1. Sets, Relations, and Functions

We begin by recalling our axioms, definitions, and concepts from Set Theory. We consider the first order logic whose objects are sets and which has two binary predicates: $=, \in$. Recall that we may apply boolean operators to create new sentences: $\vee, \wedge, \rightarrow, \neg$. We also recall the two main quantifiers: $\forall$ (universal) and $\exists$ (existential).

The sentence $x \in A$ means that x is a member of A. We write $x \notin A$ instead of $\neg(x \in A)$.

Definition 1.1. Suppose A and B are sets. We write $A \subseteq B$ (and say A *is a subset of* B) if every element of A is an element of B. This means that

$$\forall x(x \in A \rightarrow x \in B).$$

Our first axiom is the tool that allows us how to determine when two sets are equal. Notice that it says that to show two sets are equal you must demonstrate two items. Get in the habit of mentioning the axiom of extensionality.

Axiom of Extensionality Two sets A and B are equal if and only if they have the same elements, i.e. $A \subseteq B$ and $B \subseteq A$.

Axiom of Set Existence There exists a set which does not contain any elements. This set is denoted by $\emptyset$ and called the *emptyset*.

Axiom of Specification To every set A and to every well-defined formula $S(x)$ there corresponds a set B whose elements are exactly those elements $x \in A$ for which $S(x)$ holds. This set is denoted by $\{x \in A : S(x)\}$.

Axiom of Unions We will assume that every collection of sets (which contains at least one set) can be written as $\{A_i\}_{i \in I}$ where I denotes some index set. The axiom of union says that there is a set, denoted $\bigcup_{i \in I} A_i$ which consists of precisely those elements that belong to at least one of the A_i. When the index set is finite, say $i_1, \cdots, i_n$ we instead write $A_{i_1} \cup \cdots \cup A_{i_n}$.

Proposition 1.2. *For any sets A, B, C*

1. $A \cup B = B \cup A$,
2. $A \cup (B \cup C) = (A \cup B) \cup C$,
3. $A \cup A = A$,
4. $A \subseteq B$ *if and only if* $A \cup B = B$.

Lemma 1.3. *Let $\{A_i\}_{i \in I}$ be a family of sets. There is a set whose elements are precisely the elements that belong to A_i for all $i \in I$.*

Remark 1.4. The set given by Lemma 1.3 is called the intersection of sets and is denoted by $\bigcap_{i \in I} A_i$. When I is a finite set, say $i_1, \cdots, i_n$ then we instead write $A_{i_1} \cap \cdots \cap A_{i_n}$. If two sets, say A and B satisfy $A \cap B = \emptyset$, then A and B are said to be *disjoint sets*.

Proposition 1.5. *For any sets A, B, C*

1. $A \cap B = B \cap A$,
2. $A \cap (B \cap C) = (A \cap B) \cap C$,
3. $A \cap A = A$,
4. $A \subseteq B$ *if and only if* $A \cap B = A$.

Theorem 1.1 (Distributive Laws)**.** *For any three sets A, B, C the following equalities hold.*

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Definition 1.6. For any sets A, B we define $A \setminus B = \{x \in A : x \notin B\}$. This set is uniquely defined and exists by the Axiom of Specification. The set $A \setminus B$ is called the *complement of B in A*.

Proposition 1.7. *For any sets A, B, C*

(1) $A \setminus (A \setminus B) = A \cap B$,
(2) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$,
(3) *If $A \subseteq B$, then $(C \setminus B) \subseteq (C \setminus A)$.*

Axiom of Power Sets For any set A there exists a set whose elements are precisely the subsets of A. This set is called the *power set of A* and is denoted by $\mathcal{P}(A)$.

Axiom of Ordered Pairs Let A, B be sets. The *cartesian Product of A and B* is the set of ordered pairs of the form (a, b) for $a \in A$, $b \in B$; the set is denoted by $A \times B$. Ordered pairs satisfy the property that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Definition 1.8. A *relation from a set A to a set B* is a subset $R \subseteq A \times B$. If $A = B$ we call R a *relation on A*. If $(a, b) \in R$, we also write $a \sim_R b$: Let R be a relation on a set A

a. R is *reflexive* if for all $a \in A$, $a \sim_R a$,
b. R is *symmetric* if for all $a, b \in A$ $(a \sim_R b \to b \sim_R a$,
c. R is *antisymmetric* if for all $a, b \in A$ $[(a \sim_R b$ and $b \sim_R a) \to a = b]$,
d. R is *transitive* if for all $a, b, c \in A$ $[(a \sim_R b$ and $b \sim_R c) \to a \sim_R c]$,
e. R is an *equivalence relation* if R is reflexive, symmetric, and transitive.
f. R is a *partial order* if R is reflexive, antisymmetric, and transitive.

