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$$\frac{\partial f}{dy}(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} = f_y(z_0)$$

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Chapter 1

Complex Numbers

The set of complex numbers is denoted by $\mathbb{C}$ and a complex number has the form $a+bi$, for $a, b \in \mathbb{R}$. We also will sometimes view complex numbers as ordered pairs (x, y), for $x, y \in R$. This second way of looking at the complex numbers is a more geometric view in that complex numbers are points in the plane, $\mathbb{R}^2$. With regards to addition and $\mathbb{R}$-scalar multiplication (i.e. vector space operations) the advantage would lie in the plane view $\mathbb{C} = \mathbb{R}^2$. However, when we are interested in multiplication then the advantage goes to the view $\mathbb{C}$. The form $a+bi$ is known as **rectangular form**.

Let $a+bi, c+di \in \mathbb{C}$. Define addition as

$$(a+bi)+(c+di) = (a+c)+(b+d)i.$$

When equipped with this operation $\mathbb{C}$ is an abelian group; we use 0 for the additive identity of $\mathbb{C}$ and note that $0 = 0+0i$. We can also multiply complex numbers as follows.

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

This operation together with addition makes $\mathbb{C}$ into a commutative ring with identity. We view $\mathbb{R}$ as a subring of $\mathbb{C}$ by taking a real number $r \in R$ and writing it as $r+0i$. Observe that $i^2 = -1$.

Definition 1.1. Let $z, w \in \mathbb{C}$ and set $z = x+yi$ and $w = a+bi$.

1. The **modulus**/length of z is denoted by $||z|| = \sqrt{x^2+y^2}$. The modulus is always a non-negative real number.

2. We set $\arg z$ to any real number (angle) θ for which $x = ||z|| \cos\theta$ and $y = ||z|| \sin\theta$. The number $\arg z$ is called an **argument** for z. For each z there are countably many arguments. We let $\operatorname{Arg} z$ denote the unique argument $\theta \in (-\pi, \pi]$.

3. If $z \neq 0$, the **inverse** of z is $z^{-1} = \frac{x}{||z||^2} - \frac{y}{||z||^2} i$.

4. The complex **conjugate** of z is $\overline{z} = x - yi$.

5. The **real part** of z is x: $\Re(z) = x$. The **imaginary part** of z is y: $\Im(z) = y$.

6. The **distance** between the points z and w is $||z - w|| = \sqrt{(x-a)^2 + (y-b)^2}$.

Theorem 1.2. $\mathbb{C}$ *is a field.* $\mathbb{C}$ *is a* $\mathbb{R}$*-vector space of dimension; its standard basis is* $\{1, i\}$.

Definition 1.3. For $z \in \mathbb{C}$, we set $\theta = \operatorname{Arg} z$ and let $r = ||z||$. Then $z = r\cos\theta + ri\sin\theta$. We define the **polar** form of z as $z = re^{i\theta}$. The rule is that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

It is important that you notice we are assuming that $\theta \in \mathbb{R}$.

You should check that for real numbers $r, s \in \mathbb{R}$,

$$e^{ir} \cdot e^{is} = e^{i(r+s)}.$$

Also, for any $r \in R$, $||e^{ir}|| = 1$.

Definition 1.4. The set of complex number whose modulus is 1 is a very special set. We denote this by $\mathbb{S}_1$:

$$\mathbb{S}_1 = \{z \in \mathbb{C} : ||z|| = 1\}.$$

Lemma 1.5. $(\mathbb{S}_1, \cdot, 1)$ *is an abelian group under multiplication. The torsion subgroup of* $\mathbb{S}_1$ *is precisely the roots of unity.*

The exponential map should be viewed as a function from $\mathbb{R}$ onto $\mathbb{S}_1$ and since $e^{(r+s)i} = e^{ri}e^{si}$ it follows that this is a group homomorphism from $\mathbb{R}$ onto $\mathbb{S}_1$.

Some things to look up: triangle inequality, the parallelogram law, topology of the plane, DeMoivre's Formula, geometry of multiplication.

Chapter 2

Review of Linear Algebra

In this section we will review some important things from MAS 2103H. Since we didn't deal with the complexes in the class some of this will be new material as well but hopefully it is a mere generalization. We start with $M_n(\mathbb{C})$, the set of $M_n(\mathbb{C})$ (square) matrices over $\mathbb{C}$. This is a non-commutative ring with identity under entry-wise addition and matrix multiplication. The set of invertible matrices is denoted by $GL_n(\mathbb{C})$ and called the general linear group.

Each $A \in M_n(\mathbb{C})$ can be viewed as a linear transformation $T_A : \mathbb{C}^n \to \mathbb{C}^n$ when we use matrix multiplication and view elements of $\mathbb{C}^n$, which have the form $(z_1, \ldots, z_n)$ as a column matrix

$$T_A(\vec{x}) = A\vec{x}^t.$$

Conversely, every linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ has a matrix representation $A_T \in M_n(\mathbb{C})$ by taking the standard $\mathbb{C}$-basis $\{e_1, e_2, \ldots, e_n\}$ and using $T(e_i)^t$ as the i-th column vector of A_T:

$$A_T = (T(e_1)\ T(e_2)\ \ \ldots T(e_n)).$$

You should be familiar with the determinant but now $\det : M_n(\mathbb{C}) \to \mathbb{C}$. You should also be familiar with the definition of a subspace, the span of a set, a linearly independent set, a linearly dependent set, a basis, the dimension of a subspace ($\dim_C(V)$).

Theorem 2.1. *Let $A \in M_n(\mathbb{C})$. The following statements are equivalent.*

1. *The matrix A is invertible.*
2. *The linear transformation A_T is injective.*
3. *The linear transformation A_T is surjective.*
4. $\det(A) \neq 0$.
5. $\mathrm{Nul}(A) = \{0\}$.
6. $\ker \mathbb{T}_A = \{0\}$
7. $\mathrm{Rank}(A) = n$
8. $\{T_A(e_1), T_A(e_2), \ldots, T_A(e_n)\}$ *is a basis.*

Notice that since the determinant of a product is the product of determinants, the ma $\det(\cdot)$ is a group homomorphism from $GL_n(\mathbb{C})$ onto the set of non-zero real numbers (under multiplication). The set of matrices of determinant 1 is a subgroup of $GL_n(\mathbb{C})$ that we denote by $SL_n(\mathbb{C})$ and call it the special linear group. You should also know that if A is invertible, then $\det(A^{-1}) = \det(A)^{-1}$.

Proposition 2.2. *A matrix A satisfies the property that $AX = XA$ for all $X \in M_n(\mathbb{C})$ if and only if there is some $z \in \mathbb{C}$ such that $A = zI$. That is, the matrices that commute with all matrices are scalar matrices.*

Remark 2.3. You learned the above but with regards to $M_n(\mathbb{R})$, $GL_n(\mathbb{R})$, and $SL_n(\mathbb{R})$. What really makes that theorem tick is that both $\mathbb{R}$ and $\mathbb{C}$ are fields. Since $\mathbb{R}$ is a subfield of $\mathbb{C}$ it follows that $GL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{C})$.

We also remark that from this point we regard A and T_A as the same object. We know they are not but if I start with a matrix and want to talk about its properties as a linear transformation then I don't need to switch over to T_A.

Definition 2.4. Let $A \in M_n(\mathbb{C})$. The complex number λ is said to be an **eigenvalue** of A if there is a non-zero vector $\vec{x} \in \mathbb{C}^n$ such that $A(\vec{x})) = \lambda\vec{x}$. Clearly, A is not invertible if and only if 0 is a an eigenvalue of A. Generally, λ is an eigenvalue of A if $(A - \lambda I)$.

The polynomial $\det(xI - A)$ is called the **characteristic** polynomial and denoted by $c_A(x)$. The roots of this polynomial are precisely the eigenvalues of A. We denote the set of eigenvalues of A by $\mathrm{Spec}(A)$ and call it the spectrum of A. For an eigenvalue $\lambda \in \mathbb{C}$, the space $\ker(\lambda I - A)$ is called the **eigenspace** of λ. We will sometimes denote it by

E_λ. For a given $\lambda \in \text{Spec}(A)$, we can write $c_A(x) = (x-\lambda)^t f(x)$ where $f(\lambda) \neq 0$. The number t is called the **algebraic multiplicity** of λ. The **geometric multiplicity** of λ is $\dim_{\mathbb{C}}(E_\lambda)$.

Definition 2.5. Two matrices A, B are said to be **similar** if if there exists a $P \in GL_n(\mathbb{C})$ such that

$$P^{-1}AP = B.$$

The notion of similarity is an equivalence relation. Matrices which are similar to a diagonal matrix are said to be **diagonalizable**.

Theorem 2.6. *Let $A \in M_n(\mathbb{C})$. Then A is diagonalizable if and only if for all $\lambda \in$ Spec(A), the algebraic multiplicity of A equals the geometric multiplicity of A.*

The proof of the theorem goes like this. For each eigenvalue λ find a basis for E_λ. After doing this over all eigenvalues you will have at most n such linearly independent. A is diagonalizable if you get exactly n of them, in which case you use these n eigenvectors to form a matrix P whose columns are said vectors. Then you get

$$P^{-1}AP = D$$

where D is a diagonal matrix with whose diagonal entries are the eigenvalues of A.

Theorem 2.7 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$. Consequently, every polynomial splits into linear factors.*

Proof. Proof ☐

Definition 2.8. A matrix $J = (a_{ij}) \in M_k(\mathbb{C})$ is said to be in **Jordan block** if it satisfies the following:

1. there is some $z \in \mathbb{C}$ such that $a_{ii} = z$ for all $i = 1, \ldots, k$,
2. $a_{i,i+1} = 1$
3. All other entries are 0.

A 1x1 matrix is trivially a Jordan block.

A matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ is said to be in **Jordan (canonical) form** if it is comprised of Jordan blocks.

$$\begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & J_t \end{pmatrix}$$

Observe that a matrix in Jordan form is upper triangular.

Theorem 2.9. *For every matrix $A \in M_n(\mathbb{C})$ there is an invertible matrix P such that $P^{-1}AP$ is in Jordan form.*

For the interested student who wants an algorithm on how to find such a matrix look here.

JCF

Chapter 3

Orthogonality and Orthonormal

At this point, you should look up and recall the definition of a real vector space. Also, recall the definition of a linear transformation and the matrix representation of a linear transformation. A linear operator is a linear transformation from a vector space back into itself. You will need to remember the main theorem that classifies when a square matrix is invertible. The set of invertible matrices is denoted by $GL(n, \mathbb{R})$. This is called the general linear group. The set of matrices whose determinant are 1 is labelled $SL(n, \mathbb{R})$ and called the special linear group.

Our main example is $\mathbb{R}^n$ for $n \in \mathbb{N}$. By $\mathbb{R}^0$ we will mean the set $\{0\}$. Notice that $\mathbb{C}$ is an example of a real vector space. We generalize the modulus of a complex number to define the **length** of a vector $(x_1, \dots, x_n)$ as

$$||x|| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Vectors whose length is 1 are called **unit** vectors. The collection of unit vectors of $\mathbb{R}^{n+1}$ will be denoted by $\mathbb{S}_n$ and called the n**-sphere**. (E.g. $\mathbb{S}_0 = \{\pm 1\}$ and $\mathbb{S}_1$ is the unit circle.)

Recall that the **dot product** of two vectors $\vec{u} = (u_1, \dots, u_n), \vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ is defined as

$$\vec{u} \cdot \vec{v} =< \vec{u}, \vec{v} >= \sum_{i=1}^{n} u_i v_i.$$

In particular, $\vec{v} \cdot \vec{v} = ||v||^2$.

It will be useful to view the dot product as a multiplication of matrices. When we want to do this we view the element $\vec{v} \in \mathbb{R}^n$ as a column matrix (that is, a $n \times 1$ matrix).

Lemma 3.1. *Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then*

$$\vec{u} \cdot \vec{v} = u^t v$$

where the left side of the equality is the inner product, and the right side is viewed as multiplication of matrices with u, v are column matrices.

Two vectors are said to be **orthogonal** if $< \vec{u}, \vec{v} >= 0$. Formally, the dot product is an example of what is called an **inner product** (cf. Definition 6.7). A subset is called **orthogonal** if every pair of distinct elements in the set are orthogonal.

Definition 3.2. Fix a vector $\vec{u} \in \ R^n$. The set of vectors in $\mathbb{R}^n$ orthogonal to $\vec{u}$ is called the **orthogonal complement** of $\vec{u}$, and we denote this set by $\vec{u}^\perp$.

Proposition 3.3. *For any non-zero vector $\vec{u} \in \mathbb{R}^n$, $\vec{u}^\perp$ is a subspace of $\mathbb{R}^n$ and $\dim_{\mathbb{R}} \vec{u}^\perp = n-1$.*

Example 3.4. Let $\vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$. To find a vector that is orthogonal to $\vec{w}$ we solve $(w_1, w_2, w_3) \cdot (x, y, z) = w_1 x + w_2 + w_3 z = 0$. Notice that this is an equation of a plane in $\mathbb{R}^3$. Without loss of generality we assume that $w_1 \neq 0$ and so the plane becomes

$$x = -\frac{w_2}{w_1} y - \frac{w_3}{w_1} z.$$

At this point y and z are free variables. You can choose and y and z to yield a vector $\vec{u}$ and then use the information to find another vector, $\vec{v}$, which is orthogonal to both $\vec{w}$ and $\vec{x}$. In particular, the vector $\vec{u} = \begin{pmatrix} -w_2 - w_3 \\ w_1 \\ w_1 \end{pmatrix}$ works.

The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthogonal set of non-zero vectors. Moreover, the set is linearly independent and hence a basis for $\mathbb{R}^3$. <u>Sketch:</u> Suppose $\alpha_1 \vec{u} + \alpha_2 \vec{v} + \alpha_3 \vec{w} = \vec{0}$

$$\begin{aligned} 0 &= < 0, \vec{u} > \\ &= < \alpha_1 \vec{u} + \alpha_2 \vec{v} + \alpha_3 \vec{w}, \vec{u} > \\ &= \alpha_1 ||\vec{u}||^2. \end{aligned}$$

Definition 3.5. An **orthonormal** set of vectors is an orthogonal set of vectors each of which is a unit vector. The standard basis $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ is an orthonormal basis.

