MAT 4930 Foundations of Geometry
Fall 2015

The goal is for you to work through the list on your own. You should be able to prove each theorem. If you have an issue with a theorem or need help please talk to me about them. This is not a group project. You will learn things from the class and textbook that will help you, and the point of the notes is to get you used to "Discovery Learning".

I am asking you (requiring??) to keep a journal with your notes, proofs, solutions to these problems. I will let you know in class when certain problems are due or whether you will be asked to present them on the board to the class. By journal you can use a composition book or you can start learning how to latex or use scientific word; the latter has an advantage of the ability to back-it-up in case you lose the information.

I will periodically modify these notes and add more problems to the list. By the end of the semester we should be close to Theorem 100.

1 Terms

We start with a set of objects. The set will be called *the universe* and denoted $\mathcal{U}$. The elements of $\mathcal{U}$ will be called *points* and will often be denoted by the letters P, Q, and R (though the set of points may be infinite). Specific subsets of $\mathcal{U}$ will be called *lines* and we will denote lines by ℓ, ℓ_1, and ℓ_2 (though again there might be an infinite number of lines). It is usually our axioms that tell us which subsets can be lines. The only properties about the universe and its points and line are those in the axioms given to us as well as those results which we are able to prove. If ℓ is a line and P is a point of ℓ, i.e. $P \in \ell$, we instead will say that P *is on* ℓ.

We begin with "four" axioms.

Axiom A *There exists at least one point and one line.*

Axiom B *Every line contains at least two points.*

Axiom C *If P and Q are two different points, there is a line which contains both P and Q.*

Axiom D *If L and M are two different lines, there is no more than one point contained in the intersection $L \cap M$.*

Question 1 *What is the smallest possible number of points (so that axioms A,B,C are all satisfied)?*

Theorem 2 *If P and Q are two different points, there do not exist two different lines containing both P and Q.*

Definition (1): If P and Q are different points, the unique line containing points P and Q will be denoted by $\overleftrightarrow{PQ}$

Theorem 3 *If A and B are two different points of $\overleftrightarrow{PQ}$ then $\overleftrightarrow{AB}=\overleftrightarrow{PQ}$.*

Definition (2): If a set $\mathcal{T}$ of points is contained in some line, then $\mathcal{T}$ is said to be *collinear*. Otherwise, $\mathcal{T}$ is *noncollinear*. Two lines are said to be *parallel* if they are disjoint. (Notice that for this class this means that when we are speaking non-trivially by two parallel lines we shall mean two distinct lines. Some authors/courses assume that a line is parallel to itself.)

Definition (3): A geometry is said to satisfy the *elliptical property* if no two lines are parallel.

A geometry is said to satisfy the *Euclidean parallel postulate* if given a point P not on a line l there is a unique line through P that is parallel to l.

A geometry is said the *hyperbolic* if given a point P not on a line l there are at least two lines through P that are parallel to l.

Axiom E *There exists a noncollinear set of three points.*

Theorem 4 *Every point is contained in at least two lines.*

Question 5 (Richard) *What is the smallest possible number of points (so that axioms A,B,C,D are all satisfied)?*

Question 6 (Colleen) *What is the smallest possible number of points if every line contains at least three points?*

Question 7 *What is the smallest possible number of points if every line contains at least four points?*

In each of the following interpretations/models, which of the axioms are satisfied and which are not? Tell whether each model has the elliptical, Euclidean, or hyperbolic parallel property.

Example 8 *"Points" are lines in 3-space, "lines" are planes in 3-space, incidence means a line lying in a plane.*

Example 9 *"Points" are 1-dimensional subspaces of 3-space, "lines" are 2-dimensional subspaces of 3-space, incidence means a 1-dimensional subspace of a 2-dimensional space.*

Example 10 *Fix the unit circle. "Points" are the ordered pairs (x, y) satisfying $x^2 + y^2 < 1$. "Lines" are chords of the circle. Incidence is the usual sense.*

Example 11 *Fix the unit sphere. A "point" is a set consisting of two elements from the sphere that lie in a diameter of the sphere and "lines" are the set of all points whose (two) elements belong to a great circle of the sphere, that is, the diameter of the circle is 2. Incidence means both points lie on the circle.*

We now assume there is a ternary relation on certain triples of points, where if A, B, C are three *different* points, we write the relation as A–B–C, and say B *is between* A *and* C. Observe that A–A–B is false.

