $\{\pi_i\}_{i=1}^n$ where $\pi_i(x_1, \dots, x_n) = x_i$ for $i \in \{1, \dots, n\}$, and that $u = \sum\limits_{i=1}^{n} |\pi_i|$ is a strong unit in $\mathrm{FVL}(n)$. Thus $(\mathrm{FVL}(n), u) \in \mathbf{W}^*$. As discussed in [8: Part II], there is a vector-lattice embedding η: $\mathrm{FVL}(n) \to C(S^{n-1})$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$, such that $\operatorname{coz}(\eta(\mathrm{FVL}(n)))$ is a base for the topology of S^{n-1}. Dividing the functions in $\eta(\mathrm{FVL}(n))$ by u yields the Yosida representation of $\mathrm{FVL}(n)$ and gives $Y(\mathrm{FVL}(n), u) = S^{n-1}$. Then [8: Lemma 3.5] shows that $\mathrm{FVL}(n)$ obeys Theorem 2.1(5), so $\mathrm{FVL}(n) \in$ (Y).

For $n > 1$, one may check that π_1 is a weak unit that's not strong (or use [8: Lemma 3.6]), so $\mathrm{FVL}(n) \notin$ (CR) by Theorem 3.3. Hence, $\mathrm{FVL}(n) \notin$ (M) when $n > 1$. However, since S^0 is the two-point space, one may apply Lemma 3.1(R2) to see that $\mathrm{FVL}(1) \in$ (CR), and hence that $\mathrm{FVL}(1) \in$ (M).

***Example* 5.3.** (CR) $\nsubseteq$ (Y).

We give two (types of) examples.

(1) $C(K)$ for any K infinite, compact, and almost-P is (CR) not (Y).

(2) (Reconsidering Example 5.2(1)) Let $G \leq C(\alpha\mathbb{N})$ be generated by F, a, and b, where

$$a(x) = \begin{cases} \dfrac{1}{x}, & x \text{ is even}, \\ 0, & x \text{ is odd}, \end{cases} \qquad \text{and} \qquad b(x) = \begin{cases} \dfrac{1}{x^2}, & x \text{ is even}, \\ 0, & x \text{ is odd}. \end{cases}$$

Observe that $g \in G$ can be put

$$(*)\, g \doteq r + sa + tb \quad \text{for some } r, s, t \in \mathbb{R},$$

so that $\operatorname{coz}(g)$ differs by a finite set in $\mathbb{N}$ from either a clopen $U \ni \alpha$, or from the evens. Thus $G \in$ (CR). $G \notin$ (Y) since $\operatorname{coz}(a) = \operatorname{coz}(b)$ but $a \notin G(b)$.

***Example* 5.4.** Some $G \in$ (M).

Of course, (HA) $\subseteq$ (M), and any $F(K, \mathbb{R}) \in$ (HA), and many "weird" members of (HA) are exhibited in [12].

Beyond (HA), and resembling Example 5.2(1) and Example 5.3(2): let $G \leq C(\alpha\mathbb{N})$ be generated by F and

$$a(x) = \begin{cases} \dfrac{1}{x}, & x \text{ is even,} \\ 0, & x \text{ is odd.} \end{cases}$$

As before, $G \in$ (CR). Also, $G \in$ (Y) is visible from Theorem 2.1(5). So $G \in$ (M).

(This kind of example can be manufactured replacing $\alpha\mathbb{N}$ by any K compact zero-dimensional space with a proper dense cozero-set S. S will be a disjoint union $\bigcup\limits_n U_n$, with the U_n's clopen. Then, "split" $\{U_n\}$ into "evens and odds", and define a accordingly.)

A more elaborate $G \leq C(\alpha\mathbb{N})$ with (M) may be obtained as follows. Take $\mathcal{A}$ an infinite family of "almost disjoint" infinite subsets of $\mathbb{N}$ (i.e., $A_1 \neq A_2$ in $\mathcal{A}$ implies $|A_1 \cap A_2| < \omega$). For $A \in \mathcal{A}$, let $g_A \in C(\alpha\mathbb{N})$ be given by

$$g_A(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Let G be the subgroup of $C(\alpha\mathbb{N})$ generated by $\{g_A \mid A \in \mathcal{A}\} \cup F$. It can be shown that G is a sub-ℓ-group of $C(\alpha\mathbb{N})$, that $G \notin$ (HA), and that $G \in$ (M).

Question 5.5. In $\mathbf{W}^*$, what does $G \in$ (M), (Y), or (CR) entail about the Yosida space YG?

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