Any orthogonal set of vectors can be made into an orthonormal simply by taking unit vectors in the same direction.

Remark 3.6. At this point we could discuss the Gram-Schmidt process of orthonormalizing a linearly independent set using projection maps. Gram-Schmidt

In studying the space around us we find it useful to classify all of the symmetries of our space. Try to think about how you would define a symmetry. One way that you learned back in middle school is a movement to $\mathbb{R}^n$ that send an object like a triangle to a congruent object. Formally, a symmetry on $\mathbb{R}^n$ is an isometry.

Here is a discussion on isometries of $\mathbb{R}^n$. It includes a proof (for $\mathbb{R}^2$) that an isometry which fixes the origin is linear.

Definition 3.7. An **isometry** on $\mathbb{R}^n$ is a map that preserves distances. This means specifically that we have a map, say $F : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$||F(\vec{u}) - F(\vec{v})|| = ||\vec{u} - \vec{v}||.$$

Rotations, reflections, and translations are isometries. By a translation we mean adding a vector, so a map of the form $t_{\vec{v}}(\vec{u}) = \vec{u} + \vec{v}$. It is obvious that $t_{\vec{v}}$ is an isometry.

Lemma 3.8. *Let F and G be isometries. Then $F \circ G$ is an isometry.*

Proof. Let $x, y \in \mathbb{R}^n$. Then

$$\begin{aligned} ||(F \circ G)(x) - (F \circ G)(y)|| &= ||F(G(x)) - F(G(y))|| \\ &= ||G(x) - G(y)|| \\ &= ||x - y|| \end{aligned}$$

□

Let F be an isometry and set $\vec{v} = F(\vec{0})$. Let $G = t_{-\vec{v}} \circ F$. Then G is an isometry by the lemma, and

$$G(\vec{0}) = t_{-\vec{v}} \circ F)(\vec{0}) = t_{-\vec{v}}(F(\vec{0}) = \vec{0}.$$

Furthermore, $F = t_{\vec{v}} \circ G$. This is important. It follows that every isometry can be written as the composition of a translation and an isometry which sends $\vec{0}$ to $\vec{0}$. We know exactly what translation do so we are interested in isometries that send the origin to the origin.

Theorem 3.9. *The following statements are equivalent for an $F : \mathbb{R}^n \to \mathbb{R}^n$.*

1. *The function F is an isometry and $F(\vec{0}) = \vec{0}$.*
2. *The function F preserves the dot product.*
3. *The function F is a linear transformation and preserves the dot product.*

This means that we want to study linear isometries. First and foremost we can gather some other interesting information.

Proposition 3.10. *An isometry on $\mathbb{R}^n$ is a bijection. Furthermore, if F is an isometry, then so is F^{-1}.*

Proof. Clearly, the translations are bijection. Let F be isometry and as we mentioned above we can write $F = t_v \circ G$ where G is a linear isometry. Suppose that $x \neq y$. Then

$$||G(x) - G(y)|| = ||x - y|| > 0$$

from which it follows that $G(x) \neq G(y)$. So G is linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is a injection, and thus also a surjection. It follows then that F is a bijection since it is a composition of bijections.

Notice, that $F^{-1} = (t_v \circ G)^{-1} = G^{-1} \circ t_{-v}$. Thus to conclude that F^{-1} is an isometry it suffices to check that the inverse of a linear isometry is again an isometry. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$\begin{aligned} ||G^{-1}(\vec{x}) - G^{-1}(\vec{y})|| &= ||G(G^{-1}(\vec{x})) - G(G^{-1}(\vec{v}))|| \\ &= ||\vec{x} - \vec{y}|| \end{aligned}$$

This means that G^{-1} preserves distance, i.e. G^{-1} is an isometry. □

We can now conclude that the collection of isometries on $\mathbb{R}^n$ is a nice algebraic object.

Theorem 3.11. *The set of isometries on $\mathbb{R}^n$ is a group. We denote this set by $E(n)$ and call it the* **Euclidean group***. The set of translations of $\mathbb{R}^n$ is a normal subgroup of $E(n)$. Moreover, $E(n)$ is a semi-direct product of the translations with the linear isometries.*

We want to describe the linear isometries of $\mathbb{R}^n$ in detail.

Linear isometries of $\mathbb{R}^2$ are simply rotations or reflections through lines of symmetry in $\mathbb{R}^2$. What are the rotations of $\mathbb{R}^3$? Any rotation of $\mathbb{R}^3$ maps the unit ball back onto itself and also simply rotates the standard basis. This new basis must maintain its orthonormality. Any linear isometry on $\mathbb{R}^n$ must send the standard basis to a basis and since the isometry preserves distance each element in the basis is a unit vector and moreover since the isometry preserves the dot product, it must preserve orthogonality. This leads us to our next definition.

Definition 3.12. Call the matrix $A \in M_n(\mathbb{R})$ **orthogonal** if its column vectors form an orthonormal basis. Let $O(n)$ be the collection of all orthogonal matrices. The set of orthogonal matrices whose determinant is 1 will be denoted by $SO(n)$.

Proposition 3.13. *Let $A \in M_n(R)$. the following statements are equivalent.*

1. *The matrix A is orthogonal.*
2. $A^t A = I_n$.
3. *A is invertible and $A^{-1} = A^t$.*
4. $AA^t = I_n$
5. *The row vectors of A form an orthonormal set.*
6. *For all $u, v \in \mathbb{R}^n$, $< Au, Av >=< u, v >$. (Preserves inner product.)*
7. *For all $u \in \mathbb{R}^n$, $||Au|| = ||u||$. (Preserves length.)*

Proof. That the first five conditions are equivalent should be evident.

1. implies 6. Suppose A is orthogonal and let $u, v \in \mathbb{R}^n$. Then

$$< Au, Av >= (Au)^t Av = u^t A^t Av = u^t v =< u, v > .$$

6. implies 7. Let $u \in \mathbb{R}^n$. Then

$$||Au||^2 =< Au, Au >=< u, u >= ||u||^2.$$

7. implies 1. We are supposing that A preserves length. Let u_i be the i-th column vector of A. i.e. $Ae_i = u_i$. Then $1 = ||e_i|| = ||u_i||$ so every column vector of A is a unit vector. Let $1 \leq i \neq j \leq n$. You can do the calculation

$$\begin{aligned} ||Ae_i + Ae_j||^2 &= ||u_i + u_j|| \\ &= ||u_i||^2 + 2 < u_i + u_j > +||u_j||^2 \\ &= 2 + 2 < u_i, u_j > \end{aligned}$$

On the other hand

$$\begin{aligned} ||Ae_i + Ae_j||^2 &= ||A(e_i + e_j)|| \\ &= ||e_i + e_j||^2 \\ &= 2 \end{aligned}$$

It follows that $< u_i, u_j >= 0$ which means the column vectors of A are also orthogonal.

□

Corollary 3.14. *For any* $A \in O(n)$, $\det A = \pm 1$.

Theorem 3.15. *For any* $n \in \mathbb{N}$, $O(n)$ *is a subgroup of* $GL(n)$.

Proof. Clearly, if $A \in O(n)$, then $A^{-1} = A^t \in O(n)$. Show that if $A, B \in O(n)$ then so is AB. □

More Information on Rotations of $\mathbb{R}^3$

Example 3.16. Let V be the set of sequences on $\mathbb{R}$ which are eventually 0. This is a vector space. It is also a metric space with the metric

$$d((s_n),(t_n)) = \sqrt{\sum_{i=0}^{\infty}(s_i - t_i)^2}.$$

This sum makes sense since only for a finite number of i are the numbers not 0.

The "shift operator" on V, which is defined by $(a_1, a_2, \ldots) \mapsto (0, a_1, a_2, \ldots)$ is a linear isometry that is injective but not surjective.

Chapter 4

Bilinear maps

In this very short chapter I want to define some different kinds of maps that fall under the umbrella of bilinear maps. They will play a role in everything that follows and since there are some slight differences I wanted to put them here. I have defined them elsewhere as well. Through $\mathbb{F}$ is a field, either $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.1. Let V, W, X be vector spaces. A **bilinear map** is a function $f : V \times W \to X$ such that for all $\alpha \in F$, $u, v \in V$, and $z, w \in W$,

$$f((\alpha u + v, w)) = \alpha f((u, w)) + f((v, w)) \text{ and } f((v, \alpha z + w)) = \alpha f((v, z)) + f((v, w)).$$

Observe that these two conditions are equivalent to saying that for each $v \in V$ and $w \in W$ the maps $T_v : W \to X$ defined by $T_v(z) = f((v, z))$ and $S_w : V \to X$ defined by $S_w(u) = f((u, w))$ are linear transformations.

When $V = W$ then we can check whether the bilinear map $f : V \times V \to X$ is one the following three important types:

i. **symmetric**: for all $v, w \in V$, $f((v, w)) = f((w, v))$.
ii. **skew-symmetric**: for all $v, w \in V$, $f((v, w)) = -f((w, v))$.
iii. **alternating**: for all $v \in V$, $f((v, v)) = 0$.

In the case that, $V = W$ and $X = \mathbb{F}$, then a bilinear map is called a **bilinear form**. The phrases *symmetric bilinear form* or *alternating bilinear map* should now make sense.

Definition 4.2. Let V be real vector space. An **inner product** is a function $< \cdot, \cdot >: V \times V \to \mathbb{R}$ satisfying

1. $< \cdot, \cdot >$ is a symmetric bilinear form,

2. for all $v \in V$, $< v, v > \geq 0$
3. $< v, v >= 0$ if and only if $v = \vec{0}$.

Condition 3. here is referred to as **positive definite**.

Example 4.3. Let V be a vector space and consider the definition of scalar multiplication. It is a map from $\mathbb{F} \times V \to V$ and one can check that the usual distributive laws in the definition are simply saying that the scalar multiplication is a bilinear map.

When $V = \mathbb{F}$, a 1-dimensional vector space, then this scalar multiplication is simply the usual multiplication on F. Thus, the multiplication on a field is an example of a bilinear map $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$, which is symmetric since multiplication is commutative, and it is never alternating.

Example 4.4. The most common example of a bilinear form on a n-dimensional vector space V, is to fix a basis $\beta = \{\vec{b}_1, \cdots, \vec{b}_n\}$ for V, and then we define the **dot product** by

$$< v, w >= \sum_{i=1}^{n} v_i w_i$$

where $v = \sum_{i=1}^{n} v_i \vec{b}_i$ and $w = \sum_{i=1}^{n} w_i \vec{b}_i$.

Since the multiplication on $\mathbb{R}$ is commutative it follows that this dot product is symmetric. Moreover, for all $v \in V$, $< v, v > \geq 0$ and equals zero precisely when $v = \vec{0}$. Therefore, the dot product is an inner product.

Example 4.5. Let $V = C([0, 1])$ the set of real-valued continuous functions on the interval $[0, 1]$. For $f, g \in V$ define

$$< f, g >= \int_0^1 f(t)g(t)dt.$$

This defines an inner product on V.

You should notice that if you change your basis you get a different bilinear form on V. So right away we can see that there are an infinite number of inner products on a vector space. The usual dot-product on $\mathbb{R}_n$ is simply this example where the chosen basis is the standard basis.

Example 4.6. Let $A \in M_n(\mathbb{F})$. Define a map $f_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows. Consider elements of $\mathbb{R}^n$ as a $n \times 1$ column matrix. For $u, v \in \mathbb{R}^n$

$$f_A((u, v)) = u^t A v.$$

It is straightforward to check that f_A is a bilinear form on $\mathbb{R}^n$. This bilinear form is said to be determined by the matrix A. An interesting question is to classify when this is a symmetric bilinear form.

Theorem 4.7. *Every bilinear form on $\mathbb{R}^n$ is determined by some matrix $A \in M_n(\mathbb{F})$.*

Proof

Definition 4.8. Let $A \in M_n(\mathbb{R})$ and set $A = (a_{ij})$. We say that A is a **symmetric matrix** if for each $1 \leq i, j \leq n$, $a_{ij} = a_{ji}$. Observe that another way of defining a symmetric matrix is by saying that $A = A^t$.

Observe that any diagonal matrix is symmetric.

Proposition 4.9. *The collection of symmetric matrices is a subspace of $M_n(\mathbb{R})$ of dimension $\frac{n^2+n}{2}$.*

Theorem 4.10. *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form and let $A \in M_n(\mathbb{R})$ be the unique matrix such that $f(u, v) = u^t A v$. Then f is a symmetric bilinear form if and only if A is a symmetric matrix.*

Proof. Let $A = (a_{ij})$ be the matrix that determines f and recall that we denote he standard basis by $\{e_1, e_2, \ldots, e_n\}$. We view each vector of $\mathbb{R}^n$ as a column matrix. Then,

$$\begin{aligned} f(e_i, e_j) &= e_i^t A e_j \\ &= (a_{i1} a_{i2} \cdots a_{in}) e_j \\ &= a_{ij} \end{aligned}$$

If f is symmetric, then $a_{ij} = f(e_i, e_j) = f(e_j, e_i) = a_{ji}$ which shows that A is symmetric.

