Houston Journal of Mathematics
© University of Houston
Volume , No. ,

THE YOSIDA SPACE OF THE VECTOR LATTICE HULL OF AN ARCHIMEDEAN ℓ-GROUP WITH UNIT

RICHARD N. BALL, ANTHONY W. HAGER, DONALD G. JOHNSON, JAMES J. MADDEN, AND WARREN WM. MCGOVERN

Communicated by Yasunao Hattori

Abstract. $\mathbf{W}$ is the category of archimedean ℓ-groups with distinguished weak order unit. For $G \in \mathbf{W}$, we have the contravariantly functorial Yosida space $\mathcal{Y}G$. For an embedding $G \leq H$, the resulting $\mathcal{Y}G \leftarrow \mathcal{Y}H$ is surjective; when this is one-to-one, we write "$\mathcal{Y}H = \mathcal{Y}G$". This is the case with the divisible hull $G \leq dG$, where, always, $\mathcal{Y}dG = \mathcal{Y}G$; however for the vector lattice hull $G \leq vG$, we frequently have $\mathcal{Y}vG \neq \mathcal{Y}G$. *Theorem.* A compact space $\mathcal{X}$ is quasi-F if and only if: $\forall G \in \mathbf{W}$ with $\mathcal{Y}G = \mathcal{X}$, also $\mathcal{Y}vG = \mathcal{X}$. ("quasi-F" means each dense cozero set is C*-embedded.)

1. Introduction/preliminaries

In $\mathbf{W}$ (or more generally), a *hull class* $\mathbf{A}$ *with hull operator* $\mathbf{a}$ is an isomorphisn closed object class $\mathbf{A}$ in $\mathbf{W}$ together with the operator $\mathbf{a}$ which satisfies: for each $G \in \mathbf{W}$ there is $G \leq \mathbf{a}G$ which is a unique minimum essential extension to an $\mathbf{A}$-object. Then, any essential extension $G \leq A_0 \in \mathbf{A}$ contains (a model of) $\mathbf{a}G$, as $\mathbf{a}G = \cap \{A \in \mathbf{A} : G \leq A \leq A_0\}$. If, further, $\mathbf{A}$ is a (essential mono-) reflective subcategory (any $G \to A \in \mathbf{A}$ lifts (uniquely) over $\mathbf{a}G$), then any $G \leq A_0 \in \mathbf{A}$ (not assumed essential) contains $\mathbf{a}G$. (See [2], [5], [17], [15].)

This paper considers the effect of the vector lattice hull/monoreflection on Yosida spaces.

2010 *Mathematics Subject Classification.* 06F20, 46A40, 54B35, 54G05, 18A40, 46E05, 54C30.

Key words and phrases. Archimedean ℓ-group, vector lattice reflection, Yosida space, quasi-F-space.

For $G \in \mathbf{W}$, the *Yosida space* of G is the compact Hausdorff space $\mathcal{Y}G$ of ideals of G maximal for not containing the unit (the "values of the unit"), with the hull-kernel topology. The *Yosida representation* of G is the embedding of G into the lattice $D(\mathcal{Y}G)$ (extended-real (almost finite) continuous functions) carrying the unit to the constant function 1; the image of G separates the points of $\mathcal{Y}G$, and $\mathcal{Y}G$ is unique for these features. We identify G with its image. For any $G \in \mathbf{W}$, G^* denotes the set of bounded elements of G. Thus, $G^* = G \cap C(\mathcal{Y}G) \in \mathbf{W}$ and $\mathcal{Y}G^* = \mathcal{Y}G$.

Further, "$\mathcal{Y}$" is functorial: if $G \overset{\varphi}{\to} H \in \mathbf{W}$, there is the unique continuous $\mathcal{Y}G \overset{\mathcal{Y}\varphi}{\leftarrow} \mathcal{Y}H$ for which $\varphi(g) = g \circ \mathcal{Y}\varphi$, given as $\mathcal{Y}\varphi(M) = \varphi^{-1}M$. (See [18].) If φ is one-to-one, then $\mathcal{Y}\varphi$ is surjective.

The *divisible hull/monoreflection*, d, preserves $\mathbf{W}$, and for $G \in \mathbf{W}$, is $dG = \{rg : g \in G,\ r \in \mathbb{Q}\} \leq D(\mathcal{Y}G)$ (as is easily seen). It results (by the uniqueness above, or otherwise) that $\mathcal{Y}dG = \mathcal{Y}G$. (Divisible hulls exist in torsion-free abelian groups, so in abelian ℓ-groups, and hence in archimedean ℓ-groups, thence in $\mathbf{W}$. See [2], [1].)

The vector lattice hull/monoreflection, v, exists for archimedean ℓ-groups and preserves $\mathbf{W}$. (See [4], [3].) But (in contrast with d), $\{rg : g \in G,\ r \in \mathbb{R}\}$ may not be closed under addition in $D(\mathcal{Y}G)$ (see §3 below). This means that for the embedding $G \hookrightarrow vG$, the Yosida surjection $\mathcal{Y}G \overset{\tau}{\leftarrow} \mathcal{Y}vG$ need not be one-to-one: "$\mathcal{Y}vG \neq \mathcal{Y}G$". We shall reserve "$\tau$" for this surjection ($G$ being understood). This paper considers how - and the degree to which - τ can fail to be one-to-one. Our most incisive observation is the theorem in the abstract, Theorem 3.1 below, with elaboration in Theorem 3.5.

We shall need some facts about "covers" of compact spaces. See [11] and [24] for details.