Axiom F *If A–B–C, then B–C–A is false.*

Axiom G *If A–B–C, then C–B–A. (This says betweeness is symmetric.)*

Axiom H *If A–B–C, then $\{A, B, C\}$ is a collinear set.*

Axiom I *If $\{A, B, C\}$ is a collinear set of three different points, then one of the following is true: A–B–C, or B–C–A, or C–A–B.*

Theorem 12 (Chastity)

(a) If A–B–C, then B–A–C is false.
(b) If A–B–C, then A–C–B is false.
(c) If A–B–C, then C–A–B is false.

Theorem 13 (Tiffany) *If $\{A, B, C\}$ is a collinear set of three different points, then exactly one of the following is true: A–B–C, or B–C–A, or C–A–B.*

Definition (4): Suppose that A and B are two distinct points.

(a) $\overline{AB}=\{x \mid x=A \text{ or } x=B \text{ or } A\text{–}x\text{–}B\}$ ($\overline{AB}$ is called the *closed segment from A to B*);

(b) $\overset{\circ\!-\!\circ}{AB}=\{x \mid A\text{–}x\text{–}B\}$ (the $\overset{\circ\!-\!\circ}{AB}$ is called the *open segment from A to B*).

Definition (5). Suppose A and B are two different points.

(a) $\overrightarrow{AB}=\{x \mid x=A, A\text{–}x\text{–}B, x=B, \text{ or } A\text{–}B\text{–}x\}$. (a *closed ray*)

(b) $\overset{\circ\!\longrightarrow}{AB}=\overrightarrow{AB}-\{A\}$, that is, $\overrightarrow{AB}$ with A removed. Equivalently, $\overset{\circ\!\longrightarrow}{AB}=\{x \mid A\text{–}x\text{–}B, x=B, \text{ or } A\text{–}B\text{–}x\}$. (an *open ray*)

Theorem 14 (Richard) *Let A and B be two different points. $\overrightarrow{AB}\cap\overrightarrow{BA}=\overline{AB}$.*

Theorem 15 *Let A and B be two different points. $\overrightarrow{AB}\cup\overrightarrow{BA}=\overleftrightarrow{AB}$.*

Question 16 *Suppose A–C–B and $D\in\overleftrightarrow{AC}$ $(D\neq A,B,C)$. Is $D\in(\overrightarrow{CA}\cup\overrightarrow{CB})$?*

Definition (6): Suppose L is a line and A and B are points not on L. We say A and B are *on the same side of* L if either $A=B$ or if $\overline{AB}\cap L=\emptyset$. If A and B are different points and $\overline{AB}\cap L\neq\emptyset$ then we say A and B are on opposite sides. (Observe that if A and B are not on L then they are either on the same side or on opposite sides.)

Question 17 *Is being on the same side transitive?*

Axiom J *For any line L and points A,B,C:*
(i) If A and B are on the same side of L, and B and C are on the same side of L, then A and C are on the same side of L.
(ii) If A and B are on opposite sides of L, and B and C are on opposite sides, then A and C are on the same side of L.

Lemma 18 *If A–B–C and $E\notin\overleftrightarrow{AC}$, then A and B are on the same side of $\overleftrightarrow{EC}$.*

Theorem 19 *Suppose A–B–C and A–C–D. Then B–C–D. Similarly, A–B–D.*

Theorem 20 *Suppose A–C–B. Then $\overleftrightarrow{AB}=\overrightarrow{CA}\cup\overrightarrow{CB}$.*

Corollary 21 *Suppose A and B are on the same side of L and $x\in\overline{AB}$. Then x and A are on the same side of L.*

Corollary 22 *Suppose L is a line, $A\in L$, and $B\notin L$. If $C\in\overset{\circ\!\longrightarrow}{AB}$ then B and C are on the same side of L.*

Definition (7): Let ℓ be a line. A set of points S is called a *side of* ℓ if there is a point A not on ℓ such that one of the following happens: S is the set of points on the same side of ℓ as A; or S is the set of points on the opposite side of ℓ as A.