Conversely, suppose that A is symmetric. Then by the equalities above $f(e_i, e_j) = f(e_j, e_i)$. Let $x, y \in \mathbb{R}^n$ and set $x = (x_1, \ldots, c_n)$ and $y = (y_1, \ldots, y_n)$. Then $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{i=1}^n y_n e_i$. It follows that

$$\begin{aligned}
f(x,y) &= f(\sum_{i=1}^{n} x_i e_i, y) \\
&= \sum_{i=1}^{n} x_i f(e_i, y) \\
&= \sum_{i=1}^{n} x_i f(e_i, \sum_{j=1}^{n} y_j) \\
&= \sum_{i=1}^{n} x_i (\sum_{j=1}^{n} y_j f(e_i, e_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j f(e_i, e_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} y_j x_i f(e_j, e_i) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} y_j x_i f(e_i, e_j) \\
&\cdots \\
&= f(y,x).
\end{aligned}$$

□

Theorem 4.11. *The matrix A is symmetric matrix if and only if there is an orthogonal matrix $Q \in O(n)$ such that $Q^{-1}AQ$ is a diagonal matrix.*

Proof. Suppose that $Q \in O(n)$ and $Q^{-1}AQ = D$ is a diagonal matrix. Then $A = QDQ^{-1}$. Orthogonality means that $Q^{-1} = Q^t$. Then

$$\begin{aligned}
A^t &= (QDQ^{-1})^t \\
&= (Q^{-1})^t D^t Q^t \\
&= (Q^t)^{-1} D Q^t \\
&= (Q^{-1})^{-1} D Q^{-1} \\
&= QDQ^{-1} \\
&= A.
\end{aligned}$$

Therefore, A is symmetric.

For the proof of the converse we send you to Proof

□

An interesting question is for which symmetric matrices A, is the bilinear form $f_A(u,v) = u^tAv$ an inner product. (It is necessary that A be symmetric.) The only remaining piece is that for all $0 \neq v \in \mathbb{R}^n$, $f_A(v,v) > 0$.

Remark 4.12. I found online that there is a question of finding "the inner product of two vectors" given a matrix A.

What is interesting is that given any matrix $A \in M_n(\mathbb{R})$, the matrix $B = A^tA$ is a symmetric matrix:

$$\begin{aligned} B^t &= (A^tA)^t \\ &= A^t(A^t)^t \\ &= A^tA \\ &= B \end{aligned}$$

So for any matrix A, you can define a bilinear form on $\mathbb{R}^n$ by $< u, v >= u^tA^tAv$.

There is a discussion about quadratic forms on a vector space and how it relates to bilinear forms.

Sylvester's Law of Inertia

Chapter 5

Minkowski Space-time

The **Euclidean group** $E(n)$ is the set of all isometries of $\mathbb{R}^n$. It is comprised of all translations, rotations, reflections, and any finite composition of them. The translations $T(n)$ form a normal subgroup and

$$E(n) = T(n) \ltimes O(n).$$

The **Poincaré Group** is the group of Minkowski space-time isometries. That means a bijection which preserves the Minkowski metric (i.e. "inner product"– though it is degenerate). An element in space-time is a quadruple (t, x, y, z). We place the time first. For $\vec{u} = (t_u, x_u, y_u, z_u)$ and $\vec{v} = (t_v, x_v, y_v, z_v) \in \mathbb{R}^4$ define

$$< \vec{u}, \vec{v} >= x_u x_v + y_u y_v + z_u z_v - t_u t_v$$

This leads to the norm $||\vec{u}|| = x_u^2 + y_u^2 + z_u^2 - t_u^2$. The Minkowski metric is a symmetric bilinear form. So we are looking for maps f of $\mathbb{R}^4$ back into $\mathbb{R}^4$ such that for all $\vec{u}, \vec{v} \in \mathbb{R}^4$

$$< f(\vec{u}), f(\vec{v}) >=< \vec{u}, \vec{v} > .$$

Obviously, translations by any element in $\mathbb{R}^4$ is an isometry: for $v \in \mathbb{R}^4$, $T_v(x) = x + v$. We now construct more isometries of Minkowski space-time. But we do this in generality by constructing pseudo-orthogonal matrices.

Let $p, q \in \mathbb{N}$ and $n = p + q$. Let g be the diagonal matrix satisfying

$$g_{ij} = \begin{cases} -1, & \text{if } i = j \text{ and } i \leq p \\ 1, & \text{if } i = j \text{ and } i > p \\ 0, & \text{otherwise.} \end{cases}$$

On $\mathbb{R}^n$ define a symmetric bilinear form[1] $[\cdot,\cdot]_{p,q}$ by

$$[x,y]_{p,q} = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n = x^tgy.$$

A matrix $A \in M_n(\mathbb{R})$ is said to be **pseudo-orthogonal** and belong to $O(p,q)$ if for all $x, y \in \mathbb{R}^4$

$$[x,y]_{p,q} = [Ax, Ay]_{p,q}.$$

Observe that $O(n) = O(0,n)$.

Proposition 5.1. *The matrix $A \in GL_n(R)$ belongs to $O(p,q)$ if and only if $g^{-1}A^tg = A^{-1}$.*

Clearly, it follows that $A \in O(1,3)$ if and only if A is a linear isometry of Minkowski space-time. Such an object is called a Lorentz transformation and $O(1,3)$ is called the **Lorentz group**.

Corollary 5.2. *Let $A \in O(p,q)$. Then* $\det(A) = \pm 1$.

Proof.

$$\begin{aligned}\det(A) &= \det(A^t)\\ &= \det(gA^{-1}g^{-1})\\ &= \det(A^{-1})\\ &= \frac{1}{\det(A)}\end{aligned}$$

It follows that $\det(A)^2 = 1$. □

Proposition 5.3. *For any $n = p + q$, $O(p,q)$ is a subgroup of $GL_n(\mathbb{R})$.*

Proof. □

Proposition 5.4. *Let $\{v_1, \ldots, v_n\} \in \mathbb{R}^n$ be vectors and let A be the matrix whose column vectors are v_i: $A = [v_1 \ \ldots \ v_n]$. The following statements are equivalent.*

1. *The vectors satisfy $[v_i, v_j]_{p,q} = [e_i, e_j]_{p,q}$ for all $i, j = 1, \ldots, 4$.*

2. *$A \in O(p,q)$.*

3. *$[e_i, e_j]_{p,q} = [Ae_i, Ae_j]_{p,q}$ for all $i, j = 1, \ldots, 4$.*

[1] A bilinear form is a map $B : V \times V \to \mathbb{R}$ which is linear in each coordinate, that is $B(u+v, w) = B(u,w) + B(v,w)$ and dually.

Lemma 5.5. *Suppose f is an isometry of Minkowski space-time and $f(\vec{0}) = \vec{0}$. Then f is linear.*

Proof. Set $v_i = f(e_i)$ $(i = 1, \ldots, 4)$ and let $A = [v_1\ v_2\ v_3\ v_4]$. Observe that

$$[v_i, v_j]_{p,q} = [f(e_i), f(e_j)]_{p,q} = [e_i, e_j]_{p,q}$$

and so by Proposition 5.4, $A \in O(1,3)$. Set $S = A \circ f$ and take not that $S(e_i) = e_i$ for each $i = 1, \ldots, 4$. So $S = I_4$, and thus $f = A^{-1} \in O(1,3)$.

[Hint: Use that if for a fixed $u \in \mathbb{R}^n$, $[u, v] = 0$ for all $v \in \mathbb{R}^n$, then $u = 0$.]

□

Theorem 5.6. *Any isometry of Minkowski space-time is affine. That is, if f is an isometry, then there is some $A \in O(1,3)$ and some $\vec{v} \in \mathbb{R}^4$ such that for all $x \in \mathbb{R}^4$*

$$f(\vec{x}) = A\vec{x} + \vec{v}.$$

Remark 5.7. In $\mathbb{R}^4 \times O(1,3)$, define a multiplication by

$$(\vec{u}, F) \cdot (\vec{v}, G) = (\vec{u} + F(\vec{v}), FG).$$

This multiplication is a group operation.

Let $f = Ax + \vec{v}$ be an isometry. Identify f with $(\vec{v}, A) \in \mathbb{R}^4 \times O(1,3)$. The interested reader will show that this identification is a group isomorphism. Consequently, the Poincare Group is the semi-direct product of the translations with the Lorentz group.

$$\mathbb{R}^4 \ltimes O(1,3).$$

Here is a construction of examples of Lorentz transformations.

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, R_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}, R_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$B_x = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B_y = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\theta & 0 & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B_z = \begin{pmatrix} \cosh\theta & 0 & 0 & \sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix}.$$

Example 5.8. Let $g = \begin{pmatrix} -1 & 0 \\ & -1 \end{pmatrix}$. We want to classify $A \in O(1,1)$. Set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and observe that

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

. Therefore the equation $A^t = g^{-1}A^{-1}g$ is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

In this case that $\det(A) = 1$ we conclude that $a = d$ and $b = c$. Therefore,

$$A = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$$

and $u^2 - v^2 = 1$, and conversely.

In the case $\det(A) = -1$, $a = -d$, $b = -c$ so that

$$A = \begin{pmatrix} u & v \\ -v & -u \end{pmatrix}$$

and $u^2 - v^2 = 1$, and conversely.

Chapter 6

Metric Spaces

Definition 6.1. Let X be a set. A **metric** or **distance function** on X is a function $d : X \times X \to \mathbb{R}^+$ such that

1. for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,
2. for all $x, y \in X$, $d(x, y) = d(y, x)$,
3. for all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

A **metric space** is a set X equipped with a metric d; we write it as (X, d).

Example 6.2. The **usual metric** (or standard) on $\mathbb{R}^n$ is the map defined by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

Definition 6.3. Let (X, d) be a metric space.

a. The **open ball** of radius $\epsilon > 0$ centered at x is the set

$$N_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}.$$

b. A subset $O \subseteq X$ is said to be an **open set** if for each $x \in O$ there is an $\epsilon_x > 0$ such that $N_\epsilon(x) \subseteq O$.
c. The sequence $\{x_n\}$ is said to **converge to** x if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \epsilon$.
d. The subset $V \subseteq X$ is said to be **closed** if for any sequence in V, say $\{x_n\}$, if $x_n \to x$, then $x \in V$.
e. The **interior** of a set $T \subseteq X$ is the collection of points $x \in T$ satisfying there is an $\epsilon > 0$ such that $N_\epsilon(x) \subseteq T$.

f. The **closure** of a set $T \subseteq X$ is the set of $x \in X$ such that there is a sequence in T which converges to x.
g. A subset of X that is both open and closed is said to be **clopen**. Notice that both X and $\emptyset$ are clopen. If these are the only clopen subsets of X, then the space is said to be **connected**. Otherwise, X is said to be **disconnected**.

Definition 6.4. Let V be a $\mathbb{R}$-vector space. A **norm** on X is a function $||\cdot|| : V \to \mathbb{R}^+$ satisfying

1. for all $\vec{v} \in V$, $||\vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$,
2. for all $\vec{v} \in V$ and $\alpha \in \mathbb{R}$, $||\alpha\vec{v}|| = |\alpha|||\vec{v}||$,
3. for all $\vec{v}, \vec{u} \in V$, $||\vec{v} + \vec{u}|| \leq ||\vec{v}|| + ||\vec{u}||$.

A **normed vector space** is simply a vector space equipped with a norm: $(V, ||\cdot||)$.

Example 6.5. The standard norm on $\mathbb{R}^n$ is given by $||(x_1, \ldots, x_n)|| = \sqrt{x_1^2 + \ldots + x_n^2}$.

Proposition 6.6. *Suppose* $(V, ||\cdot||)$ *is a normed* $\mathbb{R}$*-vector space. Let* $d : V \times V \to \mathbb{R}^+$ *be the function defined by* $d(\vec{v}, \vec{u}) = ||\vec{v} - \vec{u}||$. *Then* d *is a metric. Therefore, every normed vector space is a metric space.*

Definition 6.7. Let V be a $\mathbb{R}$-vector space. An **inner product** on V is a function $<\cdot,\cdot>: V \times V \to \mathbb{R}^+$ satisfying the following properties

1. for all $\vec{v_1}, \vec{v_2}, \vec{u} \in V$ and $\alpha, \beta \in \mathbb{R}$, $< \alpha\vec{v_1} + \beta\vec{v_2}, \vec{u} >= \alpha < \vec{v_1}, \vec{u} > +\beta < \vec{v_2}, \vec{u} >$,
2. for all $\vec{v}, \vec{u} \in V$, $< \vec{v}, \vec{u} >=< \vec{u}, \vec{v} >$,
3. $< \vec{v}, \vec{v} >= 0$ if and only if $\vec{v} = \vec{0}$.

A space $(V, <\cdot,\cdot>)$ equipped with an inner product is called an **inner product space**.

Proposition 6.8. *Let* $(V, <\cdot,\cdot>)$ *be an inner product space. Set* $||\vec{v}|| =< \vec{v}, \vec{v} >^2$. *Then* $(V, ||\cdot||)$ *is a normed vector space.*

Example 6.9. The usual dot product on $\mathbb{R}^n$ defined by $\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i$ is an inner product. Therefore, each $\mathbb{R}^n$ is an inner product space.

One should note that a set can be equipped with different metrics. For example, the discrete metric is given by $d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{otherwise.} \end{cases}$. It is a fact that when equipped with the discrete metric every subset is open and every subset is closed.

However, there is a notion of equivalence amongst metrics. Suppose d_1 and d_2 are metrics on the set X. The metrics are said to be **equivalent** if the open subsets of (X, d_1) are precisely the open subsets of (X, d_2).

Theorem 6.10. *Let X be a set and suppose d_1, d_2 are metrics on X. The following statements are equivalent.*

1. *The metrics d_1 and d_2 are equivalent.*
2. *The closed subsets of (X, d_1) are precisely the closed subsets of (X, d_2).*
3. *For any sequence $\{x_n\}$ in X, the sequence converges to x relative to d_1 if and only if the sequence converges to x relative to d_2.*
4. *If O is an open set relative to d_1 and $x \in O$, then there is an open set U relative to d_2 such that $x \in U \subseteq O$, and vice-versa.*

Definition 6.11. A subset S of $\mathbb{R}^n$ is said to be a **bounded** set if there is some $M \in \mathbb{R}$ such that $||v|| \leq M$ for all $v \in S$.

A closed and bounded subset of $\mathbb{R}^n$ is a **compact** set. (There is a more general definition of compactness that is equivalent to the way I am defining it here (Heine-Borel).)

Definition 6.12. You should be familiar with the definition of a continuous function. We define it in general now. Let (X, d) and (Y, d) be metric spaces. A function $f : X \to Y$ is said to be **continuous** at $x \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $d(x, z) < \delta$, then $e(f(x), f(z)) < \epsilon$. A function that is continuous at every point is called a **continuous function**.