In compact Hausdorff spaces, with all maps continuous: $\mathcal{X} \overset{f}{\twoheadleftarrow} \mathcal{Y}$ is called *irreducible* if: $\mathcal{F}$ proper closed in $\mathcal{Y} \Longrightarrow f(\mathcal{F}) \neq \mathcal{X}$. Then $(\mathcal{Y}, f)$ is called a *cover* of $\mathcal{X}$. Such a map inversely preserves dense sets. The present relevance of these functions is: in $\mathbf{W}$, an extension $G \leq H$ is essential iff its Yosida map $\mathcal{Y}G \twoheadleftarrow \mathcal{Y}H$ is a cover of $\mathcal{Y}G$; also, given $G \in \mathbf{W}$ and a cover $\mathcal{Y}G \overset{f}{\leftarrow} \mathcal{X}$, G embeds into the lattice $D(\mathcal{X})$, as $g \mapsto g \circ f$. (See [19].)

We note a few items about covers. Given two covers of $\mathcal{X}$, say $(\mathcal{Y}_i, f_i)$, $i = 1, 2$, if there is $\mathcal{Y}_1 \overset{h}{\leftarrow} \mathcal{Y}_2$ with $f_2 = f_1 \circ h$, we write $(\mathcal{Y}_1, f_1) \leq (\mathcal{Y}_2, f_2)$, and say the two covers are "equivalent" if h is a homeomorphism. The collection of equivalence classes of covers of $\mathcal{X}$ is a set, denoted $\operatorname{cov} \mathcal{X}$; it is also a complete lattice. Note that, given $\mathcal{X}$ and $\mathcal{S}$ dense in $\mathcal{X}$, the unique $\mathcal{X} \overset{f}{\twoheadleftarrow} \beta\mathcal{S}$ extending the inclusion $\mathcal{S} \hookrightarrow \mathcal{X}$ has $(\beta\mathcal{S}, f) \in \operatorname{cov}\mathcal{X}$.

Various of these details will be used without explicit mention.

2. Real ideals

For $G \in \mathbf{W}$, the *real ideal space* of G is

$$\mathcal{R}G \equiv \{M \in \mathcal{Y}G : G/M \leq \mathbb{R}\}$$

So, viewing $G \leq D(\mathcal{Y}G)$,

$$\begin{aligned} \mathcal{R}G &= \{p \in \mathcal{Y}G : g(p) \in \mathbb{R}, \forall\, g \in G\} \\ &= \cap \left\{g^{-1}(\mathbb{R}) : g \in G\right\}. \end{aligned}$$

If $\mathcal{R}G$ is dense in $\mathcal{Y}G$, then G may be called an "ℓ-group of real-valued functions", because $G \ni g \mapsto g\,|_{\mathcal{R}G} \in C(\mathcal{R}G)$ is one-to-one. If $G \in \mathbf{W}^*$ (meaning the unit is strong), then $\mathcal{R}G = \mathcal{Y}G$ and $G \leq C(\mathcal{Y}G)$.

We take note of the effect of v, or of any essential monoreflection, on $\mathcal{R}G$.

Proposition 2.1. *([9]). Suppose $\mathbf{W} \overset{a}{\to} \mathbf{A}$ is an essential monoreflection.*

(a) $\forall\, G \leq aG$, the associated $\mathcal{Y}G \overset{\mu}{\twoheadleftarrow} \mathcal{Y}aG$ is a cover for which $\mu(\mathcal{Y}aG \smallsetminus \mathcal{R}aG) = \mathcal{Y}G \smallsetminus \mathcal{R}G$, and μ restricts to a homeomorphism $\mathcal{R}G \leftarrow \mathcal{R}aG$.

(b) $\forall\, \mathcal{X}$, $C(\mathcal{X}) \in \mathbf{A}$.

(c) If $\mathcal{R}G$ is dense in $\mathcal{Y}G$, then $aG \leq C(\mathcal{R}G)$, so $\mathcal{Y}aG$ is a compactification of $\mathcal{R}G$.

(d) If $G \in \mathbf{W}^*$, then $aG \leq C(\mathcal{Y}G)$, so $\mathcal{Y}aG = \mathcal{Y}G$.

We are concerned here with the particular case of $a = v$, the vector lattice reflector. Our basic question is "What is (or can be) $\mathcal{Y}\mathcal{G} \overset{\tau}{\twoheadleftarrow} \mathcal{Y}vG$?" Repeating some of the above for $a = v$:

Corollary 2.2. *Suppose $G \in \mathbf{W}$, with $G \leq vG$ and its irreducible map $\mathcal{Y}G \overset{\tau}{\twoheadleftarrow} \mathcal{Y}vG$.*

(a) If $M \in \mathcal{R}G$ then $\left|\tau^{-1}(M)\right| = 1$.

(b) If $\mathcal{R}G$ is dense in $\mathcal{Y}G$, then $vG \leq C(\mathcal{R}G)$ and $\mathcal{Y}vG$ is a compactification of $\mathcal{R}G$.

For this situation, one can pose some detailed questions, which we have not answered except in trivial cases.

Questions: *For* G as in Corollary 2.2,

(a) For any $M \in \mathcal{Y}G$, what does "$\left|\tau^{-1}(M)\right| = 1$" mean? Is there an algebraic condition on M or on G/M?

(b) With $\mathcal{R}G$ proper and dense in $\mathcal{Y}G$, when is $vG = C(\mathcal{R}G)$? (When is $\mathcal{Y}vG = \beta\mathcal{R}G$?)

