A Characterization of Commutative Clean Rings

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Abstract. A commutative ring A is said to be *clean* if every element of A can be written as a sum of a unit and an idempotent. This definition dates back to 1977 where it was introduced by W. K. Nicholson [7]. In 2002, V. P. Camillo and D. D. Anderson [1] investigated commutative clean rings and obtained several important results. In [4] Han and Nicholson show that if A is a semiperfect ring, then $A[\mathbb{Z}_2]$ is a clean ring. In this paper we generalize this argument (for commutative rings) and show that $A[\mathbb{Z}_2]$ is clean if and only A is clean. We also show that if the group ring $A[G]$ is a commutative clean ring, then G must be a torsion group. Our investigations lead us to introduce the class of 2-clean rings.

Keywords: commutative clean ring, commutative group ring

1. 2-Clean Rings

Definition 1.1. The element $a \in A$ is said to be *clean* if there exists an idempotent $e \in A$ such that $a - e$ is invertible. If every element of A is clean, then A is said to be a *clean ring.*

The set of units and idempotents of A shall be denoted by $\mathcal{U}(A)$ and $Id(A)$, respectively. As usual $Max(A)$ denotes the set of maximal ideals of A, and it is equipped with the hull-kernel topology. This means that the collection of sets of the form $V(a) = \{M \in Max(A) : a \in M\}$ for $a \in A$ form a base for the closed sets. The complement of $V(a)$ is denoted by $U(a)$.

It is well known that von Neumann regular rings, and more generally zero-dimensional rings are clean. Also, local rings are clean, and, in fact, are precisely the indecomposable clean rings. Thus, if A is an integral domain, then A is clean if and only if A is local. Some fundamental facts regarding clean rings are that every homomorphic image of a clean ring is a clean ring, and that a ring A is clean if and only if $A/n(A)$ is clean where $n(A)$ denotes the nilradical of A. Also, a direct product of rings is clean if and only if each factor is clean. Most of these facts can be found in [4] or [1]. We urge the interested reader to check there for their proofs.

All rings are commutative and with identity.

In [6] a list of several characterizations of commutative clean rings is given. Included in the list is one given by Johnstone [5] which we presently state.

Theorem 1.2 [Johnstone]**.** *A is a clean ring if and only if $Max(A)$ is zero-dimensional and every prime ideal is contained in a unique maximal ideal.*

A topological space is said to be *zero-dimensional* if it has a base of clopen sets. It is known that if $e \in Id(A)$, then $U(e)$ is a clopen subset of $Max(A)$ but the reverse is not true in general. A clopen subset of $Max(A)$ of the form $U(e)$ for some $e \in Id(A)$ will be called an *idempotent clopen.* In [6] it is shown that $Max(A)$ is zero-dimensional and every clopen is an idempotent clopen precisely when A is a clean ring.

Theorem 1.3. *The following are equivalent for a ring A*

(i) *A is a clean ring.*

(ii) *For each $a \in A$ there exists an idempotent e such that $V(a) \subseteq U(e)$ and $V(a-1) \subseteq V(e)$.*

(iii) *The collection of idempotent clopen subsets of $Max(A)$ is base for the topology on $Max(A)$.*

(iv) *$Max(A)$ is zero-dimensional and every clopen subset is of the form $U(e)$ for some $e \in Id(A)$.*

Here is another characterization of a clean ring.

Proposition 1.4. *A is clean if and only if for every $a \in A$, there is an idempotent $e \in A$ such that both $a + e$ and $a - e$ are invertible.*

Proof. The sufficiency is clear. As for the necessity, let $a \in A$. By Theorem 1.3, choose an idempotent $e \in A$ such that $V(a^2) \subseteq U(e)$ and $V(a^2 - 1) \subseteq V(e)$. Let M be any maximal ideal of A. If $a \in M$, then so is a^2 and hence $e \notin M$. It follows that $a + e, a - e \notin M$. If $a, e \notin M$, then neither $a + 1$ nor $a - 1$ belong to M. Thus, $a + e + M = a + 1 + M \neq M$ and $a - e + M = a - 1 + M \neq M$, whence $a + e, a - e$ are invertible.■

Definition 1.5. We call a commutative ring A *2-clean* if for every pair of elements $a, b \in A$ there exists an idempotent $e \in A$ such that $a+b-e$ and $a-b-e$ are both invertible. Observe that letting $b=0$ we obtain that a 2-clean ring is clean. Also, if $char A = 2$, then A is clean if and only if A is 2-clean.