Proposition 6.13. *Suppose (X, d) and (Y, d) are metric spaces and let $f : X \to Y$ be a function. The following statements are equivalent.*

1. *The function f is continuous.*
2. *For every open subset $O \subseteq Y$, $f^{-1}(O)$ is an open subset of X.*
3. *For every closed subset $C \subseteq Y$, $f^{-1}(C)$ is a closed subset of X.*
4. *If $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.*

Example 6.14. We define the i**th projection map** $\pi_i : \mathbb{R}^k \to \mathbb{R}$ by $\pi_i((x_1, \ldots, x_n) = x_i$. Each projection map is continuous.

Theorem 6.15. *Suppose (X, d) and (Y, d) are metric spaces and let $f : X \to Y$ be a continuous function. If $S \subseteq X$ is compact, then so is $f(S)$.*

We now turn to normed vector spaces.

Definition 6.16. Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be two norms on V. The norms are said to be **norm equivalent** if the distance functions induced by the norms are equivalent.

Theorem 6.17. *Let V be a finite dimensional vector space. Then any two norms on V are norm equivalent.*

Proof. Norm Equivalence for Finite dimensional spaces

□

Example 6.18. Consider $V = M_n(\mathbb{R})$. When we view V as being a copy of $\mathbb{R}^{n^2}$ then our usual metric turns into a matrix metric. Let $A = (a_{ij})$ and $B = (b_{ij})$ belong to $M_n(\mathbb{R})$. Then thus usual distance on $M_n(\mathbb{R})$ is given by

$$d(A, B) = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{n} (a_{ij} - b_{ij})^2}.$$

This induces the matrix norm

$$||A|| = \sqrt{\sum_{i=1}^{n} \sum_{i=1}^{n} a_{ij}^2}.$$

However, there is another norm on $M_n(\mathbb{R})$ which is known as the **operator norm**.

(Wait for the punch line.) Let V be a normed vector space and let $T : V \to V$ be an operator on V. Set

$$||T|| = \sup\{||Tv|| : ||v|| = 1\}.$$

The operator T is called a **bounded operator** if this supremum exists. The set of bounded operators on V is a vector subspace and, in fact, closed under composition; the space is denoted by $\mathbb{B}(V)$. The operator norm makes $\mathbb{B}(V)$ into a normed vector space.

If V is a finite dimensional real-vector space, then the set $\{v \in V : ||v|| = 1\}$ is homeomorphic to the n-sphere and hence is compact so that the set used to define the operator norm is a bounded subset of $\mathbb{R}$ and hence has a supremum. Therefore, $\mathbb{B}(\mathbb{R}^n) = M_n(\mathbb{R})$, i.e. any matrix in $M_n(\mathbb{R})$ is a bounded operator. Consequently, the usual metric on $M_n(\mathbb{R})$ is equivalent to the operator norm.

Here are some useful properties about $\mathbb{R}^n$ and hence any finite dimensional vector space.

Definition 6.19. Let $(V, ||\cdot||)$ be a normed vector space and let $\{v_n\}$ be a sequence. The sequence is called a **Cauchy sequence** if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$

$$||v_n - v_m|| < \epsilon.$$

V is said to be **complete** if every Cauchy sequence converges. A complete normed vector space is called a **Banach space**.

E.g. Every finite dimensional real vector space is a Banach space. It follows that any finite dimensional subspace of a (possibly infinite dimensional) normed vector space is closed. Every closed subspace of a Banach space is complete.

A normed vector space that is not complete, and hence not Banach.

More on Banach spaces.

Chapter 7

Differentiability

As we did in the calculus sequence once we talk about continuity we next talk about differentiability. Recall the following different kinds of derivatives. The first two sections are a recap from Calculus 3. This is all done in terms of the usual metric on Euclidean spaces.

7.1 Vector-Valued Functions / Curves

Let I be an interval of $\mathbb{R}$ and let $F : I \to \mathbb{R}^n$ be a vector-valued function. We also call this a **curve** in $\mathbb{R}^n$. First of all, we can write $F = (f_1, \ldots, f_n)$ where each $f_i : I \to \mathbb{R}$ is the ith component of F. Notice that $f_i = \pi_i \circ F$ is a function from $I \to \mathbb{R}$.

Definition 7.1. The derivative of F (if it exists) is given by

$$F'(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = (f_1'(t), \cdots, f_n'(t))$$

If the curve has derivatives of all orders, then the curve is called **smooth**

Example 7.2. The exponential map $exp : \mathbb{R} \to \mathbb{C}$ defined by $exp(t) = e^{it} = \cos t + i \sin t$ is a smooth map. What is its derivative?

7.2 Functions of Several Variables

Next, we consider a function $f : O \to \mathbb{R}$ where O is an open subset of $\mathbb{R}^n$. Let i be a fixed coordinate of $\mathbb{R}^n$. We define the i-th partial derivative of f (if it exists) as

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

Example 7.3. Let $f(x, y, z) = x^2yz + \sin x$. Then $\frac{\partial f}{\partial x} = 2xyz + \cos x$, $\frac{\partial f}{\partial y} = x^2z$, and $\frac{\partial f}{\partial z} = x^2y$.

We could go into directional derivatives, but we won't need it. We do want to remind you of the gradient

$$\nabla f(\vec{x}) = (\frac{\partial f}{\partial x_1}(\vec{x}), \ldots, \frac{\partial f}{\partial x_n}(\vec{x})).$$

Definition 7.4. Let D be an open region in $\mathbb{R}^n$ and let $f : D \subset R^n \to \mathbb{R}$ be a function of several variables. Let $\vec{v}_0 = (v_1, \ldots, v_n)$ be an interior point of D. Then f is differentiable at $\vec{v}$ means

1. Each partial derivative $\frac{\partial f}{\partial x_i}$ exists at the point $\vec{v}_0$.

2. $$\lim_{\vec{x} \to \vec{v}_0} \frac{f(\vec{x}) - f(\vec{v}_0 - (\sum_{i=1}^{n} [\frac{\partial f}{\partial x_i}(\vec{v}_0)(x_i - v_i))}{||\vec{x} - \vec{v}_0||} = 0$$

We can rewrite the limit as

$$0 = \lim_{\vec{x} \to \vec{v}_0} \frac{f(\vec{x}) - f(\vec{v}_0 - \nabla f(\vec{v}_0) \cdot (\vec{x} - \vec{v}_0)}{||\vec{x} - \vec{v}_0||}.$$

Theorem 7.5 (Sufficient condition)**.** *Suppose that the partial derivatives all exist and are continuous on a neighborhood of the point $\vec{v}_0$. Then f is differentiable at $\vec{v}_0$.*

7.3 Vector Fields

Let $O \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets, and let $F : O \to V$. Such a function is called a vector field. Set $F_i = \pi_i \circ F$ and call this the **component** function of F. Then F_i is a function of several variables and so we can talk about the partial derivatives of each component. Notice that $F = (F_1, \ldots, F_m)$. Each F_i has n many partial derivatives for a total of $n \cdot m$.

Ex. The gradient is an example of a vector field.

Definition 7.6. We say that the vector field F is **differentiable** at $\vec{x}$ if each partial derivative of each component function of F exists. The function F is called **smooth** if each component function has continuous partial derivatives of all orders.

A smooth bijection is called a **diffeomorphism**.

Given such an F we can define a matrix whose entries are the partial derivatives of F. The i,j-entry is the function $\frac{\partial F_i}{\partial x_j}$. This matrix is called the **total derivative** or the **Jacobian matrix** of F and denoted by $\mathbf{D}F$.

Example 7.7. Set $F((x,y)) = (y,x,xy)$ so that $F : \mathbb{R}^2 \to \mathbb{R}^3$. Then $F_1(x,y) = y$, $F_2(x,y) = x$, and $F_3(x,y) = xy$. Then $\frac{\partial F_1}{\partial x} = 0$, $\frac{\partial F_1}{\partial y} = 1$, $\frac{\partial F_2}{\partial x} = 1$, $\frac{\partial F_2}{\partial y} = 0$, $\frac{\partial F_3}{\partial x} = y$, and $\frac{\partial F_3}{\partial y} = x$. The total derivative of F is

$$(\mathbf{D}F)(x,y) = \begin{pmatrix} 0 & 1 & y \\ 1 & 0 & x \end{pmatrix}$$

Here is more on differentiability of vector fields: Total Derivative

The main point is the following.

Theorem 7.8. *The vector field $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{v} \in \mathbb{R}^n$ if and only if each component function f_i is differentiable at $\vec{v}$.*

7.4 Complex Functions

We are interested in functions $f : G \to \mathbb{C}$ where G is an open subset of $\mathbb{C}$. This is an exmple of a vector field and so we can talk about differentiability in terms of vector fields. However, recall the definition of the derivative on $\mathbb{R}$: $f : \mathbb{R} \to \mathbb{R}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This makes sense since R is a field. Well, so is $\mathbb{C}$.

Let $f : G \to \mathbb{C}$ and $G \subseteq \mathbb{C}$ an open set. The **complex derivative** of f at $z_0 \in G$ is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Such a function that is differentiable on an open set containing z_0 is called **holomorphic**. Such a function that is holomorphic on all of $\mathbb{C}$ is called **entire**. Notice that the derivative of an entire function $f : \mathbb{C} \to \mathbb{C}$ is again a function $f' : \mathbb{C} \to \mathbb{C}$. This is much nicer than derivatives of vector fields.

Example 7.9. Let $F : \mathbb{C} \to \mathbb{C}$ be defined by $F(z) = \overline{z}$. Then $F(x,y) = (x,-y)$. The total derivative of F is

$$\mathbf{D}F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and hence F is differentiable as a vector field. However, it is not complex differentiable.

Example 7.10. Let $n \in \mathbb{N}$ and set $F(z) = z^n$. Then $F'(z) = nz^{n-1}$.

Example 7.11. Let $exp : \mathbb{C} \to \mathbb{C}$ be defined for $z = x + iy$ by

$$exp(z) = e^z = e^x(\cos y + i \sin y).$$

Then $exp'(z) = e^z$.

Theorem 7.12. *(Cauchy Riemann Equations) Suppose that f is complex differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives of f exist and satisfy*

$$\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$$

.

Example 7.13. Let $f(z) = z^3$. Then the total derivative of f is

$$\mathbf{D}f = \begin{pmatrix} 3x^2 - 3y^2 & 6xy \\ -6xy & 3x^2 - 3y^2 \end{pmatrix}.$$

7.5 Hamiltonians

When we view $\mathbb{C}$ as $\mathbb{R}^2$ with a new multiplication, the question that arises is whether there is a multiplication on $\mathbb{R}^3$ that makes into a $\mathbb{R}$-algebra. The answer is no and in a strong way. Before we get to that, let's show that there is a way of making $\mathbb{R}^4$ into a something that resembles a field.

Let $\{e_1, e_2, e_3, e_4\}$ be the usual standard basis for $\mathbb{R}^4$. But instead let's change notation and set $\mathbf{1} = e_1$, $\mathbf{i} = e_2$, $\mathbf{j} = e_3$ and $\mathbf{k} = e_4$. Then every element of $x \in \mathbb{R}^4$ can be written as $x = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$.

Take two such elements $x = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $y = e\mathbf{1} + f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ and add component-wise and define multiplication as

$$(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(e\mathbf{1} + f\mathbf{i} + g\mathbf{j} + h\mathbf{k}) = m\mathbf{1} + n\mathbf{i} + p\mathbf{j} + q\mathbf{k},$$

where

$$\begin{aligned} m &= ae - bf - cg - dh \\ n &= af + be \\ p &= \\ q &= \end{aligned}$$

We use $\mathbb{H}$ to denote the underlying set $\mathbb{R}^4$ equipped with this multiplication and call it the division ring of **quaternions**; discovered by Hamilton. You should check that this operations make $\mathbb{H}$ into a non-commutative ring with identity so that every non-zero element has a multiplicative inverse, i.e. it is a division ring (aka skew field). Here is a different way of viewing a quaternion. We define a map $\Phi : \mathbb{H} \to M_2(\mathbb{C})$ by

$$\Phi(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = \begin{pmatrix} a + id & -b - ic \\ b - ic & a - id \end{pmatrix}$$

We leave it to the interested reader to check that Φ is an injective ring homomorphism. Therefore, the inverse of a nonzero x is the matrix inverse. Moreover, it is a normed algebra with the usual norm $||a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}|| = \sqrt{a^2 + b^2 + c^2 + d^2}$. Put nicely,

$$(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}).$$

It follows that the set of quaternions of unit length is a multiplicative subgroup of $\mathbb{H} \setminus \{0\}$. The final thing to notice is that the quaternions of unit length is $\mathbb{S}_3$. Therefore, there is a group structure on $\mathbb{S}_3$.

In case you are looking for more finite dimensional division algebras:

Finite Dimensional Division Algebras over $\mathbb{R}$

Chapter 8

Manifolds

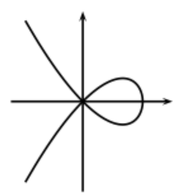


Figure 8.1: A curve that is not a manifold

In this chapter we want to briefly talk about manifolds. The nicest kinds of manifolds will be the ones that are subspaces of $\mathbb{R}^n$. But be careful this does not mean that the "dimension" of the manifold is n, e.g. We can $\mathbb{R}$ can be viewed as the x-axis in $\mathbb{R}^2$. The main property of an n-manifold is that locally at every point it looks like $\mathbb{R}^n$. Figure 8 provides an example of a closed subspace of $\mathbb{R}^2$ that is not a manifold.

Definition 8.1. Here are the properties that define a **topological n-manifold** M:

1. (M, d) is a metric space.

2. Every point in M has a neighborhood that that is homeomorphic to an open subset of $\mathbb{R}^n$.

3. M has a countable base.

Remark 8.2. An interesting example is when $n = 0$. By design $\mathbb{R}^0 = \{0\}$. Then there is a unique topological 0-manifold,up to homeomorphism. It is a countable discrete space. This is often called a **discrete manifold**.