The inscrutability here partly explains why our approach to the question "What is $\mathcal{Y}G \overset{\tau}{\twoheadleftarrow} \mathcal{Y}vG$?" has been converted by replacing $\mathcal{Y}G$ by a compact space $\mathcal{X}$, and then tailoring those Gs for which $\mathcal{Y}G = X$ - as in the theorem of the abstract, whose proof we now turn to.

3. Quasi-F-spaces and the Theorem

A space $\mathcal{X}$ is called *quasi-F (QF)* if each dense cozero-set in $\mathcal{X}$ is C^*-embedded. These spaces were introduced (without naming) in [21], to the purpose: in $D(\mathcal{X})$, the partially defined operation $+$ (and/or $\cdot$) is fully defined iff $\mathcal{X}$ is QF. Then, clearly, $D(\mathcal{X}) \in \mathbf{W}$ and is a vector lattice.. The name QF was given in [7] where it is shown: for compact $\mathcal{X}$, there is the minimum QF cover, $\mathcal{X} \overset{\sigma}{\twoheadleftarrow} QF\mathcal{X}$, namely $QF\mathcal{X} = \varprojlim \{\beta S : S \in \operatorname{dcoz}\mathcal{X}\}$, where $\operatorname{dcoz}\mathcal{X}$ denotes the collection of dense cozero sets in $\mathcal{X}$. We reserve "σ" for this surjection. (See also [25] and [22] for considerable further information about "QF".)

Thus, for $G \in \mathbf{W}$, G embeds into $D(QF\mathcal{Y}G)$ as $g \to g \circ \sigma$; we write $G \leq D(QF\mathcal{Y}G)$. We have $G \leq vG \leq D(QF\mathcal{Y}G)$, representing an upper bound for vG. By Yosida, and cover theory, σ factors uniquely as $\mathcal{Y}\mathcal{G} \overset{\tau}{\twoheadleftarrow} \mathcal{Y}vG \twoheadleftarrow QF\mathcal{Y}G$, representing an upper bound, in the sense of covers, for $\mathcal{Y}vG$. We show in Theorem 3.5 below that these upper bounds are, in a certain sense, "least".

Theorem 3.1. *Suppose $\mathcal{X}$ is a compact space. For each $G \in \mathbf{W}$ with $\mathcal{Y}G = \mathcal{X}$, also $\mathcal{Y}vG = \mathcal{X}$ if and only if $\mathcal{X}$ is QF.*

Lemma 3.2 below is, in detail, the construction for the proof of Theorem 3.1. This will find further purpose. The construction is an elaboration of [13], Example 1.

Note that if $\mathcal{X}$ is compact and $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, then there are various $f \in D(\mathcal{X})^+$ with $f^{-1}(\mathbb{R}) = \mathcal{S}$: for any $w \in C(\mathcal{X})^+$, with $\operatorname{coz} w = \mathcal{S}$, define $f \in D(\mathcal{X})$ as $1/w$ on $\mathcal{S}$ and $+\infty$ on $\mathcal{X} \smallsetminus \mathcal{S}$.

Lemma 3.2. *Suppose $\mathcal{X}$ is a compact space. Suppose $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, $f \in D(\mathcal{X})^+$ with $f^{-1}(\mathbb{R}) = \mathcal{S}$, and $u \in C^*(S)$. There is $G = G(f,u) \leq C(\mathcal{S})$ for which: $f \in G$; $G^* = C(\mathcal{X})$ (so $\mathcal{Y}G = \mathcal{X}$); and $u \in vG$. Since $vG \leq C(\mathcal{S})$, the natural* *map* $\mathcal{X} \twoheadleftarrow \beta\mathcal{S}$ *(Čech-Stone extension of* $\mathcal{S} \hookrightarrow \mathcal{X}$*) factors as* $\mathcal{X} = \mathcal{Y}G \overset{\tau}{\twoheadleftarrow} \mathcal{Y}vG \twoheadleftarrow \beta\mathcal{S}$.

We first prove Theorem 3.1 from Lemma 3.2, then prove Lemma 3.2.

Proof of Theorem 3.1. If $\mathcal{X}$ is not QF, there is $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$ that is not C^*-embedded in $\mathcal{X}$: there is $u \in C^*(\mathcal{S})$ that cannot be extended over $\mathcal{X}$. Take $f \in D(\mathcal{X})^+$ with $f^{-1}(\mathbb{R}) = \mathcal{S}$. Then, Lemma 3.2 provides a group $G = G(f,u)$, for which the mapping $\mathcal{Y}G \overset{\tau}{\twoheadleftarrow} \mathcal{Y}vG$ is not one-to-one: $\mathcal{Y}G \neq \mathcal{Y}vG$.

If $\mathcal{X}$ is QF, then $\mathcal{X} = QF\mathcal{X}$, so when $\mathcal{Y}G = \mathcal{X}$ the usual $\mathcal{Y}G \twoheadleftarrow \mathcal{Y}vG \twoheadleftarrow QF\mathcal{Y}G$ is, in fact, $\mathcal{X} = \mathcal{Y}G \leq \mathcal{Y}vG \leq QF\mathcal{X} = \mathcal{X}$. □

Proof of Lemma 3.2. Let $\mathcal{X}, \mathcal{S}$ and f be as stated. Then, $\mathcal{S} = \cup_n f^{-1}[0,n]$. For $g, h \in C(\mathcal{S})$, define:

$$g \doteq h \text{ if } \exists\, n \in \mathbb{N} \text{ with } f(x) > n \Longrightarrow g(x) = h(x).$$

(Here, we say: "g and h are eventually equal, with respect to $\mathcal{S}$.")