Definition 1.9. Let S be an equivalence relation on A. For an element a of A the equivalence class of a (relative to S) is defined to be $\{x \in A : x \sim_S a\}$ and is denoted by [a]: The set $A/S = \{[a] : a \in A\}$ is called the *quotient set of A by S* or the quotient set of A by $\sim_S$.

Definition 1.10. A partition of a set A is a set D of non-empty pairwise disjoint subsets of A with the property that for each $x \in A$ there is an element B of D with $x \in B$.

Theorem 1.2. *Let A be a non-empty set. If S is an equivalence relation on A then A/S is a partition of A.*

Definition 1.11. A function from a set A to a set B is a relation f from A to B such that

(i) For each $a \in A$ there is a $b \in B$ with $(a, b) \in f$,
(ii) If $(a, b) \in f$ and $(a, d) \in f$, then $b = d$.

We write $f : A \mapsto B$ to denote that f is a function from A to B: If $(a, b) \in f$ we instead write $f(a) = b$. The set A is called the *domain of f*, while the set B is called *codomain* of f. The *range of f* is the set of $b \in B$ such that $b = f(a)$ for some $a \in A$.

Definition 1.12. A function $f : A \mapsto B$ is *injective* (aka one-to-one) if for all $x, y \in A$ $[f(x) = f(y) \to x = y]$. The function f is *surjective* (aka onto) if B is the range of f. If f is both injective and surjective, then f is called a *bijection*.

If $f : A \mapsto B$ and $g : B \mapsto C$ are functions, the *composition* is the function $g \circ f : A \mapsto C$ given by $(g \circ f)(x) = g(f(x))$ for each $x \in A$.

For any set A let 1_A denote the function $1_A : A \mapsto A$ defined by $1_A(x) = x$ for each $x \in A$. This function is called the *identity function on* A.

Proposition 1.13. *Suppose $f : A \mapsto B$ and $g : B \mapsto C$ are functions.*

a. *If f and g are injective, then $g \circ f$ is injective.*
b. *If f and g are surjective, then $g \circ f$ is surjective.*

Proposition 1.14. *Suppose $f : A \mapsto B$. Then f is a bijection if and only if there is a function $g : B \mapsto A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. In this case g is unique and is called the inverse of f and is denote by f^{-1}.*

2. Real Numbers

We let $\mathbb{R}$ denote the set of real numbers.

Definition 2.1. For a set $S \subseteq \mathbb{R}$ and $u \in \mathbb{R}$, we call u an *upper bound of* S if $u \geq x$ for every $x \in S$. Not every subset of $\mathbb{R}$ possesses an upper bound. If there is an upper bound for S, then we say that S is bounded above. A *lower bound* and a *bounded below* set are defined analogously.

Let $S \subset \mathbb{R}$ and let $b \in \mathbb{R}$. We say that b is a *least upper bound for* S if it satisfies the following two properties:

i. b is an upper bound of S,
ii. For any other upper bound of S, say $u \in \mathbb{R}$, $b \leq u$.

A greatest lower bound for a set is defined analogously.

Lemma 2.2. *Suppose $S \subseteq \mathbb{R}$. If S has a least upper bound, then it is unique. Similarly, for greatest lower bound. (So we can use the article the instead of a.) We denote the least upper bound (resp. greatest lower bound) if S, by lub(S) (resp. glb(S)).*

Axiom of Real Numbers We assume that the set of real numbers is *Dedekind Complete*, that is, whenever a set is bounded above (resp. below), then the set in question possesses a least upper bound (resp. greatest lower bound). This statement is actually a theorem however we shall simply assume that $\mathbb{R}$ is Dedekind Complete. [Named after Richard Dedekind c. 1831-1916 who developed the foundational theory of the real numbers.]