Theorem 23 *Every line has exactly two sides. These two sides are disjoint. One of them may be empty.*

Theorem 24 *Suppose A,B,C are non-collinear and ℓ is a line intersecting $\overline{AB}$ at a point between A and B. Then ℓ intersects either $\overline{AC}$ or $\overline{BC}$. Furthermore if $C\notin\ell$ then ℓ does not intersect both segments.*

Axiom K. If A and B are two different points, then there is a point C such that A–B–C. Observe that it now follows that every line has two nonempty sides.

Definition (8): Suppose $\{A, B, C\}$ is a non-collinear set of points. Define $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$, that is, the set of points lying in $\overrightarrow{BA}$ or in $\overrightarrow{BC}$ (or in both). ($\angle ABC$ is called the *angle* ABC.) Observe by commutativity of union $\angle ABC = \angle CBA$.

Question 25 *What is the smallest possible number of points on a line?*

Theorem 26 *If $\{A, B, C\}$ is a non-collinear set, then $\angle ABC \neq \angle BAC$.*

Theorem 27 *If $\angle ABC = \angle DEF$, then either*
1. $\overrightarrow{BA} = \overrightarrow{ED}$ and $\overrightarrow{BC} = \overrightarrow{EF}$, or
2. $\overrightarrow{BA} = \overrightarrow{EF}$ and $\overrightarrow{BC} = \overrightarrow{ED}$;
in either case, $B = E$.

Theorem 28 *Let A, B, C be non-collinear. Suppose $E \in \overset{\circ}{\overrightarrow{BA}}$ and $F \in \overset{\circ}{\overrightarrow{BC}}$. Then $\angle ABC = \angle EBF$.*

Definition (9): We define the *interior of the angle* $\angle ABC$ to be the set of points D for which: a) D and C are on the same side of $\overleftrightarrow{AB}$, and b) D and A are on the same side of $\overleftrightarrow{BC}$.

Theorem 29 *Suppose D is in the interior of $\angle ABC$. Then every point in $\overset{\circ}{\overrightarrow{BD}}$ is also in the interior of $\angle ABC$.*

Theorem 30 *Suppose D is in the interior of $\angle ABC$. If D–B–E then E is not in the interior of $\angle ABC$.*

Theorem 31 *Suppose A, B, C are non-collinear and $D \in \overleftrightarrow{AC}$. D is in the interior of $\angle ABC$ if and only if A–D–C.*

Theorem 32 *If D is in the interior of $\angle ABC$ and D is also in the interior of $\angle BCA$, then D is in the interior of $\angle CAB$.*

Question 33 *If D is in the interior of $\angle ABC$, must there exist a point x on $\overset{\circ}{\overrightarrow{BA}}$ and a point y on $\overset{\circ}{\overrightarrow{BC}}$ such that x–D–y?*

Lemma 34 *Suppose D is in the interior of $\angle ABC$ and A–B–E. Then C is in the interior of $\angle DBE$.*

Theorem 35 *Suppose D is in the interior of $\angle ABC$. Then $\overline{AC} \cap \overrightarrow{BD} \neq \emptyset$.*

Theorem 36 *Every angle has nonempty interior.*

Theorem 37 *Suppose A and B are two different points. Then there is a point C such that A–C–B. Moreover, there are infinite number of points between them.*

Question 38 *Can a line be contained in the interior of an angle?*

We now assume we have an undefined relation $\cong$ on the set of all closed segments, written $\overline{AB}\cong\overline{QR}$, and said "$\overline{AB}$ is *congruent* to $\overline{QR}$".

Axiom L. The relation "$\cong$" is an equivalence relation. That is: If A and B are two different points and C and D are two different points and E and F are two different points, then all three of the following are true:
(a) $\overline{AB}\cong\overline{AB}$ (reflexive)
(b) If $\overline{AB}\cong\overline{CD}$ then $\overline{CD}\cong\overline{AB}$ (symmetric)
(c) If $\overline{AB}\cong\overline{CD}$ and $\overline{CD}\cong\overline{EF}$, then $\overline{AB}\cong\overline{EF}$. (transitive)

Axiom M. If A and B are two different points and C and D are two different points, then there is a point E on the open ray $\overset{\circ\!\longrightarrow}{CD}$ such that $\overline{AB}\cong\overline{CE}$.

Axiom N. If A–B–C then it is *not* true that $\overline{AB}\cong\overline{AC}$.

Axiom O. If A–B–C, A'–B'–C', $\overline{AB}\cong\overline{A'B'}$, and $\overline{BC}\cong\overline{B'C'}$, then $\overline{AC}\cong\overline{A'C'}$.