Now let u be as stated, and let $\gamma \in \mathbb{R} \smallsetminus \mathbb{Q}$. Define $G = G(f,u) \subseteq C(\mathcal{S})$:

$$\begin{aligned} &g \in G \text{ means } g \in C(\mathcal{S}) \text{ and for some } a, b \in \mathbb{Z} \text{ and } w \in C(\mathcal{X})\,|_{\mathcal{S}}, \\ &g \doteq af + b\gamma(f+u) + w = (a+b\gamma)f + (b\gamma u + w) \qquad (\dagger) \end{aligned}$$

(Note that this also depends upon γ, which seems immaterial.)

Note, first, that $f \in G$ ($a = 1$, $b = 0$, $w = \mathbf{0}$) and $C(\mathcal{X})\,|_{\mathcal{S}} \subseteq G$ ($a = 0 = b$). Clearly, $g \in G \Rightarrow -g \in G$, and if $g_1, g_2 \in G$ are expressed (eventually) in the form (†) then their sum eventually can be expressed in that form. So, G is a subgroup of $D(\mathcal{S})$.

Each member of G is either unbounded, in which case it is dominated, eventually, by the $(a+b\gamma)f$ term in (†) (a and b cannot both be 0), or it is bounded ($a = b = 0$) and so is in $C(\mathcal{X})\,|_{\mathcal{S}} \subseteq G$ (so, in fact, $G^* = C(\mathcal{X})\,|_{\mathcal{S}}$). It follows that $G \in \mathbf{W}$, as we now demonstrate.

Suppose $g_1, g_2 \in G$: $g_i = (a_i + b_i\gamma) f(x) + (b_i\gamma u(x) + w_i(x))$ whenever $f(x) > n_i$, for $i = 1, 2$.

Suppose g_1 is unbounded. Then, $a_1 + b_1\gamma \neq 0$, and either:

- $a_1 + b_1\gamma > a_2 + b_2\gamma$ (or the reverse) and so, eventually, $g_1 > g_2$, and $g_1 \vee g_2 \doteq (a_1 + b_1\gamma) f + (b_1\gamma u + w_1)$.

  (The details.

$$g_1 - g_2 \doteq [(a_1 + b_1\gamma) - (a_2 + b_2\gamma)] f + [(b_1 - b_2)\gamma u + (w_1 - w_2)].$$

The second term is a function in $C^*(\mathcal{S})$ so there is $n \in \mathbb{N}$ which satisfies:

$$|(b_1 - b_2)\gamma u(x) + (w_1 - w_2)(x)| < n \text{ for all } x \in \mathcal{S}.$$

Now choose $m \in \mathbb{N}$ with

$$m > n_1 \vee n_2 \vee \frac{n}{(a_1 + b_1\gamma) - (a_2 - b_2\gamma)}.$$

Now when $f(x) > m$, we have $(g_1 - g_2)(x) > 0$, so $g_1 \vee g_2 \doteq g_1$.)

Or,

- $a_1 + b_1\gamma = a_2 + b_2\gamma$, so $a_1 = a_2$ and $b_1 = b_2$; whence, eventually, $g_1 - g_2 = w_1 - w_2$ and we have: $g_1 \vee g_2 \doteq (a_1 + b_1\gamma) f + b_1\gamma u + w_1 \vee w_2$.

Suppose g_1 and g_2 are both bounded ($a_1 = b_1 = 0 = a_2 = b_2$). Then, eventually $g_1 \vee g_2 \doteq w_1 \vee w_2$.

Thus, in every case $g_1 \vee g_2 \in G$, so $G \in \mathbf{W}$.

Observe that with $g \equiv f$ and $h \equiv \gamma(f + u)$, we have $\frac{1}{\gamma}h - g = u \in vG$. □

We would like, have tried, to extend the construction in Theorem 3.1 to obtain the answer to (Q_1) $\forall\ \mathcal{X}\ \exists\ G$ with $\mathcal{Y}G = \mathcal{X}$ and $\mathcal{Y}vG = QF\mathcal{X}$? A first step might be extending Lemma 3.2 to: (Q_2) $\forall\ \mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, $\exists\ G(\mathcal{S})$ with $\mathcal{Y}G(\mathcal{S}) = \mathcal{X}$ and $\mathcal{Y}vG(\mathcal{S}) = \beta\mathcal{S}$? One tries to prove this *via* a construction like that in the proof of Lemma 3.2, but letting the u there range over **all of** $C^*(\mathcal{S})$, defining a *set* G to be all those functions $g \in G(\mathcal{S}, f, \gamma)$ for which there are $a, b \in \mathbb{Z}$, $w \in C(\mathcal{X})$ and *some* $u \in C^*(\mathcal{S})$ with $g \doteq af + b\gamma(f + u) + w$. But, even using only two functions, u_1 and u_2, this process fails to always yield a group: in many situations, the set G will contain two functions whose sum fails to be extendable over $\mathcal{X}$.

On the other hand, here is a partial answer to (Q_2).