The proof of our next proposition is straightforward.

Proposition 1.6. *1) A homomorphic image of a 2-clean ring is 2-clean.*
2) A direct product $A = \Pi_{i \in I} A_i$ is 2-clean if and only if each factor A_i is 2-clean.

Proposition 1.7. *The following are equivalent for a ring A.*

(i) A is 2-clean.

(ii) $A/n(A)$ is 2-clean.

(iii) A is clean and $A/\mathfrak{J}(A)$ is 2-clean.

(iv) $A/\mathfrak{J}(A)$ is 2-clean and idempotents lift modulo the Jacobson radical.

Proof. The proof is exactly the same as the one used to show that a ring A is clean if and only if $A/n(A)$ is clean. We leave the verification to the interested reader. ■

Our next result gives an easier way of checking whether a clean ring is 2-clean.

Proposition 1.8. *A is 2-clean if and only if A is clean and $2 \in \mathfrak{J}(A)$.*

Proof. Necessity. If A is 2-clean, then it is clean. Let $a \in A$ be an arbitrary element. Using the 2-clean condition on the pair a, a there is an idempotent $e \in A$ such that $2a - e$ and $-e$ (and hence e) are invertible. The only idempotent that is invertible is $e = 1$. Therefore, $2a - 1$ is invertible, whence $2 \in \mathfrak{J}(A)$.

Sufficiency. Let $a, b \in A$. Choose an idempotent $e \in A$ such that $a+b-e$ is invertible. We claim that $a-b-e$ is also invertible. Let M be a maximal ideal of A. By hypothesis, $2 \in M$ so that $a-b-e+M = a+b-e+M \neq M$. Therefore, $a-b-e \notin M$ for any maximal ideal of A. It follows that both $a-b-e$ and $a+b-e$ are invertible.■

Theorem 1.9. *Let A be a zero-dimensional ring. Then the following are equivalent:*

(i) A is 2-clean.

(ii) $char A = 2^k$ for some natural k.

(iii) 2 is nilpotent.

(iv) For each maximal ideal M of A, $char A/M = 2$.

Proposition 1.10. *Suppose A is a local ring. Then A is 2-clean if and only if $char A/M = 2$.*

Proof. The necessity is clear so suppose that $char A/M = 2$. This means that $a+b+M = a-b+M$ for all $a, b \in A$. Therefore, $a+b \in M$ if and only if $a-b \in M$. Thus, if both $a+b, a-b \in M$, then $a+b-1, a-b-1 \notin M$ and hence both are invertible. Otherwise, $a+b, a-b \notin M$ means that they are both invertible.■

2. Commutative Clean Group Rings

In this section we address the question of when a group ring is clean. Some work on this question was done in [4] where they showed that if A is a boolean ring and G is a torsion group, then $A[G]$ is a clean ring. We will only consider the case where the group in question is abelian. When a group ring is zero-dimensional or local has been characterized and therefore we have some sufficient conditions for a group ring to be a clean ring. We list [3] as our main reference on commutative group rings, though we warn the casual reader that most of the information in [3] is done for semi-group rings.

All groups are abelian.

Definition 2.1. Let A be a ring and $(G, +)$ a group. The group ring is the set of formal sums of the form $\Sigma_{i=1}^{n} a_i X^{g_i}$ under pointwise addition and multiplication defined by $X^g X^h = X^{g+h}$ on monomials and extending this in the obvious way.

Theorem 2.2. *$A[G]$ is zero-dimensional if and only if A is zero-dimensional and G is a torsion group.*

For commutative rings and abelian groups we obtain the following corollary.

Corollary 2.3. *If A is a boolean ring and G is a torsion group, then $A[G]$ is a clean ring.*

Theorem 2.4. *$A[G]$ is local if and only if A is local, $char A/M = p$, and G is a p-group. In particular, if $A[G]$ is local, then G is torsion.*

Our aim is to show that for $A[G]$ to be a clean ring it is necessary that A be clean and G be torsion. The first part is clear as A is a homomorphic image of $A[G]$. To obtain the second part we need a series of lemmas. The first is well known.