Given a topological n-manifold M, a **coordinate chart** is a pair (U, φ) where U is an open subset of U and $\varphi : U \to \mathbb{R}^n$, $\varphi(U)$ is open, and φ is a homeomorphism between U and $\varphi(U)$. The set U is called the domain of the chart. We say that U is**centered at** p if $\varphi(p) = 0$. An **atlas** for M is a collection $\mathcal{A}$ of charts whose domains cover M.

Two charts (U, φ) and (V, ψ) are said to be **smoothly compatible** if either the domains are disjoint or the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism (that is, continuous partial derivatives of all orders). An atlas $\mathcal{A}$ is called a **smooth atlas** if every pair of charts in the atlas are smoothly compatible.

Definition 8.3. A **smooth n-manifold** is a topological n-manifold that is equipped with a smooth atlas.

Example 8.4. Here are some examples of smooth manifolds that are subsets of $\mathbb{R}^m$.

1. Every k-dimensional subspace of $\mathbb{R}^n$ is a smooth k-manifold.

2. Any finite dimensional real vector space V is a smooth n-manifold for $n = \dim_{\mathbb{R}} V$.

3. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth map. Then the graph of $\mathbb{R}^n$ (as a subset of $\mathbb{R}^{n+1}$) is a smooth n-manifold.

4. Every open subset of $\mathbb{R}^n$ is a smooth manifold with an atlas consisting of the single chart (U, id_U).

5. $\mathbb{S}_n$ is a smooth manifold of dimension n. For each $i = 1, \ldots, n+1$, define

$$U_i^+ = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}_n : x_i > 0\}.$$

   And define U_i^- analogously. Next for each $i = 1, \ldots, n+1$ define the map $\phi_i^+ : U_i^+ \to \mathbb{R}^n$ by mapping $(x_1, \ldots, x_{n+1})$ to the n-tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Define ϕ_i^- analogously. Then $\mathcal{A} = \{(U_i^\pm, \phi_i^\pm)\}$.

Definition 8.5. Let M be a smooth n-manifold and $p \in M$. We define a **tangent vector at** p to be a vector in $\mathbb{R}^n$ of the form $(\varphi \circ \gamma)'(0)$ where $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve such that $\gamma(0) = p$ and p belongs to a chart (U, φ).

We let TM_p denote the set of tangent vectors at p.

Example 8.6. Let $M = \mathbb{S}_1$. If $p = 1$, then $T_p = \mathbb{R}$, and if $p = i$, then $T_p = \mathbb{R}$.

Proposition 8.7. *Let M be a smooth n-manifold and $p \in M$. The tangent space at p is a vector space.*

Proof. Let $\vec{v}_1, \vec{v}_2 \in T_p$. This means there are smooth curves, say $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to M$ such that $\gamma_1(0) = p = \gamma_2(0)$ and $(\varphi \circ \gamma_1)' = \vec{v}_1$ and $(\varphi \circ \gamma_2)' = \vec{v}_2$ for some chart (U, φ) centered at p. Consider the smooth curve

$$\gamma = \varphi^{-1} \circ ((\varphi \circ \gamma_1 + \varphi \circ \gamma_2)).$$

We leave it to the interested reader to check that $\gamma : (-\epsilon, \epsilon) \to M$, $(\varphi \circ \gamma)(0) = p$, and $(\varphi \circ \gamma)'(0) = \vec{v}_1 + \vec{v}_2$.

□

If $M \subseteq \mathbb{R}^m$ is a smooth n-manifold, then we can also view the tangent space at p as those vectors in $\mathbb{R}^n$ satisfying $\vec{v} = \gamma'(t)$ for some $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(t) = 0$. Then the tangent vectors form a subspace of $\mathbb{R}^m$ of dimension n. For example, in the case of $\mathbb{S}_1 \subseteq \mathbb{R}^2$, then tangent space at $p = 1$ is $T_p = \mathbb{R} \cdot i$.

Chapter 9

Lie Groups

9.1 Lie Groups

Definition 9.1. A **Lie group** is a group $(G, \cdot, e)$ that is also a smooth n-manifold such that multiplication and inversion are smooth maps.

The interest in Lie groups stems from the observation that many sets of symmetries of physical objects that occur end up being Lie groups. Now, in studying Lie groups, it turns out that group theory is not as easy as linear algebra. This is where the tangent space comes into play. The tangent space at the identity of G is a vector space, and there is a way of transferring information from the tangent space back to the Lie group.

Example 9.2. The simplest examples of Lie groups are $\mathbb{R}^n$ and therefore also $M_n(\mathbb{R})$. However, for a vector space we need to replace the word multiplication with addition.

Proposition 9.3. *The determinant* $\det(\cdot) : M_n(R) \to \mathbb{R}$*, is a smooth surjection. When restricted to* $GL_n(\mathbb{R})$*, it is a group homomorphism onto* $(\mathbb{R}^*, \cdot, 1)$.

Proof. The formal determinant is a polynomial in n^2-variables:

$$\det((a_i j)) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \Pi_{i=1}^n a_{i\sigma(i)}.$$

Recall that S_n is the group of permutations on the set $\{1, 2, \ldots, n\}$ and that the signature of a permutation is defined as $\pm$ depending on whether the permutation is even ($\operatorname{sgn}(\sigma) = 1$) or odd ($\operatorname{sgn}(\sigma) = -1$). Since polynomials are smooth, the determinant is continuous. Clearly, the determinant map is a surjection.

Observe that $GL_n(\mathbb{R}) = \det^{-1}((-\infty, 0)\cup(0,\infty)$ and hence is an open subset of $M_n(\mathbb{R})$. The second sentence in the proposition is just recognition that the determinant of a product is the product of the determinants.

□

Let's look at matrix multiplication in a more algorithmic way. Define the function $F : \mathbb{R}^{2n^2} \to \mathbb{R}^{n^2}$ by $F((a_i)) = (x_i)$ as follows. For $i \in \{1, \ldots, n^2\}$ write $i = kn + j$ for unique $k \in \{0, 1, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$. Then

$$x_i = x_{kn+j} = \sum_{t=1}^{n} a_{kn+t}a_{n^2+(t-1)n+j}.$$

Next, take two matrices $A = (a_{ij})$ and matrix $B = (b_{ij})$ and view them together as one $2n^2$-tuple

$$(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, ..., a_{2n}, \ldots, a_{nn}, b_{11}, \ldots, b_{1n}, b_{21}, \ldots, b_{2n}, \ldots, b_{nn}).$$

Denote this assignment as $m : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$; m is a bijection and thus has an inverse. Matrix multiplication is the composition $m^{-1} \circ F \circ m$. Clearly, m, m^{-1}, and F are smooth maps, and hence matrix multiplication is a smooth map.

Next, consider inversion defined on $GL_n(\mathbb{R})$. Since the computation of the inverse via cofactor expansion is algorithmic and depends only on determinants of smaller matrices and transposition it follows that inversion is smooth.

Theorem 9.4. *For any $n \in \mathbb{N}$, $GL_n(\mathbb{R})$ is a matrix Lie group.*

Notice that for a subgroup of $GL_n(\mathbb{R})$ to be a Lie group we also need it to be a real n-manifold. When $n = 0$ this means we are dealing with a **discrete group**. When $n > 0$, then this is what is usually termed as a **continuous matrix group**. Incidentally, any discrete matrix group happens to be a closed subgroup of $GL(n, \mathbb{R})$. There is a heavy duty theorem of Cartan known as the Closed Subgroup Theorem which leads us to define a **matrix Lie group** as a closed subgroup of $GL_n(\mathbb{C})$. (Recall that by closed we mean topologically. This means that if $\{A_n\} \subseteq H$ and $A_n \to A$ and $A \in GL_n(\mathbb{R})$, then $A \in H$.)

Some examples are in order. Let $G = SL(\mathbb{Z})$ and observe that G is a discrete matrix group. We know that the circle group $\mathbb{S}_1$ is a group and it is a subgroup of $GL_1(\mathbb{C})$. Also, we can view $\mathbb{S}_3$ as a subgroup of the non-zero quaternions which can be viewed as inside $GL_2(\mathbb{C})$. Since both $\mathbb{S}_1$ and $\mathbb{S}_3$ are compact, they are therefore both continuous matrix groups.

Given a matrix Lie group G, we want to study the tangent space of G at its identity I_n denoted T_1. Tangent vectors in T_1 are vectors of the form $\gamma'(0)$ where $\gamma : (\epsilon, \epsilon) \to G$ is a smooth curve and $\gamma(0) = I_n$.

Example 9.5. Let $\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Clearly, γ is a smooth curve into $M_2(\mathbb{R})$ and in fact we know that γ maps into $SO(2)$. So, what is $\gamma'(0)$. We know that $\gamma'(t) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix}$ so

$$\gamma'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a tangent vector. Recall that $SO(2)$ is the circle group $\mathbb{S}_1$ and that a tangent vector of 1 is one of the form $\mathbb{R}i$. As matrices go we get for any θ

$$\gamma_\theta(t) = \begin{pmatrix} \cos\theta t & -\sin\theta t \\ \sin\theta t & \cos\theta t \end{pmatrix}$$

is a smooth curve satisfying

$$\gamma_\theta'(t) = \begin{pmatrix} -\theta\sin\theta t & -\theta\cos\theta t \\ \theta\cos\theta t & -\theta\sin\theta t \end{pmatrix}$$

so that T_1 consists of matrices of the form

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

i.e. $\mathbb{R}i$.

Example 9.6. In this example we aim to study the tangent vectors of $O(n)$. Let $\gamma(t) = A(t)$ be a smooth curve in $O(n)$ such that $\gamma(0) = I$. Then for every $s \in (-\epsilon, \epsilon)$, $A(s)$ satisfies $A(s)A^t(s) = I_n$. Taking the derivative of both sides and recognizing that $\frac{d}{ds}I_n = 0$ yields

$$0 = A'(s)A^t(s) + A(s)(A^t(s))' = A'(s)A^t(s) + A(s)A'(s)^t.$$

Now, $\gamma'(0) = A'(0)$ which by the equation Since $0 = A'(0) + A'(0)^t$. It follows that a tangent vector satisfies the matrix equation $X + X^t = 0$, which is equivalent to saying that $X^t = -X$. Any matrix satisfying this is called **skew-symmetric**. Therefore, the tangent space T_1 of $O(n)$ is a subspace of the space of skew-symmetric matrices.

We denote the space of skew-symmetric matrices by $\mathrm{Skew}_n(\mathbb{R})$, and the space of symmetric matrices by $\mathrm{Symm}_n(\mathbb{R})$.

We now turn to another important example of a Lie group. On $\mathbb{C}^n$ we define an inner product as follows: let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$.

$$< z, w >= \sum_{i=1}^{n} z_i \cdot \overline{w_i}.$$

Using the transpose guarantees that $< z, z >$ is a real non-negative number. This generalizes the inner product on $\mathbb{R}^n$, but this is an example of a **Hermitian inner product**:

1. $< z, w >= \overline{< w, z >}$,

2. $< cz_1 + z_2, w >= c < z_1, w > + < z_2, w >$,

3. $< z, cw_1 + w_2 >= \overline{c} < z, w_1 > + < z, w_2 >$,

4. $< z, z >$ is non-negative real number and $< z, z >= 0$ if and only if $z = 0$.

In matrix form the inner product is given by $< z, w >= z^t \cdot \overline{w}$.

We let $U(n)$ be the set of matrices that preserve the inner product, and call this the **unitary group**. As before $SU(n)$ is the subgroup of unitary matrices with determinant equal to 1. The matrix A is unitary if and only if $A\overline{A}^t = I$. Therefore, if $A \in U(n)$, then $\det(A) = \pm 1$.

Definition 9.7. Let $A \in M_n(\mathbb{C})$. The conjugate transpose of A, $\overline{A}^t$, is called the **adjoint** of A and instead denote it by A^*. (So $A^* = \overline{A}^t$.) We call a matrix **normal** if $AA^* = A^*A$.

9.2 Lie Algebras

Recall that a **bilinear map** on a vector space V is a function $< \cdot, \cdot >: V \to V$ that satisfies the following property: for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$

$$< \alpha u + v, w >= \alpha < u, w > + < v, w >$$

and

$$< u, \alpha v + w >= \alpha < u, v > + < u, w > .$$

The bilinear map is said to be **alternating** if $< u, u >= 0$ for all $u \in V$. (One will also find the term skew-symmetric in place of alternating, but I would rather use that term for matrices .)

Definition 9.8. A **Lie algebra** is a (finite-dimensional) vector space $\mathfrak{g}$ that is equipped with an alternating bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$$

for all $u, v, w \in \mathfrak{g}$. This equation is called the **Jacobi identity**, and such a map is called a **Lie bracket**.

Example 9.9. Consider the zero-lie bracket defined on $\mathbb{R}^n$: $[u,v] = 0$ for all $u, v \in \mathbb{R}^n$. This makes $\mathbb{R}^n$ into a trivial Lie algebra. A Lie algebra is called *abelian* if its Lie bracket is identically 0.

Example 9.10. On $\mathbb{R}^3$ we define the Lie bracket to be the cross product. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Define:

$$[\vec{u}, \vec{v}] = \vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Example 9.11. On $M_n(\mathbb{C})$ define a Lie bracket by

$$[A, B] = AB - BA.$$

Then $M_n(\mathbb{C})$ is a Lie algebra. Furthermore, any vector subspace of $M_n(\mathbb{C})$ which is closed under the Lie bracket is also a Lie algebra.

Example 9.12. Let $\mathfrak{o}(3) = \mathrm{Skew}_3(\mathbb{R})$. An arbitrary matrix has the form $\begin{pmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{pmatrix}$, whence $\mathfrak{o}(3)$ is a 3-dimensional Lie algebra. This Lie algebra is isomorphic to the Lie algebra $\mathbb{R}^3$ equipped with the cross product.

Remark 9.13. Notice that on $\mathbb{R}^n$ we can discuss a Lie bracket by saying what it does to the standard basis. Alternating says that $[e_i, e_i] = 0$ and if $i < j$, then $[e_j, e_i] = -[e_i, e_j]$ so it suffices to say what happens to pairs of basis elements $[e_i, e_j]$ with $i < j$.