Theorem 3.3. *Suppose $\mathcal{X}$ is compact, $\mathcal{S} \in \operatorname{dcoz}(\mathcal{X})$, and the cardinal of $C^*(\mathcal{S})$ is c (cardinal of the reals). Then there is $G \in W$ with $G \leq C(\mathcal{S})$, $\mathcal{Y}G = \mathcal{X}$, and $(vG)^* = C^*(\mathcal{S})$. Thus $\mathcal{Y}vG = \beta\mathcal{S}$.*

Proof. $\mathcal{S}$ is locally compact, so has its one-point compactification $\alpha\mathcal{S} = \mathcal{S} \cup \{\alpha\}$, and $\mathcal{S}$ is σ-compact, so $\mathcal{S} \in \operatorname{dcoz}\alpha\mathcal{S}$. As explained at the end, the result for the general $\mathcal{X}$ follows from the result for just $\mathcal{X} = \alpha\mathcal{S}$, which we now prove.

The method is that of Lemma 3.2, elaborated with:

Let $\mathcal{H}$ be a Hamel basis for the reals with $1 \in \mathcal{H}$, and let $b : \mathcal{H} \to C^*(\mathcal{S})$ be a bijection with $b(1) = 0$.

Now take $f \in C(\mathcal{S})^+$ with extension in $D(\alpha\mathbb{N})$ (again "f"), having $f(\alpha) = +\infty$. For $h \in \mathcal{H}$, put $u(h) \equiv h \cdot (f + b(h)) \in C(\mathcal{S})$. Note that $u(h) = 0$ iff $h = 0$ iff $u(h)$ is bounded; and all $u(h) \in D(\alpha\mathbb{N})$ (by extension); $u(1) = f$.

Let $\mathcal{U}$ consist of all finite sums $\sum f_i u(h_i)$ with the $r_i \in \mathbb{Q}$. This is the $\mathbb{Q}$-vector lattice in $C(\mathcal{S})$ generated by the $u(h)$'s. Let $u \in \mathcal{U}$:

$$u = \sum r_i u(h_i) = \sum r_i h_i (f + b(h_i)) = \left(\sum r_i h_i\right) \cdot f + \sum r_i h_i b(h_i) = \gamma_u \cdot f + w.$$

Note that w is bounded, and that u is bounded iff $\gamma_u = 0$ iff all $r_i = 0$ (by the $\mathbb{Q}$-linear independence) iff $u = 0$. Thus $\mathcal{U} \subseteq D(\alpha\mathcal{S})$ (by extension).

Now define $G \subseteq C(\mathcal{S})$ as in Lemma 3.2: $g \in G$ means $g \doteq u + c$ for some $u \in \mathcal{U}$ and $c \in \mathbb{R}$; g is bounded iff $g \doteq c$. Evidently, G is a group, $G \subseteq D(\alpha\mathcal{S})$ (by extension), and G is a lattice as in the proof of Lemma 3.2. So we have $YG = \alpha\mathcal{S}$.

For $vG \leq C(\mathcal{S})$, we have: For $h \neq 0$, $\frac{1}{h}u(h) - f = b(h) \in vG$. I.e., $C^*(\mathcal{S}) \leq vG \leq C(\mathcal{S})$. Thus $\mathcal{Y}vG \leq \beta\mathcal{S}$.

Finally: Suppose $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, i.e., $\mathcal{X}$ is an arbitrary compactification of $\mathcal{S}$, instead of the above αS. Let $\alpha\mathcal{S} \overset{\tau}{\leftarrow} \mathcal{X}$ extend the identity on S. Let G be as constructed above, using $\alpha\mathcal{S}$, $G \approx G \circ \tau \leq D(\mathcal{X}) \cap C(\mathcal{S})$. Let $H \equiv jm(G \circ \tau + C(\mathcal{X})) \leq D(\mathcal{X}) \cap C(\mathcal{S})$. (See §4 about "$jm$".) We have $H^* = C(\mathcal{X})$, so that $\mathcal{Y}H = \mathcal{X}$. (If this is not obvious, see [6], 2.6.) Since $G \leq H \leq C(\mathcal{S})$, we have $vG \leq vH \leq C(\mathcal{S})$ (see §1); $\mathcal{Y}vH = \beta\mathcal{S}$ follows. □

Remark 3.4. **a):** *Theorem 3.3 applies to any compactification $\mathcal{X}$ of metrizable $\mathcal{S}$ which is infinite, locally compact and σ-compact (for then $\mathcal{S}$ is separable, so $|C^*(\mathcal{S})| = c$).*

b): *One wonders if Theorem 3.3 can be extended to $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$ (with $|C^*(\mathcal{S})| = c$). A particular case of this is $\mathcal{X} = [0,1]$, $\mathcal{S}$ its irrational points, where $QF\mathcal{X}$ is the projective cover of $[0,1]$.*

c): *Here is a very weak partial answer "yes" to (Q_1): "yes" for $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, $|C^*(\mathcal{S})| = c$, $\mathcal{S}$ itself QF, because the G produced in Theorem 3.3 has $\mathcal{Y}vG = \beta\mathcal{S}$ and $\beta\mathcal{S} = QF\mathcal{X}$. Examples of such $\mathcal{S}$ are $\mathcal{S} = \sum \mathcal{Y}_n$, with $\forall n$ $\mathcal{Y}_n$ infinite compact QF with $|C(\mathcal{Y}_n)| \leq c$ (for then $|C^*(\mathcal{S})| = \prod_n |C(\mathcal{Y}_n)| = c^{\aleph_0} = c$). Compact QF $\mathcal{Y}$ with $|C(\mathcal{Y})| \leq c$ include $\mathcal{Y} = \beta\mathbb{N}$ and $\mathcal{Y} = \alpha D(m)$, where $D(m)$ is discrete of cardinal $m \leq c$.*

On the other hand again, another close look at Lemma 3.2 reveals what might be called a weak answer to (Q_1) of a different sort, *per* the last sentence before Theorem 3.1. Given $\mathcal{X}$, we make G "closely tied" to $\mathcal{X}$, with vG "very large" in $D(QF\mathcal{X})$; in particular, with $\mathcal{Y}vG = QF\mathcal{X}$. Probably, refinement of this is possible; here, we only sketch the details.