Lemma 2.5. *$A[G]$ is an integral domain if and only if A is an integral domain and G is torsion free.*

Lemma 2.6. *For any ring A and group G, $A[G]$ is clean if and only if $A/n(A)[G]$ is clean.*

Proof. The necessity is obvious since $A/n(A)[G]$ is a homomorphic image of $A[G]$. As for the sufficiency it is known that $n(A)[G] \leq n(A[G])$, which means that $A[G]/n(A[G])$ is a homomorphic image of $A/n(A)[G]$ and therefore $A[G]$ is clean.■

Proposition 2.7. *If $A[G]$ is a clean ring, then A is clean and G is a torsion group.*

Proof. Clearly, A is a clean ring. Let $t(G)$ denote the torsion subgroup of G. Let P be a prime ideal of A. Then $(A/P)[G/t(G)]$ is a homomorphic image of $A[G]$ and hence is clean. Since $G/t(G)$ is a torsion-free group and A/P is an integral domain it follows by the Lemma 2.5 that $(A/P)[G/t(G)]$ is a clean integral domain and hence is local. By Theorem 2.4 it follows that $G/t(G)$ is torsion, whence $G = t(G)$.■

Remark 2.8. The converse of the previous proposition is certainly not true. In [4] it is shown that if A is the localization of $\mathbb{Z}$ at the prime $7\mathbb{Z}$ (A is clean because it is local) and $G = \mathbb{Z}_3$ (clearly torsion), then $A[G]$ is not clean. In general, discovering when $A[G]$ is clean has eluded us except for in certain cases. In the next section we consider the specific case when $G = \mathbb{Z}_2$.

Proposition 2.9. *Suppose A is zero-dimensional. Then the following are equivalent:*

(i) $A[G]$ *is clean.*

(ii) $A[G]$ *is zero-dimensional.*

(iii) G *is a torsion group.*

Proof. This is in fact a corollary to Proposition 2.7 and Theorem 2.2.■

Corollary 2.10. *Let F be a field. Then $F[G]$ is clean if and only if G is torsion.*

3. When $G = \mathbb{Z}_2$

In this section we investigate the cleanliness of the group ring $A[\mathbb{Z}_2]$.

Definition 3.1. Let e be an idempotent of A. The element $eX^0 \in A[G]$ is an idempotent of $A[G]$. Any idempotent of $A[G]$ of this form is said to be an *idempotent in A*. If every idempotent of $A[G]$ is in A, then we say that the *idempotents of $A[G]$ are in A*.

Lemma 3.2. *The element $rX^0 + sX^1 \in A[\mathbb{Z}_2]$ is invertible if and only if $r+s, r-s \in \mathcal{U}(A)$.*

Proof. Suppose $rX^0+sX^1 \in \mathcal{U}(A[\mathbb{Z}_2])$. Then there are $f, g \in A$ such that $(rX^0+sX^1)(fX^0+gX^1) = 1$. This means that $rf + sg = 1$ and $rg + sf = 0$. Adding these two equations gives us that $(r + s)(f + g) =$. Subtracting yields $(r - s)(f - g) = 1$. Therefore, $r + s, r - s \in \mathcal{U}(A)$.

Conversely, if $r + s, r - s \in \mathcal{U}(A)$, then so is $r^2 - s^2$. Let $f = \frac{r}{r^2-s^2}$ and $g = \frac{-a}{r^2-s^2}$. A quick check shows that $rf + sg = 1$ and $rg + sf = 0$. It follows that $rX^0 + sX^1$ is an invertible element of $A[\mathbb{Z}_2]$.■

Lemma 3.3. *The element $cX^0 + dX^1 \in A[\mathbb{Z}_2]$ is an idempotent if and only if $c^2 + d^2 = c$ and $2cd = d$. In this case, $c + d, c - d \in Id(A)$.*

Proof. The first statement follows from the equation $cX^0 + dX^1 = (cX^0 + dX^1)^2 = (c^2 + d^2)X^0 + 2cdX^1$. The second is obtained by adding (and subtracting) the equations.■

Proposition 3.4. *The following are equivalent for the ring A:*

(i) A *is clean.*

(ii) For every $a \in A$, the element $aX^0 \in A[\mathbb{Z}_2]$ can be written as the sum of a unit and an idempotent in A.

(iii) For every $a \in A$, the element $aX^1 \in A[\mathbb{Z}_2]$ can be written as the sum of a unit and an idempotent in A.

Proof. (i) ⇒ (ii) If A is clean, then for each $a \in A$ let $u \in \mathcal{U}(A)$ and $e \in Id(A)$ such that $a = u + e$. Then

$$aX^0 = uX^0 + eX^0$$

is a representation of aX^0 as a sum of a unit of and an idempotent in A.