Theorem 39 *(Segment Subtraction) If A–B–C, D–E–F, $\overline{AB}\cong\overline{DE}$, and $\overline{AC}\cong\overline{DF}$, then $\overline{BC}\cong\overline{EF}$.*

Theorem 40 *If A–B–C and $\overline{AC}\cong\overline{DF}$, then there is a point E such that D–E–F and $\overline{AB}\cong\overline{DE}$.*

Definition (14) $\overline{AB}<\overline{CD}$ means there is a point X such that $\overline{AB}\cong\overline{CX}$ and C–X–D.

Theorem 41 *It is false that $\overline{AB}<\overline{AB}$.*

Theorem 42 *If $\overline{AB}\cong\overline{CD}$ and $\overline{CD}<\overline{EF}$ then $\overline{AB}<\overline{EF}$.*

Theorem 43 *If $\overline{AB}<\overline{CD}$ and $\overline{CD}\cong\overline{EF}$ then $\overline{AB}<\overline{EF}$.*

Theorem 44 *If $\overline{AB}<\overline{CD}$ and $\overline{CD}<\overline{EF}$ then $\overline{AB}<\overline{EF}$.*

Theorem 45 *If A and B are two different points and C and D are two different points, then one and only one of the following is true:*
(a) $\overline{AB}<\overline{CD}$,
(b) $\overline{AB}\cong\overline{CD}$, or
(c) $\overline{CD}<\overline{AB}$.

We are now given another undefined relation on pairs of angles, also called "is congruent to", and written $\angle ABC \cong \angle DEF$.

Axiom L$'$. The relation $\cong$ for angles is symmetric, reflexive, and transitive. (See Axiom L.)

Axiom M$'$. If A, B, C are non-collinear, and D, E, F are non-collinear, then there is a point G on the same side of $\overleftrightarrow{DE}$ as F such that $\angle ABC \cong \angle DEG$. (See Axiom M.)

Axiom N$'$. If D is in the interior of $\angle ABC$, then it is *not* true that $\angle ABC \cong \angle ABD$. (See Axiom N.)

Axiom O$'$. If D is in the interior of $\angle ABC$ and D' is in the interior of $\angle A'B'C'$, and $\angle ABD \cong \angle A'B'D'$ and $\angle DBC \cong \angle D'B'C'$, then $\angle ABC \cong \angle A'B'C'$. (See Axiom O.)

Axiom 40$'$. If D is in the interior of $\angle ABC$, and $\angle ABC \cong \angle A'B'C'$, then there is a point D' in the interior of $\angle A'B'C'$ such that $\angle ABD \cong \angle A'B'D'$.

Lemma 46 *Suppose D, E, F are non-collinear and G is on the same side of $\overleftrightarrow{DE}$ as F. If D and G are on opposite sides of $\overleftrightarrow{EF}$, then F is in the interior of $\angle DEG$.*

Lemma 47 *i) Suppose D is in the interior of $\angle ABC$ and E is in the interior of $\angle ABD$. Then E is in the interior of $\angle ABC$.*
ii) Suppose D is in the interior of $\angle ABC$ and E is in the interior of $\angle DBC$. Then E is in the interior of $\angle ABC$.

Theorem 48 **(39′)** *If D is in the interior of $\angle ABC$ and D' is in the interior of $\angle A'B'C'$, and $\angle ABC \cong \angle A'B'C'$ and $\angle ABD \cong \angle A'B'D'$, then $\angle DBC \cong \angle D'B'C'$.*

Definition (15) $\angle ABC < \angle DEF$ means that there is a point X in the interior of $\angle DEF$ such that $\angle ABC \cong \angle DEX$.

Theorem 49 **(41′)** *It is false that $\angle ABC < \angle ABC$.*

Theorem 50 **(42′)** *If $\angle ABC \cong \angle DEF$ and $\angle DEF < \angle GHI$, then $\angle ABC < \angle GHI$.*

Theorem 51 **(43′)** *If $\angle ABC < \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle ABC < \angle GHI$.*

Theorem 52 **(44′)** *If $\angle ABC < \angle DEF$ and $\angle DEF < \angle GHI$, then $\angle ABC < \angle GHI$.*

Theorem 53 **(45′)** *If A, B, C are non-collinear and D, E, F are non-collinear then one and only one of the following is true:*
(a) $\angle ABC < \angle DEF$,
(b) $\angle DEF < \angle ABC$, or
(c) $\angle ABC \cong \angle DEF$.