Theorem 3.5. *Suppose $\mathcal{X}$ is a compact space. There is a family $\{G_i\}_{i\in I} \subseteq \mathbf{W}$ with the following properties.*

- **a):** *For each $i \in I$, $\mathcal{Y}G_i = \mathcal{X}$; consequently, $G_i \leq vG_i \leq D(QF\mathcal{X})$.*
  Let G be the sub-ℓ-group of $D(QF\mathcal{X})$ generated by $\cup_{i\in I} G_i$.
- **b):** *G is order-cofinal in $D(QF\mathcal{X})$.*
- **c):** *$(vG)^*$ is uniformly dense in $C(QF\mathcal{X})$. Thus, $\mathcal{Y}vG = QF\mathcal{X}$.*
- **d):** *$QF\mathcal{X} = \vee_{i\in I}\mathcal{Y}vG_i = \mathcal{Y}vG$ (the* sup *in the sense of* $\operatorname{cov}\mathcal{X}$*).*

Proof. Given $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, Lemma 3.2 provides the groups $G(f,u) \leq C(\mathcal{S})$; relabel this $G(\mathcal{S},f,u)$. Our index set, I, will consist of all such triples $(\mathcal{S},f,u)$: $i \in I$ means $i = (\mathcal{S},f,u)$ and $G_i = G(\mathcal{S},f,u)$ with $\mathcal{Y}G_i = \mathcal{X}$. Thus, we have a).

We will use some details from [7], §3. For each $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, we have the cover $\mathcal{X} \twoheadleftarrow \beta\mathcal{S} \twoheadleftarrow QF\mathcal{X}$. Now, $QF\mathcal{X} = \varprojlim\{\beta\mathcal{S} : \mathcal{S} \in \operatorname{dcoz}\mathcal{X}\}$, and

$$QF\mathcal{X} = \vee\{\beta\mathcal{S} : \mathcal{S} \in \operatorname{dcoz}\mathcal{X}\}$$

expresses $QF\mathcal{X}$ in the lattice $\operatorname{cov}\mathcal{X}$ (abusing notation). If $\mathcal{S} \in \operatorname{dcoz}\mathcal{X}$, then $\mathcal{X} \twoheadleftarrow \beta\mathcal{S}$ gives an embedding $C(\mathcal{S}) \leq D(QF\mathcal{X})$, and $\cup_{\mathcal{S}\in \operatorname{dcoz}\mathcal{X}} C(\mathcal{S})$ is uniformly dense in $D(QF\mathcal{X})$, thus order-cofinal, and $\cup_{\mathcal{S}\in \operatorname{dcoz}\mathcal{X}} C^*(S)$ is uniformly dense in $C(QF\mathcal{X})$.

From these facts, and Lemma 3.2, our assertions will follow.

Fix $\mathcal{S}$. Set $\mathcal{S}' = \left\{f \in D(\mathcal{X})^+ : f^{-1}\mathbb{R} = \mathcal{S}\right\}$. Then $\mathcal{S}'$ "$\subseteq$" $C(\mathcal{S})$ and is order cofinal there. (Choose $f \in \mathcal{S}'$; then $\forall\ g \in C(\mathcal{S})$, $f \vee g$ (really, $f\ |_{\mathcal{S}} \vee g$) extends over $\mathcal{X} \setminus \mathcal{S}$ by defining $f \vee g(x) = +\infty$ there, so $f \vee g \in \mathcal{S}'$.) Since each $f \in G(\mathcal{S},f,u)$, we have $G_{\mathcal{S}} \equiv \underset{(f,u)}{\cup} G(\mathcal{S},f,u)$, and thus $G = \vee_i G_i = \vee_{\mathcal{S}} G_{\mathcal{S}}$ is order cofinal in $D(QF\mathcal{X})$. Hence, b).

Since $u \in vG(\mathcal{S},f,u)$ always, we have $C^*(\mathcal{S}) \leq vG_{\mathcal{S}}$, and since $\cup_{\mathcal{S}} C^*(\mathcal{S})$ is uniformly dense in $C(QF\mathcal{X})$, so is $\cup_{\mathcal{S}}(vG_{\mathcal{S}})^*$, and hence also the larger $(vG)^*$. Thus, c).

Finally, $C^*(\mathcal{S}) \leq vG_{\mathcal{S}} \leq D(QF\mathcal{X})$ yields $\beta\mathcal{S} \twoheadleftarrow \mathcal{Y}vG_{\mathcal{S}} \leq QF\mathcal{X}$ and, taking suprema in $\operatorname{cov}\mathcal{X}$, we get $QF\mathcal{X} = \vee_{\mathcal{S}}\mathcal{Y}vG_{\mathcal{S}} \leq \mathcal{Y}vG \leq QF\mathcal{X}$. Thus, d). □

4. Addenda

We take this opportunity to correct the following situation. In [13], Theorem 2(a) states: If A is a reduced archimedean f-ring ("**frA** object"), then its vector lattice hull, vA, is a reduced archimedean f-algebra ("**frAa** object"). The proof there was inadequate.