(ii) ⇒ (i) Let $a \in A$. If
$$aX^0 = (rX^0 + sX^1) + eX^0$$
is a representation of aX^0 as a sum of a unit and an idempotent in A, then $s = 0$ and thus $r = r + s \in \mathcal{U}(A)$. Therefore, A is a clean ring.

(iii) ⇒ (i) Let $a \in A$. Suppose that aX^1 is the sum of a unit and an idempotent in A, say
$$aX^1 = (rX^0 + aX^1) + eX^0.$$
for some $e \in Id(A)$. By Lemma 3.2 $r + a, a - r \in \mathcal{U}(A)$. Since $r = -e$ it follows that $a - e$ and $a + e$ are both invertible and so by Proposition 1.4 A is clean.

(i) ⇒ (iii) Let $a \in A$. By Proposition 1.4 there exists an $e \in Id(A)$ such that $a - e, a + e$ are invertible. Set $r = -e$ so that
$$aX^1 = (rX^0 + aX^1) + eX^0.$$
Since $r + a = a - e$ and $r - a = -(a + e)$ are invertible it follows that aX^1 is the sum of a unit and an idempotent in A.■

Proposition 3.5. *For every $a \in A$, the element $aX^0 + aX^1 \in A[\mathbb{Z}_2]$ can be written as the sum of a unit and an idempotent in A if and only if $2 \in \mathfrak{J}(A)$. In this case every idempotent of $A[\mathbb{Z}_2]$ belongs to A.*

Proof. Suppose $aX^0 + aX^1 \in A[\mathbb{Z}_2]$ can be written as the sum of a unit and an idempotent in A for every $a \in A$, say
$$ax^0 + aX^1 = (rX^0 + sX^1) + eX^0.$$
Since $s = a$ it follows that $r + a, r - a \in \mathcal{U}(A)$ by Lemma 3.2. Notice that $e = a - r$ which is invertible and hence $e = 1$ and $r = a - 1$. Thus $r + a = 2a - 1 \in \mathcal{U}(A)$ for every $a \in A$, whence $2 \in \mathfrak{J}(A)$.

Conversely, let $a \in A$ and $2 \in \mathfrak{J}(A)$. Since
$$aX^0 + aX^1 = ((a - 1)X^0 + aX^1) + X^0$$
and $(a - 1) + a = 2a - 1$ and $(a - 1) - a = -1$ are both invertible, the result follows from Lemma 3.2.

Finally, suppose $2 \in \mathfrak{J}(A)$ and let $aX^0 + bX^1$ be idempotent. This means that $a^2 + b^2 = a$ and $2ab = b$. The latter equality implies that $(2a - 1)b = 0$. By hypothesis, $2 \in \mathfrak{J}(A)$ and therefore $2a - 1$ is invertible. It follows that $b = 0$ and so the idempotents of $A[\mathbb{Z}_2]$ belong to A.■

Proposition 3.6. *The following are equivalent.*

(a) For each $a \in A$, the element $aX^0 + aX^1$ is clean.

(b) For each $a \in A$ there exists an $e \in Id(A)$ and $t, u \in A$ such that $(2a - 1)t = (1 - e)$, and $2au = e$.

(c) For each $a \in A$, there exists an $e \in Id(A)$ such that $V(2a) \subseteq V(e)$ and $V(2a - 1) \subseteq U(e)$.

Proof. (b)$\Rightarrow$(a). Let $a \in A$ and choose $e \in Id(A)$ and $t, u \in A$ satisfying (b). Let $f = 1 - e$ so that $ef = 0$ and $e + f = 1$. Observe that since both e and f are idempotent we may without loss of generality assume that $ue = u$, and $tf = t$. Let $x = aue$ and observe that $2x = e$ and $ex = x$. Let $y = (a-1)f$, $z = (a-x)e$, and $r = y + z$. Next let $d = -x$ and $c = d + 1 = 1 - x$.