Definition (16) Given three non-collinear points A, B, C the *triangle*, denoted $\triangle ABC$, is the union of the segments $\overline{AB} \cup \overline{BC} \cup \overline{AC}$. Observe that $\triangle ABC = \triangle BAC = \triangle BCA$ and so forth. We say $\triangle ABC \cong \triangle DEF$ if corresponding sides are congruent and corresponding angles are congruent.

Axiom (SAS). Given triangles $\triangle ABC$ and $\triangle DEF$ if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.

Theorem 54 *Given $\triangle ABC$ and $\overline{DE} \cong \overline{AB}$ there is a point F such that D, E, F are non-collinear and $\triangle ABC \cong \triangle DEF$.*

Definition (17) If the angles $\angle BAD$ and $\angle CAD$ satisfy that $\{B, A, C\}$ is a collinear set and $\overrightarrow{AC} \neq \overrightarrow{AB}$, then the angles are called *supplementary*. If an angle is congruent to a supplementary angle then the angle is called a *right angle*. Finally, two lines ℓ and m are said to be *perpendicular* if they intersect at a point A and there are other points $B \in \ell$, $C \in m$ such that $\angle BAC$ is a right angle.

Theorem 55 *If in $\triangle ABC$, we have $\overline{AB} \cong \overline{BC}$, then $\angle BAC \cong \angle BCA$.*

Theorem 56 *Supplements of congruent angles are congruent.*

Proposition 57 *A supplement of a right angle is a right angle.*

Theorem 58 *An angle congruent to a right angle is a right angle.*

Definition (18) Let A–X–B and D–X–F. Also suppose that $\overleftrightarrow{AB} \neq \overleftrightarrow{DF}$. The angles $\angle AXD$ and $\angle FXB$ are called *vertical angles*.

Corollary 59 *Vertical angles are congruent.*

Theorem 60 *For every line ℓ and a point P not on ℓ there is a line through P which is perpendicular to ℓ. Therefore, right angles exist.*

Theorem 61 *(ASA) Given triangles* $\triangle ABC$ *and* $\triangle DEF$ *suppose that* $\angle BAC \cong \angle EDF$, $\angle BCA \cong \angle EFD$, *and* $\overline{AC} \cong \overline{DF}$. *Then* $\triangle ABC \cong \triangle DEF$.

Theorem 62 *If in* $\triangle ABC$, $\angle ABC \cong \angle ACB$, *then* $\overline{AB} \cong \overline{AC}$.

Theorem 63 *(SSS)*

Theorem 64 *Suppose* A–B–C *and* A'–B'–C' *and* D *is a point not on* $\overleftrightarrow{AB}$ *(similarly for* D'*). If* $\overline{AD} \cong \overline{A'D'}$, $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{BD} \cong \overline{B'D'}$, *then* $\overline{CD} \cong \overline{C'D'}$.

Theorem 65 *Any two right angles are congruent.*

Definition (19) Given two distinct lines ℓ_1 and ℓ_2 a third line ℓ which intersects both ℓ_1 and ℓ_2 in distinct points is called a *transversal*. We also say ℓ_2 and ℓ_2 are *cut* by the transversal ℓ.

Definition (20) Given two distinct lines ℓ_1 and ℓ_2 and a transversal ℓ let A_1 be the point where ℓ crosses ℓ_1 and A_2 be the point where ℓ crosses ℓ_2.

Definition (21) Given the triangle $\triangle ABC$ the angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ are called the *interior angles of* $\triangle ABC$. A triangle is called a *right triangle* if one of its interior angles is a right angle. Suppose $\triangle ABC$ is a right triangle and $\angle ABC$ is a right angle. The segment $\overline{AC}$ is called a *hypotenuse* of the right triangle.