In this way we can describe the cross-product on $\mathbb{R}^3$ by saying that $[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$, and $[e_2, e_3] = e_1$.

Example 9.14. On $\mathbb{R}^2$ define a Lie bracket by

$$[(x_1, y_2), (x_2, y_2)] = (x_1y_2 - y_1x_2, 0).$$

In other words, $[e_1, e_2] = e_1$.

Notice that if we had instead defined $[e_1, e_2] = e_2$, then this different Lie algebra is not given by a matrix A and Lie bracket $[\cdot,\cdot]_A$ for any $A \in M_2(\mathbb{R})$. However,

Example 9.15. On $\mathbb{R}^3$, define a bracket by

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0 \quad [e_2, e_3] = 0$$

and extend to all of $\mathbb{R}^3$ by bilinearity and alternating. This Lie algebra is called the **Heisenberg algebra**.

Definition 9.16. Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a **Lie subalgebra** if for all $u_1, u_2 \in \mathfrak{h}$, $[u_1, u_2] in \mathfrak{h}$.

The Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called an **ideal** if for all $u \in \mathfrak{g}$ and $v \in \mathfrak{h}$, $[u, v] \in \mathfrak{h}$.

Definition 9.17. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be two Lie algebras. A **Lie algebra homomorphism** is a linear transformation $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that for all $u, v \in \mathfrak{g}$

$$[\varphi(u), \varphi(v)]_{\mathfrak{h}} = \varphi([u, v]_{\mathfrak{g}}).$$

A Lie algebra homomorphism that is a bijection is a Lie algebra isomorphism.

Proposition 9.18. *The kernel of a Lie algebra homomorphism is a Lie ideal. Conversely, any Lie ideal of a Lie algebra is the kernel of a Lie algebra homomorphism.*

9.3 Exponentiation

In this section we define the exponent of a matrix. Recall that over $\mathbb{R}$ the function e^x can be written as a MacLaurin/Taylor Series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We extend this definition to sets of matrices. Namely, for $A \in M_n(\mathbb{R})$ define

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

What is fascinating is that this series converges. In fact, we show that it converges absolutely. To see this We first state a nice result regarding the usual norm on $M_n(\mathbb{R})$. A matrix norm satisfying this property is called **submultiplicative**.

Proposition 9.19. *For any $A, B \in M_n(\mathbb{R})$*

$$||AB|| \leq ||A||||B||.$$

Proof. □

Returning to exponential function, we want the output e^A to be a matrix. So we need to determine what is its ij-th entry. Let $1 \leq i, j \leq n$. We let c_{ij}^n denote the ij-th entry of the matrix A^n. This is well-defined for every natural n. We want to show that for each pair i, j the series

$$\sum_{n=0}^{\infty} \frac{c_{ij}^n}{n!}$$

converges. We show it absolutely converges, and therefore converges. Since $||c_{ij}^n|| \leq ||A^n|| \leq ||A||^n$, we know that

$$\sum_{n=0}^{\infty} ||\frac{c_{ij}^n}{n!}|| \leq \sum_{n=0}^{\infty} \frac{||A||^n}{n!} = e^{||A||}.$$

Therefore, the series absolutely converges for each pair i, j and e^A makes sense.

Proposition 9.20. *Let $\gamma : \mathbb{R} \to M_n(\mathbb{R})$ be defined by $\gamma(t) = e^{tA}$. Then*

$$\frac{d}{dt}\gamma(t) = Ae^{tA}.$$

Lemma 9.21.

$$e^0 = I$$

Proposition 9.22. *Suppose that $AB = BA$. Then*

$$e^{A+B} = e^A e^B.$$

In particular, $(e^A)^{-1} = e^{-A}$. It follows that for any matrix $A \in M_n(\mathbb{R})$, $e^A \in GL_n(\mathbb{R})$.

Corollary 9.23. *For any matrix $A \in M_n(\mathbb{R})$,*

$$\det e^A > 0.$$

Proof.

$$e^A = (e^{\frac{1}{2}A})^2 \geq 0.$$

□

Example 9.24. Let X be a skew-symmetric matrix. What kind of matrix is e^X? Well since $X^t = -X$ it follows that X and X^t commute so that

$$I = e^0 = e^{X+X^t} = e^X e^{X^t} = e^X (e^X)^t.$$

Therefore, e^X is an orthogonal matrix. So the curve $\gamma(t) = e^{tX}$ is a smooth curve in $O(n)$ for which $\gamma(0) = I$ and $\gamma'(0) = X$. So every skew-symmetric matrix is a tangent vector of $O(n)$. It follows that the Lie algebra associated the Lie Group $O(n)$ is in fact $\mathfrak{o}(n) = \text{Skew}(n)$. Furthermore, we now argue that e^X is actually a special orthogonal matrix. This follows from the fact that since the determinant of an orthogonal matrix is ± 1 and e^X has positive determinant, that for every $X \in \mathfrak{o}(n)$, $e^X \in SO(n)$. Therefore, $\mathfrak{o}(n) = \mathfrak{so}(n)$.

Here is an example where one subgroup of another Lie group has the same tangent space. It also means that the exponential map is not onto.

Example 9.25. We calculate what the exponential of a complex number is. Recall that any complex number $z = a + bi$ can be viewed as the matrix

$$Z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Set $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$, so that $Z = A + B$. Since A is a scalar matrix it commutes with B which means that $e^Z = e^{A+B} = e^A e^B$. Clearly,

$$e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix}$$

so we only need determine e^B. Since $B = b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Now $B^2 = b^2\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $B^3 = b^3\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and then starts to repeat.

$$e^B = I + \frac{b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{1!} + \frac{b^2\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}{2!} + \frac{b^3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}{1!} + \cdots$$

Notice that the 1,1-entry and 2,2-entry of this sum is given by

$$1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{b^{2n}}{(2n)!}.$$

After checking the other entries we get that

$$e^B = \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}.$$

Finally, it follows that

$$e^Z = \begin{pmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{pmatrix}.$$

which means that the exponential function restricts to an exponential function defined on the complexes:

$$e^{a+bi} = e^a \cos b + ie^a \sin b.$$

You can now prove that this function is complex differentiable, and that $\frac{de^z}{dz} = e^z$. It is also onto but not injective.

Lemma 9.26. *Let G be a matrix group and let $A(t)$ be a curve in G. Since inversion is a smooth operation the composition of γ followed by inversion is smooth: we represent this by $A(t)^{-1}$. Then*

$$\frac{dA(t)^{-1}}{dt} = -A(t)^{-1}\frac{dA}{dt}A(t)^{-1}.$$

Proof. Take the derivative of both sides of the equation $A(t)A(t)^{-1} = I$. □

Lemma 9.27. *If $Y \in T_1(G)$, then for any $M \in G$, $MTM^{-1} \in T_1(G)$.*

Proof. By definition, there is a curve $A(t)$ in G such that $A(0) = I$ and $A'(0) = Y$. Define the curve

$$B(t) = MA(t)M^{-1}$$

and notice that this is a curve in G. It is smooth since G is a Lie group and hence multiplication and inversion are smooth curves, and therefore so is their product. Clearly, $B(0) = I$ and $B'(t) = MA'(t)M^{-1} = MYM^{-1}$.

□

Theorem 9.28. *Let G be a matrix Lie group. Then the tangent space $T_1(G)$ is closed under the Lie bracket. Consequently, $T_1(G)$ is a Lie algebra.*

Proof. Let $X, Y \in T_1(G)$. This means that there are curves in G, say $A(t)$ and $B(t)$ such that $A(0) = I$ and $A'(t) = X$, and $B(0) = I$ and $B'(t) = Y$.

Define the curve $D(t) = A(t)YA(t)^{-1}$ in the tangent space and realize that this makes sense since all matrices in G are invertible. Differentiating with respect to t

$$D'(t) = A'(t)YA(t)^{-1} + A(t)Y(-A(t)A'(t)A(t))$$

yields that

$$\begin{aligned} D'(0) &= A'(0)YA(0)^{-1} - A(0)YA(0)A'(0)A(0) \\ &= XY - YX \end{aligned}$$

Alternatively, you can view $XY - YX$ as the limit (as $t \to 0$) of the tangent vectors $A'(t)YA(t)^{-1} + A(t)Y(-A(t)A'(t)A(t))$. Since $T_1(G)$ is a finite dimensional subspace of $M_n(\mathbb{R})$ it is a closed subspace and hence complete. Therefore, it is closed under limits. □

Example 9.29. We finish our discussion of Lie groups by discussing the difference between $U(n)$ and $SU(n)$. Recall that $U(n)$ denotes those matrices in $M_n(\mathbb{C})$ that preserve the (Hermitian) inner product. Specifically, these are those A for which $AA^* = I$.

As we saw before a tangent vector of $U(n)$ must satisfy $X + X^* = 0$. (Take derivative of equation $A(t)A^*(t) = I$.) Such matrices are call **skew-Hermitian**. We let $\mathfrak{u}(n)$ denote the collection of all skew-Hermitian matrices. If $X \in \mathfrak{u}(n)$, then

$$1 = e^0 = e^{X+X^*} = e^X e^{X^*} = e^X (e^X)^*$$

which shows that $e^X \in U(n)$. Since the map $\gamma(t) = e^{tX}$ is a smooth curve in $U(n)$ and $\gamma'(0) = X$ it follows that the Lie algebra of $U(n)$ is $\mathfrak{u}(n)$.

Here is some use of eigenvalues and eigenvectors. Over $\mathbb{C}$ every characteristic polynomial splits into linear factors. It follows that for every $A \in M_n(C)$ there is some invertible matrix P such that $P^{-1}AP$ is upper triangular; denote it by T. So $A = PTP^{-1}$. Then since $A^m = (PTP^{-1})^m = PT^mP^{-1}$ the calculation of e^A produces

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{A^m}{m!} \\ &= \sum_{k=0}^{\infty} \frac{(PTP^{-1})^m}{m!} \\ &= \sum_{k=0}^{\infty} \frac{PT^mP^{-1}}{m!} \\ &= P\sum_{k=0}^{\infty} \frac{T^m}{m!})P^{-1} \\ &= Pe^TP^{-1} \end{aligned}$$

It follows that $\det(e^A) = \det(e^T)$. Now every power of T is upper triangular so that e^T is also upper triangular. Furthermore, a quick checks produces that the ii-entry of T^m is

ii-th entry of T raised to the m. Therefore, the ii-th entry of e^T is of the form e^{λ_i} where λ_i is the ii-entry of T (and an eigenvalue of A). The determinant of e^T, being upper triangular, is the product of the e^{λ_i}.

$$\det(e^A) = \det(e^T) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

So, now let's see what happens when X is a tangent vector of $SU(n)$. We know that X must be skew-Hermitian and that $\det(e^X) = 1$. Is the collection of such matrices a vector space of $M_n(\mathbb{C})$? The answer is no. For example, let $X = \begin{pmatrix} 2\pi i & 0 \\ 0 & 0 \end{pmatrix}$. Then $\det(e^X) = 1$. But what about $tX = \begin{pmatrix} t2\pi i & 0 \\ 0 & 0 \end{pmatrix}$? If $t = 1/2$, then $\det(e^{\frac{1}{2}X}) = -1$. (Compare to 9.23).

So we need that for all $t \in \mathbb{R}$, $\det(e^{tX}) = 1$. Let $\mathfrak{su}(n)$ denote the collection of all skew-hermitian matrices X for which $\det(e^{tX}) = 1$ for all $t \in \mathbb{R}$.

Definition 9.30. For a matrix $A \in M_n(\mathbb{C})$, define the **trace** of A as the quantity

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_i i.$$

So the trace produces the sum of the diagonal entries. Notice that when viewed as a map $\operatorname{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ it is a linear transformation. There are two more useful properties about the trace map.

Lemma 9.31. *For all $A, B \in M_n(\mathbb{C})$,*

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Proposition 9.32. *Let $A \in M_n(\mathbb{C})$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then*

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i.$$

Proof. Let P be an invertible matrix such that $P^{-1}AP = T$ is upper triangular (Jordan canonical form). We know that $\operatorname{tr}(T) = \sum_{i=1}^{n} \lambda_i$. On the other hand

$$\operatorname{tr}(T) = \operatorname{tr}(P^{-1}AP) = \operatorname{tr}(AP^{-1}P) = \operatorname{tr}(A).$$

□

Theorem 9.33. *The Lie algebra associated to $SU(n)$ is*

$$\mathfrak{su}(n) = \{X \in \mathfrak{u}(n) : \mathrm{tr}(X) = 0\}.$$

Such a matrix is called a traceless skew-Hermitian matrix.

Chapter 10

Integration

The goal of this chapter is to define integration of complex numbers.

Recall that any continuous function $f : [a, b] \to \mathbb{R}$ is integrable. In Real Analysis you learned that there are some non-continuous functions which are also integrable. In this course, we could just worry about continuous functions as our models though I would like to define things as general as possible.

10.1 Riemann/Darboux Integral

I. $f : \mathbb{R} \to \mathbb{R}$

Let f be a bounded function on $[a, b]$. Take a finite partition P of $[a, b]$, say $a = a_0 < a_1 < \ldots < a_n = b$ and set $M_i = \sup\{f(x) : x \in [a_{i-1}, a_i]\}$ and $m_i = \inf\{f(x) : x \in [a_{i-1}, a_i]\}$. Define

$$U_{f,P} = \sum_{i=0}^{n-1} (a_{i+1} - a_i) \cdot M_i$$

and

$$L_{f,P} = \sum_{i=0}^{n-1} (a_{i+1} - a_i) \cdot m_i.$$

The function f is said to be **integrable** if

$$\inf\{U_{f,P}\} = \sup\{L_{f,P}\}$$

and this common value is denoted by

$$\int_a^b f(x)dx.$$

For a continuous function we can instead use the Riemann method. Take a partition P of $[a, b]$, say $a = a_0 < a_1 < \ldots < a_n = b$ and let $x_i \in [a_{i-1}, a_i]$ and compute the sum of the areas of rectangles:

$$\sum_{i=0}^{n-1} f(x_i)(a_{i+1} - a_i).$$

When the partition is into equal pieces then each piece has length $\Delta x = \frac{b-a}{n}$ and we get that

$$\int_a^b f(x)dx = \lim_{n\to\infty} f(x_i)\Delta x$$

is a real value.

