Name:  

Exam 1– MAS 4301H Fall 2019

**Remark:** Throughout the test $n$ denotes a positive integer greater than 1. We consider the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ under addition.

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1. Definitions:
   (a) State the definition of a group.

   (b) What does it mean for the group $(G, \cdot, e)$ to be an *abelian* group?

   (c) What does it mean for the group $(G, \cdot, e)$ to be a *cyclic* group?

   (d) The center of a group is
   
   \[
   Z(G) = \{ \}
   \]

   (e) List the elements of $U(14) = \{}$

   (f) For a group $G$ and $g \in G$, define the *order of g* $o(g)$

2. True or false:
   i. Every subgroup of a cyclic group is cyclic.

   ii. Every group of order $p$ (prime) is cyclic.

   iii. Given a group $G$, the subset of elements of finite order is a subgroup of $G$.

   iv. For a group $G$ and $a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$.

   v. If $\sigma \in S_n$ is a $k$-cycle and $\tau \in S_n$ is a $j$-cycle, then $o(\sigma \circ \tau) = l.c.m(j, k)$.

3. Give an example of an abelian group which is not cyclic.
4. State Lagrange’s Theorem

5. Explain why $S_3$ is not abelian. [Hint: just need a counter-example.]

6. Let $G = S_4$ and set $r = (1\ 2\ 3\ 4)$ and $s = (24)$.

   i) Find the cycle decomposition of $s \circ r \circ s^{-1}$.

   ii) List the elements of $<r>$. What is the order of $r$?

7. Let $G = S_3$ and set $\sigma = (1\ 2\ 3)$.

   i) What is $o(\sigma)$?

   iii) List the elements of $\text{Conj}(\sigma)$. Show your work. [Hint: you may use that $o(h) = o(ghg^{-1})$.]

8. Draw the lattice of subgroups of $\mathbb{Z}/12$. 
9. Let \((G, \cdot, e_G)\) be a group, \(H \leq G\) and take a fixed \(g \in G\). Prove that if \(x \in gH\), then \(xH = gH\).

10. Let \((G, \cdot, e_G)\) be a group, \(H \leq G\) and take a fixed \(g \in G\). Prove that the function

\[
\psi : H \to gHg^{-1}
\]

defined by \(\psi(h) = ghg^{-1}\) is one-to-one.

11. How many rigid motions are there for a regular octahedron? Take a guess: what is the group of symmetries of the regular octahedron. [Hint: there is a rigid motion which will map any vertex to any other vertex.]
12. Choose one of the following and prove it. Let $G$ be a group, $H \leq G$ and $g \in G$.
   i. The centralizer of $g$, $C_G(g) = \{x \in G : xgx^{-1} = g\}$ is a subgroup of $G$.
   ii. If $\{H_i\}_{i \in I}$ is a family of subgroup of $G$, then $\bigcap H_i$ is a subgroup of $G$.
   iii. The cyclic subgroup generated by $g < g > = \{x \in G : \exists n \in \mathbb{Z}, x = g^n\}$, is a subgroup of $G$.
   iv. The set $gHg^{-1}$ is a subgroup of $G$.
   v. for any $s \in \{1, \ldots, n\}$, the set $G_s = \{\sigma \in S_n : \sigma(s) = s\}$ is a subgroup of $S_n$. 