Theorem 66 *If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.*

Theorem 67 *Two different lines perpendicular to the same line are parallel. Hence, in Theorem 57, the line through* P *and perpendicular to* ℓ *is unique.*

Theorem 68 *Given a line* ℓ *and a point* P *not on* ℓ *there exists at least one line* ℓ' *through* P *which is parallel to* ℓ. *(In other words (SAS) implies our geometry is either Euclidean or Hyperbolic.)*

Theorem 69 *Let* $\triangle ABC$ *and* D *be a point such that* B–C–D. *Then* $\angle ACD > \angle ABC$. *Also,* $\angle ACD > \angle BAC$. *Is it true that* $\angle ACD > \angle ACB$*?*

Theorem 70 *(SAA)*

Theorem 71 *A right triangle has a unique right angle. Furthermore, it has a unique hypotenuse. The other sides of the triangle are called legs of the right triangle.*

Theorem 72 *Two right triangles are congruent if the hypotenuses and a leg of one are congruent respectively to the hypotenuse and a leg of the other.*

Theorem 73 *Every segment has a unique midpoint.*

Theorem 74 *Every angle has a unique bisector. (A bisector of the angle* $\angle ABC$ *is a ray* $\overrightarrow{BD}$ *where* D *is in the interior of the angle and* $\angle ABD \cong \angle DBC$.*)*

Theorem 75 *In a triangle* $\triangle ABC$, $\overline{AB} \leq \overline{BC}$ *if and only if* $\angle C \leq \angle A$.

Theorem 76 *Given triangles* $\triangle ABC$ *and* $\triangle DEF$, *if* $\overline{AB} \cong \overline{DE}$ *and* $\overline{BC} \cong \overline{EF}$, *then* $\angle B < \angle E$ *if and only if* $\overline{AC} < \overline{DF}$.

Theorem 77 *Suppose* A–B–C *and* D *is not on* $\overleftrightarrow{AC}$. *If* $\overleftrightarrow{DC}$ *is perpendicular to* $\overleftrightarrow{AC}$, *then* $\overline{CD} < \overline{BD} < \overline{AD}$.

Theorem 78 *Given triangle $\triangle PQR$ and any point S between Q and R show that either $\overline{PS}<\overline{PQ}$ or $\overline{PS}<\overline{PR}$.*

Theorem 79 *Construct a triangle which is not isosceles.*

Definition (22) A set T of points is *convex* if for every pair of points $A, B \in T$, the segment $\overline{AB}$ is contained in T.

Theorem 80 *Define a half-plane and prove that a half-plane is convex. Prove that a line is convex.*

Theorem 81 *The interior of every angle is convex.*

Theorem 82 *Let S and T be convex sets. Show that $S \cap T$ is convex.*

Question 83 *True or False: If C and K are each convex sets of points, then $C \cup K$ is convex.*

Theorem 84 *The interior of a triangle is convex.*

Question 85 *True or False: A triangle is a convex set.*

Theorem 86 *If $\mathcal{C}$ and $\mathcal{K}$ are convex sets, let*

$$\mathcal{T} = \{x \mid \exists c \in \mathcal{C}, k \in \mathcal{K} \text{ such that } c - x - k\}.$$

Then $\mathcal{T} \cup \mathcal{C} \cup \mathcal{K}$ is convex.

Definition (23) Let O be a point. Given a segment $\overline{AB}$ we define the *circle centered at O and of radius $\overline{AB}$* to be the following set:

$$\gamma = \{D : \overline{OD} \cong \overline{AB}\}.$$

The *interior of* γ is defined as

$$\text{int}\gamma = \{D : \overline{OD} < \overline{AB}\}.$$

The *exterior* of γ is

$$ext\gamma = \{D : \overline{AB} < \overline{OD}\}.$$

Two points $A, B \in \gamma$ are said to form a *diameter* if A–O–B.

Theorem 87 *The interior of a circle is convex. Is the exterior of a circle convex?*

Theorem 88 *Suppose γ is a circle centered at O of radius $\overline{AB}$. If $X, Y \in \gamma$ form a diameter then O is the midpoint of $\overline{XY}$. Furthermore every circle has a diameter.*

Theorem 89 *Suppose γ is a circle centered at O of radius $\overline{AB}$. Let $X, Y \in \gamma$ be distinct points so that X, Y do not form a diameter. Let M be the midpoint of $\overline{XY}$. Show that $\overleftrightarrow{OM}$ is perpendicular to $\overleftrightarrow{XY}$.*

Theorem 90 *Suppose γ is a circle centered at O of radius $\overline{AB}$. Let $X, Y \in \gamma$ be distinct points and let M be the midpoint of $\overline{XY}$. Let ℓ be any line through M which is perpendicular to $\overline{XY}$. Prove that $O \in \ell$.*

Theorem 91 *Prove in Euclidean geometry that a triangle inscribed in a semi-circle is a right triangle.*

Definition (24). The *exterior* of $\angle ABC$ is the set of all points which are neither on $\angle ABC$ nor in the interior of $\angle ABC$.