Now,

$$c^2 + d^2 = d^2 + 2d + 1 + d^2 = 2d^2 + 2d + 1 = 2x^2 - 2x + 1 = x + f.$$

Notice that $2x = e = 1 - f$ so that $c = 1 - x = x + f$ and therefore $c^2 + d^2 = c$. Next,

$$2cd = 2c(-x) = -ce = -(1-x)e = ex - e = x - e = x - 2x = -x = d$$

and so $cX^0 + dX^1$ is an idempotent. Now,

$$\begin{aligned} r + c &= y + z + c \\ &= (a-1)f + (a-x)e + 1 - x \\ &= af - f + ae - ex + 1 - x \\ &= a - f - ex + 1 - x \\ &= a + e - 2x \\ &= a \end{aligned}$$

so that

$$aX^0 + aX^1 = (rX^0 + (r+1)X^1) + (cX^0 + dX^1).$$

What is left to be shown is that $(rX^0 + (r+1)X^1)$ is invertible. To that end, first notice that

$$\begin{aligned} 2r + 1 &= 2y + 2z + 1 \\ &= 2(a-1)f + 2(a-x)e + e + f \\ &= (2a-1)f + 2ae. \end{aligned}$$

Next,

$$\begin{aligned} (2r+1)(t+u) &= ((2a-1)f + 2ae)(t+u) \\ &= (2a-1)t + 2aue \\ &= f + e \\ &= 1. \end{aligned}$$

Therefore $rX^0 + (r+1)X^1$ is invertible by Lemma 3.2, whence $aX^0 + aX^1$ is clean.

(a)$\Rightarrow$(c). Suppose $aX^0 + aX^1$ is clean and write it as a sum of a unit and idempotent, say

$$aX^0 + aX^1 = (rX^0 + sX^1) + (cX^0 + dX^1).$$

Note that $a = r + c = s + d$. Since $(rX^0 + sX^1)$ is invertible, $r + s \in \mathcal{U}(A)$, and since $(cX^0 + dX^1)$ is an idempotent, it follows that $c + d \in Id(A)$. Now $2a = a + a = r + c + s + d = (r+s) + (c+d)$ and so $2a$ is a clean element. It follows that there exists an idempotent e such that $V(2a) \subseteq V(e)$ and $V(2a-1) \subseteq U(e)$.

The equivalence of (b) and (c) is straighforward.

■

Theorem 3.7. *For a ring A, the following are equivalent:*

(i) A is 2-clean.

(ii) $A[\mathbb{Z}_2]$ *is clean and the idempotents of* $A[\mathbb{Z}_2]$ *belong to* A.

(iii) $A[\mathbb{Z}_2]$ *is clean and* $2 \in \mathfrak{J}(A)$.

(iv) Every element of $A[\mathbb{Z}_2]$ *can be written as the sum of a unit and an idempotent in* A.

(v) A *is clean and for every* $a, b \in A$, $V(a+b) \cap V(a-b-1) = \emptyset$.

Proof. (i)⇒ (ii) If A is 2-clean, then by Proposition 1.8 and Proposition 3.5 every idempotent of $A[\mathbb{Z}_2]$ is in A. Let $aX^0 + bX^1 \in A[G]$. By the hypothesis there is an idempotent e such that $a+b-e$ and $a-b-e$ are both invertible. Set $r = a-e$ so that $r+b$ and $r-b$ are invertible. It follows that

$$aX^0 + bX^1 = (rX^0 + bX^1) + eX^0.$$

is a clean representation,whence $A[\mathbb{Z}_2]$ is a clean ring.

(ii) ⇒ (iii) Suppose that $A[\mathbb{Z}_2]$ is clean and that idempotent elements are in A. Let $a \in A$ and write

$$aX^0 + aX^1 = (rX^0 + aX^1) + eX^0$$

where $e^2 = e$, $r + e = a$, and $rX^0 + aX^0$ is invertible. As previously noted it follows that $r+a, r-a$ are both invertible. Since $r - a = -e$ is invertible it follows that $e = 1$. Therefore, the invertibility of $2a - 1 = r + a$ and arbitrariness of a imply that $2 \in \mathfrak{J}(A)$.

(iii) ⇒ (ii) This is patent.

(ii) ⇒ (i) Let $a, b \in A$. Then the element $aX^0 + bX^1$ is clean and thus we can write it as

$$aX^0 + bX^1 = (rX^0 + bX^1) + eX^0$$

where $e \in Id(A)$ and $(rX^0 + bX^1) \in \mathcal{U}(A[\mathbb{Z}_2])$. Observe that $r + e = a$ and $r+b, r-b \in \mathcal{U}(A)$. Since

$$a + b - e = r + b \text{ and } a - b - e = r - b$$

we are done.

(ii) ⇒ (iv). This is clear.