If G is an archimedean ℓ-group ("**Arch** object"), its "essential completion" takes the form $G \leq D(\mathcal{X})$, where $\mathcal{X}$ is extremally disconnected (so $D(\mathcal{X})$ is a vector lattice). A model of vG is the sub-vector lattice of $D(\mathcal{X})$, generated by G. (See [4], [3].)

Note that when $D(\mathcal{X})$ is a group, it is also a ring, so is in $|\mathbf{frA}|$). When $G \in |\mathbf{frA}|$, the embedding $G \leq D(\mathcal{X})$ can be an **frA**-embedding; that this is so can be demonstrated using a representation by Bernau or Johnson - see the discussion in §§3.4 and 5.3 in [12].

Proof of the theorem. Suppose $H \subseteq D(\mathcal{X})$, with $\mathcal{X}$ extremally disconnected. Let sH denote the sub-vector space of $D(\mathcal{X})$ generated by H :

$$sH \equiv \left\{\sum r_i h_i : 1 \leq i \leq n \in \mathbb{N},\ r_i \in \mathbb{R},\ h_i \in H\right\}.$$

Note that if H is a subring of $D(\mathcal{X})$, then so is sH.

The sublattice of $D(\mathcal{X})$ generated by H is:

$$mjH \equiv \left\{\bigwedge_j \bigvee_i h_{ij} : m, n \in \mathbb{N},\ 1 \leq i \leq m,\ 1 \leq j \leq n,\ h_{ij} \in H\right\}.$$

If H is a subgroup of $D(\mathcal{X})$, then mjH is a subgroup; also, if H is a subring of $D(\mathcal{X})$, then mjH is a subring. (See Theorem 2.1 in [14].) In any vector lattice L it is true that ([23], Theorem 11.5(vi)): for any $a, b \in L$ and $r \in \mathbb{R}$, the following identities, and their duals, hold:

$$r(a \vee b) = ra \vee rb \text{ when } r \geq 0,$$
$$r(a \vee b) = ra \wedge rb \text{ when } r \leq 0.$$

It follows that mjH is a vector lattice if H is a sub-vector space of $D(\mathcal{X})$ and it is a sub-f-ring of $D(\mathcal{X})$ when H is a sub-algebra of $D(\mathcal{X})$.

Thus, if $A \in |frA|$, with $A \leq D(\mathcal{X})$, then

$$vA = mj(sA),$$

so $vA \in |\mathbf{frA}|$. □

The proof of the theorem in [13] was inadequate in that we mis-interpreted [4] to say: for $G \leq D(\mathcal{X})$, $vG = sG$. We thank G. Buskes for questioning this.

Note that, in the argument above, the requirement that $\mathcal{X}$ be extremally disconnected is stronger than necessary: "$\mathcal{X}$ is QF" will do and - as we have seen - less than that will not do.

In another vein, we have recently been considering questions concerning subsets of $A \subseteq D(\mathcal{X})$ and functions $f \in C(\mathbb{R})$ for which $f \circ A \subseteq A$ (which means $g \in A \Longrightarrow g^{-1}(\mathbb{R}) \xrightarrow{g} \mathbb{R} \xrightarrow{f} R$ extends to a function in A (one says that A is *closed under composition* with f)). This notion, along with its analogs for $f \in C(\mathbb{R}^n)$, $n = 2, 3, \cdots, \omega$, has been used to good effect in similar settings (e.g.,[20], [9], [10], [16]). Here, we include the following very easy result, which complements Theorem 3.1 above.

Theorem 4.1. *For $\mathcal{X}$ compact, $C^*(\mathbb{R}) \circ D(\mathcal{X}) \subseteq D(\mathcal{X})$ iff $\mathcal{X}$ is QF.*

Proof. Suppose $C^*(\mathbb{R}) \circ D(\mathcal{X}) \subseteq D(\mathcal{X})$ and let $\mathcal{S} \in \operatorname{dcoz} \mathcal{X}$. If $g \in C^*(\mathcal{S})$ and $\mathbb{N} \ni n \geq |g(x)|\ \forall\ x \in \mathcal{S}$, set $f \equiv [(-\mathbf{n}) \vee \mathbf{1}_{\mathbb{R}}] \wedge \mathbf{n} \in \mathbf{C}^*(\mathbb{R})$, where $\mathbf{1}_{\mathbb{R}}(r) = r$ for each $r \in \mathbb{R}$. Then $f \circ g$ extends to a function on $D(\mathcal{X})$ which must be g, since $f \circ g = g$ on $\mathcal{S}$.

Now suppose $\mathcal{X}$ is QF, $g \in D(\mathcal{X})$, and $f \in C^*(\mathbb{R})$. Then $g^{-1}(\mathbb{R}) \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R} \in C^*\left(g^{-1}(\mathbb{R})\right)$, so extends over $\mathcal{X}$, since $g^{-1}(\mathbb{R}) \in \operatorname{dcoz} \mathcal{X}$, so is C^*-embedded in $\mathcal{X}$. □

Acknowledgements. We thank the referee for a careful reading and valuable suggestions.

References

[1] Anderson, Marlow; Conrad, Paul *Epicomplete ℓ-groups.* Algebra Universalis 12 (1981), no. 2, 224-241.