Proposition 2.3. *Let $\emptyset \neq B \subseteq A \subseteq \mathbb{R}$. If A is bounded above, then so is B and lub(B)$\leq$ lub(A). Analogously for glb.*

Proposition 2.4. *The absolute vale function has the following properties for all $x, y \in \mathbb{R}$:*

(1) $0 \leq |x|$,
(2) $0 = |x|$ *if and only if* $x = 0$,
(3) $x \leq |x|$,
(4) $|xy| = |x||y|$,
(5) $|x + y| \leq |x| + |y|$,

(6) $|x - y| \leq |x| + |y|$,

Definition 2.5. Let $S : \mathbb{N} \to Y$ be a sequence. We say $T : \mathbb{N} \to Y$ is a *subsequence of* S if there is a function $F : \mathbb{N} \to \mathbb{N}$ such that $F(i) < F(j)$ whenever $i < j$ and $T = S \circ F$. If (x_n) is a sequence, we denote a subsequence by (x_{n_k}).

Proposition 2.6. *Let (x_n) be a sequence in Y which converges to a point $x \in Y$. Then every subsequence (x_{n_k}) of (x_n) converges to x.*

Definition 2.7. Let B be a subset of Y. Then B is called sequentially compact if every sequence in B has a subsequence which converges to a point of B.

Proposition 2.8. *A sequentially compact subset of Y is closed.*

Proposition 2.9. *Let B be a sequentially compact subset of Y: Suppose $D \subseteq B$ and D is closed. Then D is sequentially compact.*

Definition 2.10. A sequence (A_n) of subsets of Y is said to be nested if $A_{n+1} \subseteq A_n$ for each natural number n.

Theorem 2.1. *If (A_n) is a nested sequence of non-empty closed subsets of Y and A_1 is sequentially compact then $\cap_{n=1}^{\infty} A_n$ is not empty.*

Example 2.11. Give an example to show that the previous theorem is false if we do not assume that A_1 is sequentially compact.

Theorem 2.2. *For any real numbers $a < b \in \mathbb{R}$, $[a, b]$ is sequentially compact.*

Definition 2.12. A subset $A \subseteq R$ is *bounded* if A is a subset of some interval $[a, b]$.

Theorem 2.3. *A subset $B \subseteq \mathbb{R}$ is sequentially compact if and only if B is closed and bounded.*

Definition 2.13. Let (Y, d) be a metric space. A non-empty subset $B \subseteq Y$ is said to be *bounded* if the set

$$S(B) = \{r \in \mathbb{R} : r = d(x, y) \text{ for some } x, y \in B\}$$

is bounded above. If B is bounded, the least upper bound of $S(B)$ is called the *diameter of* B and is denoted by $d(B)$. By convention we say the emptyset is bounded and set $d(\emptyset) = 0$.

Proposition 2.14. *Let (Y, d) be a metric space and B be a bounded subset of Y. Then $cl(B)$ is bounded and $d(B) = d(cl(B))$.*

Proposition 2.15. *A sequentially compact subset of Y is bounded.*

Definition 2.16. Let $D \subseteq Y$ and let $C = \{A_\alpha\}_{\alpha \in I}$ be a collection of subsets of Y. We say that C is a *cover of D* if for every $x \in D$ there exists some $A_\alpha \in C$ such that $x \in A_\alpha$. If in addition, each element of C is an open set, C is called an *open cover of D*.

Theorem 2.4. *Suppose (Y, d) is a metric space. Let $D \subseteq Y$: Suppose that D is non-empty and sequentially compact. Let C be an open cover of D. Then there is a real number $\epsilon > 0$ such that if $E \subseteq D$ with $d(E) < \epsilon$, then $E \subseteq A_\alpha$ for some $A_\alpha \in C$.*

Definition 2.17. Suppose (Y, d) is a metric space and let $D \subseteq Y$. D is called *totally bounded* if for every $r > 0$ there are points $x_1,\ cdots, x_n \in D$ such that $D \subseteq (B_r(x_1) \cup \cdots \cup B_r(x_n))$.

Proposition 2.18. *Suppose (Y, d) is a metric space and let $D \subseteq Y$. If D is totally bounded, then D is bounded.*

Example 2.19. Give an example of a metric space Y which is bounded, but not totally bounded.

Proposition 2.20. *Suppose (Y, d) is a metric space and let $D \subseteq Y$. If D is sequentially compact, then D is totally bounded.*

Example 2.21. Give an example of a metric space Y which is totally bounded, but not sequentially compact.