(iv) ⇒ (i). Let $a, b \in A$. By hypothesis there is a unit $rX^0 + sX^1 \in A[\mathbb{Z}_2]$ and an idempotent $e \in A$ such that

$$aX^0 + bX^1 = (rX^0 + sX^1) + eX^0.$$

It follows that $s = b$ and that $a - e = r$. It is straighforward from here to show that A is 2-clean.

(i) ⇒ (v). If A is 2-clean, then $V(a-b-1) = V(a+b-1)$ which is always disjoint from $V(a+b)$.

(v) ⇒ (1). For any $a \in A$ we have $\emptyset = V(a-a) \cap V(a+a-1) = V(0) \cap V(2a-1) = V(2a-1)$ and therefore $2a - 1$ is invertible. It follows that $2 \in \mathfrak{J}(A)$.■

Corollary 3.8. *Let* $char A = 2$. *Then* $A[\mathbb{Z}_2]$ *is clean if and only if* A *is clean.*

The next result was shown in [4]. We include a sketch of proof for completeness sake.

Proposition 3.9. *Let $2 \in \mathcal{U}(A)$. Then $A[\mathbb{Z}_2] \cong A \times A$. In this case A is clean if and only if $A[G]$ is clean.*

Proof. Define $\psi : A[G] \to A \times A$ by $\psi(aX^0 + bX^1) = (a + b, a - b)$. Then ψ is a ring homomorphism. When 2 is a regular element this homomorphism is injective, and when 2 is invertible the map is a surjection.■

Remark 3.10. If $V(2)$ is a clopen subset of $Max(A)$, then there is a decomposition of $A = A_1 \oplus A_2$ where the rings A_1 and A_2 satisfy that $2 \in \mathcal{U}(A_1)$ and $2 \in \mathfrak{J}(A_2)$. It follows that $A[\mathbb{Z}_2]$ is a clean ring, but not every element of $A[\mathbb{Z}_2]$ is the sum of a unit and a normal idemptotent.

Lemma 3.11. *Suppose $cX^0 + dX^1 \in Id(A[\mathbb{Z}_2])$. Then both $V(c)$, $V(d) \smallsetminus V(c)$ are idempotent clopen subsets of $Max(A)$. Moreover, $V(d)$ is an idempotent clopen subset of $Max(A)$.*

Proof. The second statement clearly follows from the first. By hypothesis we know that $c^2 + d^2 = c$, $2cd = d$, and $c + d, c - d \in Id(A)$. The second equation implies that $V(2), V(c) \subseteq V(d)$. In particular, we obtain that $V(c) \subseteq V(c+d) \cap V(c-d)$. Now, if $M \in V(c+d) \cap V(c-d)$, then both $2c \in M$ and $c^2 - d^2 \in M$. If $2 \in M$, then the residue field has characteristic 2 and so

$$c + M = c^2 + d^2 + M = c^2 - d^2 + M = M$$

so that $c \in M$. It follows that $V(c) = V(c+d) \cap V(c-d)$, and so $V(c)$ is an idempotent clopen subset of $Max(A)$.

Next we show that $V(c+d) \cup V(c-d) = U(d) \cup V(c)$ from which it will follow that $V(d) \smallsetminus V(c)$ is an idempotent clopen subset of $Max(A)$. Suppose that $d \notin M$. Then $2c + M = 1 + M$. From this and $c^2 + d^2 = c$ we conclude that $2d + M = 1 + M$ or $2d + M = -1 + M$. (Observe that in any field of characteristic different from 2 we have that $\frac{1}{2} - \frac{1}{2}^2 = \frac{1}{2}^2$). Therefore, $c + M = d + M$ or $c + M = -d + M$ in which case $M \in V(c - d)$ or $M \in V(c + d)$, whence $U(d) \cup V(c) \subseteq V(c + d) \cap V(c - d)$. The reverse is the easy case.■

Theorem 3.12. *For a ring A, $A[\mathbb{Z}_2]$ is clean if and only if for each $a, b \in A$ there exist idempotents $e_1, e_2, e_3, e_4 \in Id(A)$ such that $U(e_i) \cap U(e_j) = \emptyset$, $i \neq j$,*

$$\begin{aligned} 1 &= e_1 + e_2 + e_3 + e_4 && (1)\\ V(2) &\subseteq U(e_3) \cup U(e_4) && (2)\\ V(a+b) &\subseteq U(e_3) \cup U(e_1) && (3)\\ V(a+b-1) &\subseteq U(e_4) \cup U(e_2) && (4)\\ V(a-b) &\subseteq U(e_3) \cup U(e_2) && (5)\\ V(a-b-1) &\subseteq U(e_4) \cup U(e_1) && (6) \end{aligned}$$