Example 10.1. Recall that the work done by a force on an object being moved along a staight line from a to b is given by

$$W = \int_a^b F(x)dx.$$

The units of work are in newton-meters or Joules J.

Now, for $f : \mathbb{R} \to \mathbb{R}$ we want to be able to discuss $\int_{-\infty}^{\infty} f(x)dx$. Recall that this is an example of an indefinite integral. Such an integral might be proper or improper. One way to compute this is

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)dx.$$

The problem is the indefinite integral might not be a real value for some continuous functions, e.g. $f(x) = 1$. Other times the above limit integral exists but the integral is improper, e.g. $\int_{-\infty}^{\infty} xdx$. So we should actually take the integral to be over all such closed intervals and see if the limits are all the same. A simple way is to first define for $a \in \mathbb{R}$

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

and

$$\int_{-\infty}^{a} f(x)dx = \lim_{b\to-\infty} \int_{b}^{a} f(x)dx$$

And then check that for all $a \in \mathbb{R}$ both integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are proper, and then we can define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$

which will be well-defined (i.e. doesn't matter which a we take).

II. $f : \mathbb{R}^2 \to \mathbb{R}$

Now if $f : R \to \mathbb{R}$ where R is some bounded rectangle in $\mathbb{R}^2$, say $R = [a, b] \times [c, d]$, then we break up both $[a, b]$ and $[c, d]$ into equal pieces $a = a_0 < a_1 < \ldots < a_n = b$ and $c = c_0 < c_1 < \ldots c_n = d$, and set $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{n}$. Then choose a point $x_i \in [a_{i+1}, a_i]$ and $y_i \in [c_{i+1}, c_i]$ and compute the sum of the volumes of rectangular boxes and we get

$$\int_R f(x, y)dA = \lim_{n\to\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(x_i, y_j)\Delta x\Delta y.$$

(In MAC 2313 Honors Calculus 3, we called this the double integral and calculated it as

$$\int_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx.$$

Then to discuss $\int_{\mathbb{R}}^2 f(x, y)dA$ we could take rectangles in both directions getting larger and in that sense compute $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy$.

Remark 10.2. At this point we can generalize the case of the double integral to multiple integrals for functions $f : \mathbb{R}^m \to \mathbb{R}$. In this case we simply write

$$\int_{\mathbb{R}^m} f(x_1, \ldots, x_n)dx_1dx_2 \ldots dx_n = \int_{\mathbb{R}^m} f(x)dx.$$

III. $f : \mathbb{R} \to \mathbb{R}^2$

If $f : [a, b] \to \mathbb{R}^2$, then f is a vector-valued function and so we define the integral is component-wise. Recall that π_i denotes the projection map from $\mathbb{R}^2$ onto the i-th component. Therefore, $f_i = \pi_i \circ f : \mathbb{R} \to \mathbb{R}$.

$$\int_a^b f(t)dt = (\int_a^b f_1(t)dt, \int_a^b f_2(t)dt).$$

Example 10.3. Let $f(t) = (t^2, t^3)$. Then over $[0,1]$

$$\int_0^1 f(t)dt = (\int_0^1 t^2 dt, \int_0^1 t^3 dt) = (\frac{1}{3}, \frac{1}{4}).$$

Thus to talk about $\int_{-\infty}^{\infty} f(t)dt$ we need the integrals of the component functions to exist. Thus, we may write

$$\int_{-\infty}^{\infty} f(t)dt = (\int_{-\infty}^{\infty} f_1(t)dt, \int_{-\infty}^{\infty} f_2(t)dt).$$

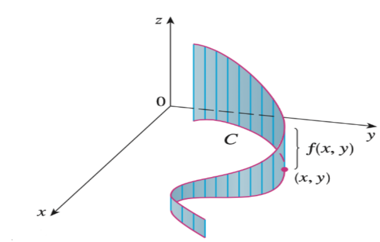
10.2 Line Integrals

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and C a curve in $\mathbb{R}^2$ with a given parametrization $\gamma(t) = (x(t), y(t))$ for $t \in [a,b]$.

The line integral of f over the curve C is

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}dt = \int_a^b f(\gamma(t))||\gamma'(t)||dt.$$

Notice then that we would need C to be a be differentiable curve.... later we will assume that it is smooth or piecewise smooth.



The line integral a function of two variables over a smooth curve is the lateral surface area.

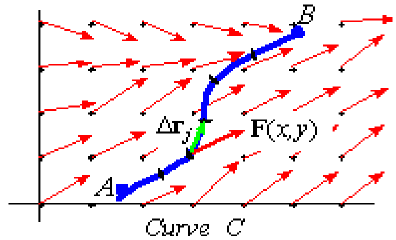
10.3 Line Integrals over Vector Fields

Now we turn to something that is getting really close to complex integration. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a force field and we can write $F(x,y) = (P(x,y), Q(x,y))$. We assume that

P and Q have continuous partial derivative on an open connected domain $R \subseteq \mathbb{R}^2$. We let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ be a piecewise smooth curve C.

The **work** done by the force field F in moving an object along the curve C is given by

$$W = \int_C F \cdot d\mathbf{r} = \int_C (P dx + Q dy) = \int_a^b F \cdot \mathbf{r}'(t) dt.$$



Vector Field generator

Example 10.4. Let $F(x, y) = (-y, x)$ and C the curve parametrized by $\mathbf{r}(t) = (\cos t, \sin t)$ over $[0, \pi]$.

On C, $F(x(t), y(t)) = (-y(t), (x(t)) = (-\sin t, \cos t)$, while $\mathbf{r}'(t) = (-\sin t, \cos t)$. Then

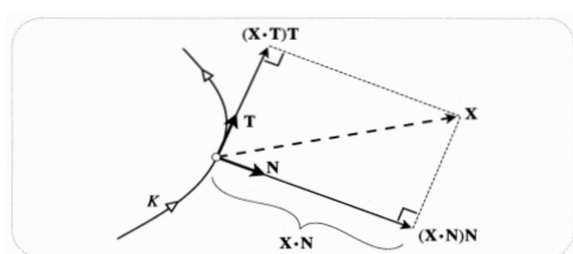
$$F \cdot \mathbf{r}' = (-\sin t, \cos t) \cdot (-\sin t, \cos t) = \sin^2 t + \cos^2 t = 1.$$

So

$$W = \int_C F \cdot d\mathbf{r} = \int_0^\pi dt = \pi.$$

Example 10.5. Here is another example. The **flux** of a flow field is the quantity of a fluid flowing across a curve per unit of time. In other words it is the rate of flow.

Let X be a force field and K a smooth curve parametrized by $\gamma(t) = x(t), y(t))$ over $[a, b]$. We let T be the tangent vector $\gamma'(t)$ and we let $N(t)$ be the normal vector to the tangent vector at $\gamma(t)$. Since the tangent vector is given by $\gamma'(t) = (x'(t), y'(t))$ we choose $N(t)$ to point to the right as we travel along C. Notice that this is $N(t) = i\gamma'(t) = (-y'(t), x'(t))$.



We can decompose X as

$$X = (X \cdot T)T + (X \cdot N)N$$

and only $(X \cdot N)X$ carries fluid across K. So we define the total **flux** of X across C to be

$$\mathcal{F}[X, K] = \int_C X \cdot N ds$$

Observe that in this example if X is a force field then $\int_C X \cdot T ds$ is precisely work looked at in Example 10.4 which we could label as $\mathcal{W}[X, K]$.

10.4 Complex Integration

In all of the examples above the output was a real number. We now define the complex integral. These are often times called path integrals or contour integrals. Let $f : \mathbb{C} \to \mathbb{C}$ and C a smooth curve in $\mathbb{C}$ parametrized by $\gamma(t) = (x(t), y(t))$ over $[a, b]$. The complex integral is defined as

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Observe that both $f(\gamma(t))$ and $\gamma'(t)$ are complex numbers and that in the expression we are asking you to multiply the complex numbers. It follows that $f(\gamma(t))\gamma'(t)$ is vector valued and so we push forward and get that

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \Re(f(\gamma(t))\gamma'(t))dt + i\int_a^b \Im(f(\gamma(t))\gamma'(t))dt.$$

Example 10.6. Let $F(z) = z + \overline{z}$ and $\gamma(t) = 2\cos t + i2\sin t$ over $[0, 2\pi]$. Then $\gamma'(t) = -2\sin t + i2\cos t$ and $F(\gamma(t)) = 2\cos t + i2\sin t + 2\cos t - i2\sin t = 4\cos t$. The product is $(4\cos t)(-2\sin t + i2\cos t) = -8\cos t\sin t + i8\cos^2 t$ so

$$\begin{aligned}
\int_\gamma \overline{z}dz &= \int_0^{2\pi} (-8\cos t\sin t + i8\cos^2 t)dt \\
&= \int_0^{2\pi} -8\cos t\sin tdt + i\int_0^{2\pi} 8\cos^2 tdt \\
&= 8(\frac{\sin^2 t}{2})|_0^{2\pi} + 8i\int_0^{2\pi} (\frac{1}{2} + \cos(2t))dt \\
&= 8i(\frac{t}{2} + \frac{\sin(2t)}{2})|_0^{2\pi} \\
&= 8\pi i.
\end{aligned}$$

Example 10.7. Let $F(z) = e^{3z}$ and γ the straight line from 1 to i. Then

$$\int_\gamma e^{3z} dz = \frac{e^{3z}}{3}|_1^i = \frac{1}{3}(e^{3i} - e^3).$$

Maybe the most helpful to take away from this section is through the following equation (c.f Examples 10.4 and 10.5):

$$\int_C F(z)dz = \mathcal{W}[\overline{F}, C] + i\mathcal{F}[\overline{F}, C].$$

Chapter 11

Function Spaces

The goal of this chapter to have you think of functions as elements or vectors. And that we can discuss inner products, norms, and distances of functions.

Let (X, d) be a metric space and let $C(X, \mathbb{C}) = \{f : X \to \mathbb{C}\}$ be the set of all continuous complex-valued functions on X. We equip $C(X, \mathbb{C})$ with two binary operations, addition and multiplication, defined by for $f, g \in C(X)$

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x) \cdot g(x).$$

Further, we make $C(X, \mathbb{C})$ a $\mathbb{C}$-vector space by defining scalar multiplication $\alpha \in \mathbb{C}$, $f \in C(X, \mathbb{C})$

$$(\alpha \cdot f)(x) = \alpha \cdot f(x).$$

Under these operations $C(X, \mathbb{C})$ is a commutative ring with identity and a $\mathbb{C}$-vector space. The subset of real-valued continuous functions is denoted by $C(X)$ and it follows that $C(X)$ is also a ring and a $\mathbb{R}$-vector space.

Definition 11.1. Let $f \in C(\mathbb{R}, \mathbb{C})$. We say that f is a C^∞ map or belongs to $C^\infty(\mathbb{R}, \mathbb{C})$ if it is infinitely differentiable. This class contains all polynomials, most of the nice trig functions, and exponential functions.

Let $\mathcal{L}(\mathbb{R})$ be the collection of all complex-valued continuous functions on $\mathbb{R}$ for which

$$\int_{-\infty}^{\infty} |f| dx < \infty.$$

Since f here is complex-valued by $|f|$ we mean the function $|f|(x) = ||f(x)||$, i.e. the composition of f followed by the modulus function so that $|f| : \mathbb{R} \to \mathbb{R}$.

Observe that

$$\int_{-\infty}^{\infty} |\alpha f - g| dx \leq \int_{-\infty}^{\infty} (|\alpha f| + |g|) dx = ||\alpha|| \int_{-\infty}^{\infty} |f| dx + \int_{-\infty}^{\infty} |g| dx < \infty$$

from which it follows that $\mathcal{L}(\mathbb{R})$ is a $\mathbb{C}$-vector space. Moreover, it is a ring.

Proposition 11.2. *Let $f : \mathbb{R} \to \mathbb{C}$. The following statements are equivalent.*

1. $f \in \mathcal{L}(\mathbb{R})$.
2. $f_1, f_2 \in \mathcal{L}(\mathbb{R})$
3. $|f| \in \mathcal{L}(\mathbb{R})$.

Next, $\mathcal{L}^2(\mathbb{R})$ consists of those complex-valued functions on $\mathbb{R}$ for which

$$\int_{\infty}^{\infty} |f|^2 du < \infty.$$

This is also a $\mathbb{C}$-vector space.

Let ℓ^p denote the collection of complex-valued sequences, say (z_n) for which

$$\sum_{n=1}^{\infty} ||z_n||^p < \infty$$

We are only interested in the case that $p = 1$ and $p = 2$. These spaces are called *little ℓ-1* and *little ℓ-2*. ℓ^1 is the space of sequences whose series is absolutely convergent, and ℓ^2 is the space of square-summable sequences. Finally, ℓ^∞ is the space of bounded sequences, and ℓ^0 is the space of sequences which ar eventually 0. Note that

$$\ell^0 \subset \ell^1 \subset \ell^2 \subset \ell^\infty.$$

Definition 11.3. Let $f \in C(\mathbb{R}, \mathbb{C})$. We say f is of **moderate decrease** if there is some constant $N \in \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$||f(x)|| \leq \frac{N}{1 + x^2}.$$

The collection of functions of moderate decrease will be denoted by $\mathcal{M}(\mathbb{R})$. It is a $\mathbb{C}$-vector space.

Example 11.4. The function $f(x) = \frac{\sin x}{1+x^2}$ is of moderate decrease.