Theorem 92 *No angle has a convex exterior.*

Theorem 93 *If a line L is the union of two convex sets J and K, then either J contains a ray or K contains a ray.*

Theorem 94 *If J is a convex set and X, Y, Z are non-collinear points of J, then the interior of triangle XYZ is contained in J.*

Definition (24) The four points A, B, C, D form a *quadrilateral* if A, B, C are non-collinear, A, C, D are non-collinear, and D is in the interior of $\angle ABC$. The quadrilateral $\square ABCD$ is the union of the segments $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$. The *interior* of the quadrilateral $\square ABCD$ is the union of the interior of triangle ABD, the interior of triangle BDC, and $\overset{\circ\!-\!\circ}{BD}$. We call the quadrilateral a *convex quadrilateral* if C is in the interior of $\angle BAD$.

Lemma 91.5 *Suppose the quadrilateral $\square ABCD$ is convex. Prove that the interior of quadrilateral $\square ABCD$ is the same as the $\square BADC$.*

Theorem 95 *If A, B, C, D form a convex quadrilateral, then the interior of the quadrilateral is convex.*

Lemma 92.5 *Given an example of a quadrilateral $\square ABCD$ that is not convex.*

Theorem 96 *Suppose A, B, C, D form a quadrilateral. If the interior of the quadrilateral is convex, then the quadrilateral is a convex quadrilateral.*

Theorem 97 *If line L contains a point between two vertices of quadrilateral A, B, C, D, then either L contains a vertex of the quadrilateral, or L contains* another *point between two vertices of the quadrilateral.*

Theorem 98 *If A, B, C, D is a quadrilateral and x is in the interior of each of the angles $\angle ABC$, $\angle BCD$, and $\angle CDA$, then x is also in the interior of $\angle DAB$.*

Theorem 99 *Suppose that the diagonals of the quadrilateral $\square ABCD$ intersect. Prove that the quadrilateral is convex.*

Definition (25) Suppose that quadrilateral $\square ABCD$ satisfies the following properties: i) $\angle B$ is a right angle, ii) $\angle C$ is a right angle, iii) $\overline{AB} \cong \overline{CD}$. In this case we call the quadrilateral a *Saccheri Quadrilateral.*

Theorem 100 *A Saccheri quadrilateral is convex.*

Theorem 101 *The diagonals of a Saccheri quadrilateral are congruent.*

Theorem 102 *Let $\square ABCD$ and $\square A'B'C'D'$ be two Saccheri quadrilateral such that $\overline{BC} \cong \overline{B'C'}$ and $\overline{AB} \cong \overline{A'B'}$. Then $\overline{AD} \cong \overline{A'D'}$, $\angle A \cong \angle A'$, and $\angle D \cong \angle D'$*

Corollary 103 *Given a Saccheri quadrilateral $\square ABCD$, $\angle A \cong \angle D$.*

Theorem 104 (HARD) *Given a Saccheri quadrilateral $\square ABCD$, $\overline{BC} \leq \overline{AD}$.*

Theorem 105 *Given a Saccheri quadrilateral $\square ABCD$, $\angle BAC \leq \angle ACD$.*

Definition (22) Observe that until now we have not associated a real number to any segment or angle. Thus, we need to be careful when we define the "sum" of two segments or angles. For any segments $\overline{AB}$ and $\overline{CD}$ we define the sum to be

$$\overline{AB} + \overline{CD} \ \cong \overline{AF}$$

where A–B–F and $\overline{CD} \cong \overline{BF}$. It is apparent what $2\ \overline{AB} = \overline{AB} + \overline{AB}$ means.

Theorem 106 *Show that the definition of sums of segments is well defined and is commutative.*

Theorem 107 *Prove that for any segments $\overline{AB}, \overline{CD}$, if $\overline{AB} < \overline{CD}$ then $2\ \overline{AB} < 2\ \overline{CD}$*

Axiom P. We now turn to archimedean geometry. In principal the idea is to assign real numbers our lengths of segments and angle measures. To this end we suppose that there is an assignments l on the collection of segments of our geometry so that the following hold:

(1) $l(\overline{AB}) \in \mathbb{R}$ for every segment $\overline{AB}$.