Proof. Necessity. Suppose $A[\mathbb{Z}_2]$ is clean and let $a, b \in A$. Then we can write

$$aX^0 + bX^1 = (rX^0 + sX^1) + (cX^0 + dX^1)$$

where $rX^0 + sX^1 \in U(A[\mathbb{Z}_2])$ and $(cX^0 + dX^1) \in Id(A[\mathbb{Z}_2])$. This means that $c^2 + d^2 = c$, $2cd = d$, $c + d, c - d \in Id(A)$ and both $r + s$ and $r - s$ are invertible. By Lemma 3.11 it follows that there are idempotents e_1, e_2, e_3, e_4 such that $U(e_4) = V(c), U(e_3) = V(d) \smallsetminus V(c), U(e_1) = V(c-d) \smallsetminus V(c)$ and $U(e_2) = V(c+d) \smallsetminus V(c)$. Observe that $U(e_i) \cap U(e_j) = \emptyset$ for each $i \neq j$ and that $\bigcup_{1 \leq i \leq 4} U(e_i) = Max(A)$. Without loss of generality we can assume that $e_i e_j = 0$ so that $e_1 + e_2 + e_3 + e_4$ is an invertible idempotent and hence (1) is satisfied. That (2) is satisfied follows from the fact that $V(2) \subseteq V(d) = U(e_3) \cup U(e_4)$.

Next, notice that $a + b - (c + d) = r + s$ and so (3) $V(a+b) \subseteq U(c+d) = U(e_1) \cup U(e_3)$. Similarly, $a - b - (c - d) = r - s$ and so (5) $V(a-b) \subseteq U(c-d) = U(e_2) \cup U(e_3)$. Let $M \in V(a+b-1)$. If $M \in U(e_1) = V(c-d) \smallsetminus V(c)$, then $c + d \notin M$ (otherwise $2c \in M$ and $2 \notin M$ would imply $c \in M$.) Since $c + d$ is idempotent, $c + d + M = 1 + M$. Hence

$$M = a + b - 1 + M = a + b - (c + d) + M$$

contradicting that $r + s$ is invertible. Similarly, if $M \in U(e_3) = V(d) \smallsetminus V(c)$ then $d + M = M$ and so $c + M = 1 + M$. Therefore

$$M = a + b - 1 + M = a + b - (c + d) + M.$$

Therefore, (4) is satisfied. A similar argument yields (6).

Sufficiency. Let $aX^0 + bX^1 \in A[\mathbb{Z}_2]$ and suppose that e_1, e_2, e_3, e_4 are idempotents satisfying the hypothesis. Observe that $e_i e_j = 0$ for all $i \neq j$. We shall produce elements $r, s, c, d \in A$ such that $c^2 + d^2 = c$, $2cd = d$, and both $r + s = a + b - (c + d)$ and $r - s = a - b - (c - d)$ are invertible. Thus, it follows that

$$aX^0 + bX^1 = (rX^0 + sX^1) + (cX^0 + dX^1)$$

is a clean representation. To that end observe that (1) implies there is an $x \in (e_1 + e_2)A$ such that $2x = e_1 + e_2$. We further suppose that $x(e_1 + e_2) = x$. Let $y_1 = xe_1$ and $y_2 = -xe_2$. Set $c = x + e_3$ and $d = y_1 + y_2$. Observe that

$$V(c) = U(e_4) \text{ and } V(d) = U(e_4) \cup U(e_3).$$

Since

$$\begin{aligned} c^2 + d^2 &= (x + e_3)^2 + (y_1 + y_2)^2 \\ &= x^2 + e_3 + x^2 e_1 + x^2 e_2 \\ &= 2x^2 + e_3 \\ &= x + e_3 \\ &= c \end{aligned}$$

and

$$\begin{aligned} 2cd &= 2(x + e_3)(xe_1 - xe_2) \\ &= 2(x^2 e_1 - x^2 e_2) \\ &= 2x^2(e_1 - e_2) \\ &= x(e_1 + e_2)(e_1 - e_2) \\ &= x(e_1 - e_2) \\ &= d \end{aligned}$$

it follows that $cX^0 + dX^1 \in Id(A[\mathbb{Z}_2])$. By construction $c + d = 2xe_1 + e_3$ so that

$$V(c+d) = U(e_2) \cup U(e_4).$$

Let $r = a - c$ and $s = b - d$. Then $r + s = a + b - (c + d)$ and $r - s = a + b - (c - d)$.