Definition 11.5. Let f be a C^∞ map on $\mathbb{R}$. We say that f is **rapidly decreasing** if for all $k, l \leq 0$

$$\sup_{x \in \mathbb{R}} ||x||^k ||f^{(l)}(x)|| < \infty.$$

The collection of all rapidly decreasing functions is called the **Schwartz space** on $\mathbb{R}$ and is denoted by $\mathcal{S}(\mathbb{R})$. You should be able to argue as above that $\mathcal{S}(\mathbb{R})$ is a $\mathbb{C}$-vector space. Moreover, it is closed under derivative: if $f \in \mathcal{S}(\mathbb{R})$, then $f' \in \mathcal{S}(\mathbb{R})$. Notice that no polynomial belongs to $\mathcal{S}(\mathbb{R})$ but for any polynomial $p(x)$ and $f \in \mathcal{S}(\mathbb{R})$, $p(x)f(x) \in \mathcal{S}(\mathbb{R})$.

Example 11.6. The function $f(x) = e^{-x^2}$ belongs to the Schwartz space, while neither $g(x) = e^{-x}$ nor $h(x) = e^{-|x|}$ belong to the Schwartz space.

The function $f(x) = \frac{\sin x}{1+x^2}$ is not rapidly decreasing.

Chapter 12

Fourier Transform

Recall that $f \in \mathcal{M}(\mathbb{R})$ means that f is a complex-valued function on $\mathbb{R}$ and so its integral is a complex number, provided it exists.

Lemma 12.1. *If $f \in \mathcal{M}(\mathbb{R})$, then $f \in L(\mathbb{R})$. Consequently, $\int_{-\infty}^{\infty} f(t)dt$ exists.*

Proof. (Sketch.) Use Cauchy sequences. Observe that on an interval of the form $[0, b)$ The integral satisfies

$$\int_0^b |f|(x)dx \leq N \int_0^b \frac{dx}{1+x^2}$$

for some positive N, and the latter is convergent, hence the former is Cauchy and hence convergent.

□

If $f \in \mathcal{M}(\mathbb{R})$, we define the **Fourier Transform** to be the function $\hat{f} : \mathbb{R} \to \mathbb{C}$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx.$$

The first thing you should notice is that the input here is $\xi \in \ R$. The second thing to notice is that the output is a complex number. Third, fix $\xi \in \mathbb{R}$ and so for any $x \in \mathbb{R}$

$$||f(x)e^{-2\pi i x\xi}|| = ||f(x)|| \cdot ||e^{-2\pi i x\xi}|| = ||f(x)||.$$

It follows that if $f(x) \in \mathcal{M}(\mathbb{R})$, then so is $f(x)e^{-2\pi i x\xi}$ for each ξ, whence the integral exists for each $\xi \in \mathbb{R}$.

Finally the important thing that we need to point out is that the function $\hat{f}(\xi)$ might not be of moderate decrease. In particular, we might not be able to take the Fourier transform of the Fourier transform.

Chapter 13

Exercises

1. Let $z = 3 + 4i$ and $w = 2 - 2i$. Find

   (a) $2z + 3w$,
   (b) z^2
   (c) zw,
   (d) z^{-1},
   (e) $||z + w||$,
   (f) $||z - w||$,
   (g) $||z|| - ||w||$,
   (h) $\overline{w}$,
   (i) $\operatorname{Arg} w$

2. Prove that $||z|| = ||\overline{z}||$ and $||z^{-1}|| = \frac{1}{||z||}$ for all nonzero $z \in \mathbb{C}$.

3. Prove the for all $z, w \in \mathbb{C}$, $||z \cdot w|| = ||z|| \cdot ||w||$.

4. Find the polar form of $z = -5 + 5i$.

5. i) Compute $e^{i\pi}$. ii) Prove that $\overline{e^{i\theta}} = e^{-i\theta}$.

6. Let $z = a + bi$ and $w = c + di$. First compute the product zw in rectangular coordinates and then convert to polar form of zw. Second, take the polar forms of z and w and then multiply, the polar forms. Is the result the same? Use the trig identities involving sums

7. Prove that the map $T : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto \overline{z}$ is a $\mathbb{R}$-linear transformation and also a field isomorphism. What is $T^2 := T \circ T$? Is conjugation a $\mathbb{C}$-linear map?

8. Let $z = a + bi$ and $T_z : \mathbb{C} \to \mathbb{C}$ defined by $T_z(w) = zw$. Prove that T_z is a $\mathbb{R}$-linear transformation and find the matrix representation of T_z. Prove that for any $z_1, z_2 \in \mathbb{C}$, $T_{z_2} \circ T_{z_1} = T_{z_1 z_2}$.

9. Prove that the map $e^{(\cdot)i} : \mathbb{R} \to \mathbb{S}_1$ is a group homomorphism between the additive reals and the circle group. What is the kernel of this map?

10. Prove that similarity of matrices is an equivalence relation.

11. Let V be a $\mathbb{R}$-vector space. Prove that the space of operators on V, $\mathrm{Hom}_{\mathbb{R}}(V)$, is a $\mathbb{R}$-vector space and also a ring under composition. Be sure to define the binary operations. For multiplication look to Exercise 7..

12. i) In $\mathbb{R}^3$, what is the general form of a line through the origin? ii) What is the general form of a plane?

13. Prove that the dot product on $\mathbb{R}^3$ is an inner product.

14. (Pythgorean Theorem) Prove that if $\vec{u}, \vec{v}$ are orthogonal, then

$$||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2.$$

15. What is the matrix representation of the linear transformation that rotates $\mathbb{R}^2$ counter-clockwise θ degrees around the origin.

16. What is the matrix representation of the linear transformation that reflects points through/across the line $y = 2x$. Can you generalize to the line $y = kx$ where $k \in \mathbb{R}$. (Also, what about through $x = 0$.)

17. Let $A \in GL_3(\mathbb{R})$. Explain why A must have an eigenvalue.

18. What is the matrix representation for rotating $\mathbb{R}^3$, $180°$ through the line $y = x$.

19. Prove that the map $F : \mathbb{C} \to M_2(\mathbb{R})$ defined by $F(z) = \begin{pmatrix} \Re(z) & -\Im(z) \\ \Im(z) & \Re(z) \end{pmatrix}$ is a ring homomorphism.

20. Prove that the restriction of F in 19. to $\mathbb{S}_1$ is the map $z \mapsto \begin{pmatrix} \cos \arg z & -\sin \arg z \\ \sin \arg z & \cos \arg z \end{pmatrix}$

21. Let $E = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 4 \\ 4 & 2 & 4 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 3 \\ 4 & 3 & 3 \end{pmatrix}$, and $G = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$. Find

    (a) E^t,
    (b) EF,
    (c) FE,
    (d) $\det(E)$
    (e) $\det(G)$
    (f) G^{-1}

22. Prove that the subgroup of translations on $\mathbb{R}^n$ is a normal subgroup of the Euclidean group. Prove that $E(n) = T(n) \rtimes O(n)$.

23. Let $A \in O(n)$ be an orthogonal matrix. Prove that $\det(A) = \pm 1$. [Hint: recognize A as multiplication by a complex number.]

24. Here are some true-false questions on orthogonal matrices: True/False

25. Let $A \in SO(2)$ be the real matrix given by rotation by θ degrees. Find the characteristic equation of A and the eigenvalues of A (over $\mathbb{C}$).

26. Prove that the map $f : V \times V \to \mathbb{R}$ given in Example 4.6 is a bilinear form.

27. Show that the function $e : \mathbb{R}^2 \to \mathbb{R}$ defined by $e(x, y) = |x| + |y|$ is a distance function. Describe $N_1(\bar{0})$ geometrically. Are $(\mathbb{R}^2, e)$ and $(\mathbb{R}^2, || \cdot ||)$ equivalent as metrics? Are they isometric?

28. Prove that conjugation on $\mathbb{C}$ is continuous but nowhere complex differentiable.

29. Prove that the function $f(z) = \frac{z}{1-|z|^2}$ is a diffeomorphism between the unit ball $N_1(\vec{0})$ and all of $\mathbb{R}^n$.

30. Prove Proposition 5.4.

31. Show that the identification defined in Remark 5.7 is a group isomorphism.

32. Prove that $A \in O(p, q)$ if and only if $g^{-1} A^t g = A^{-1}$. Conclude that if $A \in O(p, q)$, then $\det A = \pm 1$. Characterize the $A \in O(1, 1)$.

33. Let (X, d) be a metric space. Prove that a set $O \subseteq X$ is open if and only if $X \setminus O$ is closed.

34. Prove that a metric space is Hausdorff, that is, every pair of distinct points can be separated by disjoint open sets.

35. Prove that image of a connected set is connected.

36. Prove that a function between two metric spaces if and only if the inverse image of an open set is an open set.

37. Let d denote the discrete metric on $\mathbb{R}$. Prove that any function from $\mathbb{R}$ into any metric space is continuous. Prove that the identity map from (R, d) in $\mathbb{R}$ with the usual metric is a continuous bijection whose inverse is not continuous.

38. Let V be the set of real sequences which are finitely non-zero. This is an infinite dimensional vector space. Let $T : V \to V$ be the function defined by $T((v_k)) = (kv_k)$. Prove that T is a linear operator. Is T a bounded operator?

39. Prove that if $\gamma : (-\epsilon, \epsilon) \to GL_2(R)$ is a smooth map $\gamma(t) = A(t)B(t)$, then the product rule holds: $\gamma'(t) = A(t)B'(t) + A'(t)B(t)$.

40. Prove that the matrix bracket $[A, B]$ defined on $M_n(\mathbb{C})$ is a Lie bracket. (Can you do for $n = 3$?)

41. Prove that the collection of skew-symmetric matrices form a (vector) subspace of $M_n(\mathbb{R})$. Show that the matrix Lie bracket of two skew symmetric matrices is again skew symmetric. Conclude that the skew symmetric matrices form a Lie algebra.

42. Let $N \in \mathrm{Skew}_n(N)$ be a skew-symmetric matrix. Define a bracket on $\mathrm{Symm}_n(\mathbb{R})$ by $[A, B]_N = ANB - BNA$. Prove that this bracket is a Lie bracket and makes $\mathrm{Symm}_n(\mathbb{R})$ into a Lie algebra.

43. Prove that a matrix $A \in M_3(\mathbb{R})$ is skew-symmetric if and only if $< \vec{x}, A\vec{x} >= 0$ for all $\vec{x} \in \mathbb{R}^3$.

44. Prove that the cross product is an alternating bilinear map. Show that it satisfies the Jacobi identity.

45. Prove that the tangent space of $G = O(n)$ is the collection of all skew-symmetric matrices. Also, conclude that the tangent space of $SO(n)$ is also the collection of all skew-symmetric matrices.

46. Prove that the function $exp(\cdot) : \mathbb{C} \to \mathbb{C}^*$ is complex differentiable and onto, but not injective. Show that $\frac{de^z}{dz} = e^z$.

47. Prove that $\mathfrak{u}(n)$ is a vector space that is closed under the Lie bracket.

48. Prove that a skew-Hermitian matrix of $M_2(\mathbb{C})$ has the form $\begin{pmatrix} bi & c+di \\ -c+di & hi \end{pmatrix}$. What is the dimension of $\mathfrak{su}(2)$?

49. Prove Theorem 9.33.

50. Prove that the set of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{pmatrix}$ is a Lie group and its Lie Algebra is the set of matrices of the form $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. This Lie group is called the Heisenberg Lie group since its Lie algebra is isomorphic to the Heisenberg algebra. (Extra credit. Show that the exponential map is a bijection between its Lie algebra and its Lie group.)

51. Give an example of an element belonging to $C(\mathbb{R})$ which does not belong to $L^1(\mathbb{R})$. Give an example of an element belonging to $L^1(\mathbb{R})$ that does not belong to $L^2(\mathbb{R})$.

52. Explain why the Cauchy-Schwartz Inequality for integrable functions implies that $L^1(\mathbb{R})$ and $L_2(\mathbb{R})$ are rings.

53. Construct three counter examples in order to demonstrate that the subset relations in Section 11 are strict.

54. Is the Schwartz space a ring? Explain why $\mathcal{S}(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R})$.

Bibliography

[1] Sanyal, A. K., *Geometrical Transformations in Higher Dimensional Euclidean Spaces*, Masters thesis, Department of Aerospace Engineering, Texas A&M University, College Station, TX, May 2001.

[2] Mortari, D.. (2004). *Ortho-Skew and Ortho-Sym Matrix Trigonometry.* Advances in the Astronautical Sciences. 115.

[3] https://math.uchicago.edu/ may/REU2015/REUPapers/Yoo.pdf

[4] http://scipp.ucsc.edu/∼haber/ph216/rotation_12.pdf

[5] https://kconrad.math.uconn.edu/blurbs/grouptheory/isometryRn.pdf

[6] https://www2.kenyon.edu/Depts/Math/Paquin/Orthogonal.pdf

[7] http://math.colorado.edu/∼nita/Differentiability2.pdf

[8] https://en.wikipedia.org/wiki/Frobenius_theorem_(real_division_algebras)#Proof

[9] https://www.mathcounterexamples.net/non-complete-normed-vector-space/

[10] https://www.math.ucla.edu/∼jlindquist/115B/JCFBases.pdf

[11] https://www.math.ucdavis.edu/∼anne/WQ2007/mat67-Ld-FTA.pdf

[12] https://www.math.ucdavis.edu/∼hunter/m218a_09/ch5A.pdf

[13] Needham, Virtual Complex Analysis,
https://umv.science.upjs.sk/hutnik/NeedhamVCA.pdf

[14] https://www.cymath.com/answer?q=integrate

[15] https://www.math.unm.edu/∼crisp/courses/wavelets/fall09/chap7.pdf

[16] http://www.physics.fau.edu/ cbeetle/PHY6938.07F/lecture1.pdf , http://www.physics.fau.edu/ cbeetle/PHY6938.07F/

[17] https://people.bath.ac.uk/feb/ma20216/notes/node-16.html

[18] http://indico.ictp.it/event/a08167/session/124/contribution/85/material/0/0.pdf

https://www.cis.upenn.edu/c̃is610/cis61005lie1.pdf

http://people.math.harvard.edu/ãuroux/131f19/Math_131_Notes_Beckham_Myers.pdf

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