(2) $l(\overline{AB}) = l(\overline{CD})$ if and only if $\overline{AB} \cong \overline{CD}$.

(3) A–B–C if and only if $l(\overline{AC}) = l(\overline{AB}) + l(\overline{BC})$.

(4) $l(\overline{AB}) < l(\overline{CD})$ if and only if $\overline{AB} < \overline{CD}$.

l is called the *length*. Similarly we suppose there is an assignment defined on the angles of our geometry $\circ$ called the degree.

(5) $0 < \angle A^\circ < 180$ for every angle $\angle A$.

(6) $\angle A^\circ = 90$ if and only $\angle$ is a right angle.

(7) $\angle A^\circ = \angle B^\circ$ if and only if $\angle A \cong \angle B$.

(8) $\angle A^\circ < \angle B^\circ$ if and only if $\angle A < \angle B$.

(9) Given angle $\angle ABC$ and D in its interior, $\angle ABD^\circ + \angle CBD^\circ = \angle ABC^\circ$

Furthermore we assume that a line has a degree measure of 180 and that the sum of supplementary angles equals 180. If $\angle A^\circ < 90$ then we say it is an *acute angle*. If its degree is greater than 90 we call it an *obtuse angle*.

Theorem 108 *Given $\overline{AB}$ let X be its midpoint. Show that $l(\overline{AX}) = \frac{1}{2} l(\overline{AB})$.*

Theorem 109 *If $\triangle ABC \cong \triangle DEF$ then $\angle A^\circ + \angle B^\circ + \angle C^\circ = \angle D^\circ + \angle E^\circ + \angle F^\circ$*

Theorem 110 *The sum of the degrees of any two angles of a triangle is less than 180. Thus no triangle has two obtuse angles. Nor can there exist a triangle with two right angles.*

Theorem 111 *In any $\triangle ABC$ we have that*

$$l(\overline{AB}) < \quad l(\overline{AC}) + l(\overline{BC}).$$

Theorem 112 *Given the triangle $\triangle ABC$ let D be the unique midpoint of $\overline{BC}$. Pick a point $E \in \overset{\circ\longrightarrow}{AD}$ which has D as a midpoint of $\overline{AE}$. Prove that the sum of the angles of triangle $\triangle ABC$ equals the sum of the angles of triangle $\triangle AEC$. Moreover, either $\angle EAC^\circ$ or $\angle AEC^\circ$ is less than $\frac{1}{2}\angle BAC^\circ$.*

Theorem 113 *In any $\triangle ABC$ we have that*

$$\angle A^\circ + \angle B^\circ + \angle C^\circ \quad \leq 180.$$

Definition (22) Given triangle $\triangle ABC$ we define $\delta\triangle ABC$ to be the difference of 180 and the sum of the degrees of its 3 angles. This number is called the *defect* of the triangle as it is measuring how defective the triangle is. Obviously the defect is preserved under congruences.

Theorem 114 *Given triangle $\triangle ABC$ and a point D between A and B we have*

$$\delta\triangle ABC = \delta\triangle ACD + \delta\triangle CDB.$$

Corollary 115 *Given the same hypothesis as the previous theorem, $\delta\triangle ABC = 0$ if and only if $\delta\triangle ACD = 0 = \delta\triangle CDB$.*

Theorem 116 *If there exists a triangle with defect equal to 0, then there is a right triangle whose defect is 0.*

Definition (18). A set S of points is *linearly connected* if for every pair of points $X, Y \in S$ there is a finite sequence of points $X = X_0, X_1, X_2, \ldots, X_n = Y$ such that for each $i = 1, 2, \ldots, n$, the segment $\overline{X_{i-1}X_i}$ is contained in S.

Theorem 117 *Every convex set is linearly connected.*

Theorem 118 *Every angle is linearly connected.*

Question 119 *If S and T are linearly connected sets, then $S \cap T$ is linearly connected.*

Theorem 120 *If S and T are each linearly connected sets and the intersection of S and T is non-empty, then the union of S and T is linearly connected.*

Theorem 121 *If S is a linearly connected set and L is a line which contains no points of S, then all of the points of S lie on the same side of L.*

Theorem 122 *The exterior of every angle is linearly connected.*