If $r + s$ is not invertible, then it lies in some maximal ideal M. If $M \in V(a + b)$, then $M \in V(c + d)$ contradicting (3). Therefore, $M \in U(a + b)$ and hence $M \in U(c + d)$. So $a + b + M = c + d + M = 1 + M$, i.e. $M \in V(a + b - 1)$. By (4) $M \in U(e_2)$ (since $M \notin V(c) = U(e_4)$). Our contradiction is

$$1 + M = c + d + M = xe_1 + xe_2 + e_3 + xe_1 - xe_2 + M = 2xe_1 + xe_3 + M = M$$

since $M \in V(e_1) \cap V(e_3)$. Therefore, $r + s$ is invertible. A similar argument shows that $r - s$ is invertible.■

Theorem 3.13. *$A[\mathbb{Z}_2]$ is clean if and only if A is clean.*

Proof. As we have pointed so many times already, if $A[\mathbb{Z}_2]$ is clean, then so is A. So suppose that A is clean and let $a, b \in A$. Choose idempotents $f_1, f_2 \in A$ such that $V(a + b) \subseteq U(f_1), V(a + b - 1) \subseteq V(f_1)$ and $V(a - b) \subseteq U(f_2), V(a - b - 1) \subseteq V(f_2)$. Let e_1, e_2, e_3, e_4 be idempotents such that

$$U(e_1) = U(f_1) \cap V(f_2),$$
$$U(e_2) = U(f_2) \cap V(f_1),$$
$$U(e_3) = U(f_1) \cap U(f_2),$$
$$U(e_4) = Max(A) \setminus (U(e_1) \cup U(e_2) \cup U(e_3)).$$

Observe that $U(e_i) \cap U(e_j) = \emptyset$ whenever $i \neq j$. Without loss of generality $e_ie_j = 0$ whenever $i \neq j$. Therefore, all the conditions of Theorem 3.12 are satisfied except for possibly (2).

We claim that $V(2) \cap U(e_1)$ and $V(a + b) \cup V(a - b - 1)$ are disjoint closed sets. To see this let $M \in V(2) \cap U(e_1)$. Suppose that $M \in V(a + b)$. Since $M \in V(2)$, $M \in V(a - b)$ and hence $M \in U(f_2)$, contradicting that $M \in U(e_1)$. If $M \in V(a - b - 1)$, then $M \in V(a + b - 1)$ which implies that $M \in V(f_1)$, again contradicting that $M \in U(e_1)$. Similarly, $V(2) \cap U(e_2)$ and $V(a - b) \cap V(a + b - 1)$ are disjoint closed sets.

Since A is clean it follows that the collection of idempotent clopen sets form a base for the topology. It follows that there is an idempotent clopen subset $K_1 \subseteq U(e_1)$ such that $V(2) \cap U(e_1) \subseteq K_1$ and $(V(a + b) \cup V(a - b - 1)) \cap K_1 = \emptyset$. Similarly, there is an idempotent clopen subset $K_2 \subseteq U(e_2)$ such that $V(2) \cap U(e_2) \subseteq K_2$ and $(V(a-b) \cup V(a+b-1)) \cap K_2 = \emptyset$. By combining $U(e_4)$ together with K_1 and K_2 and shrinking down $U(e_1)$ and $U(e_2)$ in the appropriate manner, we obtain idempotent clopen subsets which satisfy conditions (1)-(6) of Theorem 3.12. Therefore, $A[\mathbb{Z}_2]$ is clean.■

Corollary 3.14. *Suppose G is an elementary 2-group. Then $A[G]$ is clean if and only if A is clean.*

Proof. Since G is an elementary 2-group it follows that for each $f \in A[G]$ there exists a finite subgroup H of G such that $f \in A[H]$ and H is a finite direct product of copies of $\mathbb{Z}_2$. It follows that $A[H]$ is a finite product of copies of $A[\mathbb{Z}_2]$ which is clean. Therefore, $f \in A[H]$ can be written as a s um of a unit and an idempotent in $A[H]$, and hence can be written as a sum of a unit and an idempotent in $A[G]$.■

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