[2] Bigard, Alain; Keimel, Klaus; Wolfenstein, Samuel *Groupes et Anneaux Reticules.* Lecture Notes in Mathematics, Vol. 608, Springer-Verlag, Berlin-New York (1977)

[3] Bleier, Roger D. *Minimal vector lattice covers.* Bull. Austral. Math. Soc. 5 (1971), 331-335

[4] Conrad, Paul F. *Minimal vector lattice covers.* Bull. Austral. Math. Soc. 4 (1971), 35-39

[5] Conrad, P. F. *The hulls of representable ℓ-groups and f-rings.* J. Austral. Math. Soc. 26 (1973), 385-415

[6] Carrera, R. E.; Hager, A. W. *B-saturated hull classes in ℓ-groups and covering classes of spaces.* Appl. Categor. Struct. 23 (2015), 709 - 723

[7] Dashiell, F.; Hager, A.; Henriksen, M. *Order-Cauchy completions of rings and vector lattices of continuous functions.* Canad J. Math.2 (1980), no. 3, 657-685

[8] Gillman, Leonard, Jerison, Meyer, *Rings of Continuous Functions,* Reprint of the 1960 edition. Graduate Texts in Mathematics, No. 43. Springer-Verlag, New York-Heidelberg, 1976

[9] Hager, Anthony W. *$C(\mathcal{X})$ has no proper functorial hulls.* Rings of Continuous Functions (Cincinnati, Ohio, 1982), 149-164, Lecture Notes in Pure and Appl. Math., 95, Dekker, New York, 1985

[10] Hager, Anthony W. *Algebraic closures of ℓ-groups of continuous functions.* Rings of Continuous Functions (Cincinnati, Ohio, 1982), 165-194, Lecture Notes in Pure and Appl. Math., 95, Dekker, New York, 1985

[11] Hager, Anthony W. *Minimal covers of topological spaces.* Papers on general topology and related category theory and topological algebra (New York, 1985-1987), 44-59, Ann. New York Acad. Sci., 552. New York Acad. Sci., New York, 1989

[12] Hager, Anthony W. *Some unusual epicomplete archimedean lattce-ordered groups,* Proc. Amer. Math. Soc., Vol. 143 (2015), No. 5, 1969-1980

[13] Hager, A. W.; Johnson, D. G. *Adjoining an identity to a reduced archimedean f-ring. II.* Appl. Categ. Structures 15 (2007), no. 1-2, 35-47

[14] Hager, A. W.; Johnson, D. G. *Some comments and examples on generation of (hyper-)archimedean ℓ-groups and f-rings.* Ann. Fac. Sci. Toulouse Math. (6) 19 (2010), Fascicule Special. 75-100

[15] Hager, A. W.; Madden, James J. *Essential reflections versus minimal embeddings.* J. Pure Appl. Algebra 37 (1985), no. 1, 27-32

[16] Hager, Anthony W.; Madden, James J. *The H-closed monoreflections, implicit operations, and countable composition, in archimedean lattice-ordered groups with weak unit.* Applied Categorical Structures, 24 (2016), 605 - 617

[17] Hager, Anthony W.; Martinez, Jorge *Hulls for various kinds of α-completeness in Archimedean lattice-ordered groups.* Order 16 (1999), no. 1, 89-103.

[18] Hager, Anthony W.; Robertson, Lewis C. *Representing and ringifying a Riesz space.* Symposia Mathematica, Vol. XXI (Convegno sulle Misure su Gruppi e su Spazi Vettoriali, Convegno sui Gruppi e Anelli Ordinati, INDAM, Rome, 1975), 411-431. Academic Pres, London, 1977

[19] Hager, Anthony W.; Robertson, Lewis C. *On the embedding into a ring of an Archimedean* ℓ-group. Canad. J. Math. 31 (1979), no. 1, 1-8

[20] Henriksen, M.; Isbell, J. R.; Johnson, D. G. *Residue class fields of lattice-ordered algebras.* Fund. Math. 50 (1961/1962), 107-117

[21] Henriksen, M.; Johnson, D. G. *On the structure of a class of archimedean lattice-ordered algebras.* Fund. Math. 50 (1961/1962),73-94

[22] Henriksen, M.; Vermeer, J.; Woods, R. G. *Quasi F-covers of Tychonoff spaces.* Trans. Amer. Math. Soc. 303 (1987), no. 2, 779-803

[23] Luxemburg, W. A. J.; Zaanen, A. C. *Riesz Spaces I.* North Holland, Amsterdam, The Netherlands (1971)

[24] Porter, Jack R.; Woods, R. Grant *Extensions and Absolutes of Hausdorff Spaces.* Springer-Verlag, New York, 1988. xiv+856 pp. ISBN: 0-387-96212-3

[25] Zaharov, V. K.; Koldunov, A. V. *Sequential absolute and its characterization.* Dokl. Akad. Nauk SSSR 253 (1980), no. 2, 280-284

Received May 19, 2016
Revised version received August 3, 2016
Second revision received September 5, 2016

Dept. of Math, Univ. of Denver, Denver, CO 80208, USA
E-mail address: rball@du.edu

Dept. of Math and CS, Wesleyan Univ. Middletown, CT 06459, USA
E-mail address: ahager@wesleyan.edu

Current address: 4351 Nambe Arc, Las Cruces, NM 88011, USA
E-mail address: dgjohnson@member.ams.org

Dept. of Math, LSU, Baton Rouge, LA 70803, USA
E-mail address: jamesjmadden@gmail.com

Wilkes Honors College, Florida Atlantic Univ., Jupiter, FL 33458, USA
E-mail address: warren.mcgovern@fau.edu