Name: ____

Exam 1– MAS 4301H Fall 2019

Remark: Throughout the test n denotes a positive integer greater than 1. We consider the group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ under addition.

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1. Definitions:
   (a) State the definition of a group.

   (b) What does it mean for the group $(G, \cdot, e)$ to be an *abelian* group?

   (c) What does it mean for the group $(G, \cdot, e)$ to be a *cyclic* group?

   (d) The center of a group is

   $Z(G) = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$

   (e) List the elements of $U(14) = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$

   (f) For a group G and $g \in G$, define the *order of* g: $o(g)$

2. True or false:
   i. Every subgroup of a cyclic group is cyclic.

   ii. Every group of order p (prime) is cyclic.

   iii. Given a group G, the subset of elements of finite order is a subgroup of G.

   iv. For a group G and $a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$.

   v. If $\sigma \in S_n$ is a k-cycle and $\tau \in S_n$ is a j-cycle, then $o(\sigma \circ \tau) = l.c.m(j, k)$.

3. Give an example of an abelian group which is not cyclic.

4. State Lagrange's Theorem

5. Explain why S_3 is not abelian. [Hint: just need a counter-example.]

6. Let $G = S_4$ and set $r = (1\ 2\ 3\ 4)$ and $s = (24)$.

   i) Find the cycle decomposition of $s \circ r \circ s^{-1}$.

   ii) List the elements of $< r >$. What is the order of r?

7. Let $G = S_3$ and set $\sigma = (1\ 2\ 3)$.

   i) What is $o(\sigma)$?

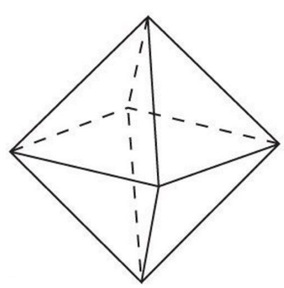
   iii) List the elements of $\text{Conj}(\sigma)$. Show your work. [Hint: you may use that $o(h) = o(ghg^{-1})$.]

8. Draw the lattice of subgroups of $\mathbb{Z}/12$.

9. Let $(G, \cdot, e_G)$ be a group, $H \leq G$ and take a fixed $g \in G$. Prove that if $x \in gH$, then $xH = gH$.

10. Let $(G, \cdot, e_G)$ be a group, $H \leq G$ and take a fixed $g \in G$. Prove that the function
$$\psi : H \to gHg^{-1}$$
defined by $\psi(h) = ghg^{-1}$ is one-to-one.

11. How many rigid motions are there for a regular octahedron? Take a guess: what is the group of symmetries of the regular octahedron. [Hint: there is a rigid motion which will map any vertex to any other vertex.]



12. Choose one of the following and prove it. Let G be a group, $H \leq G$ and $g \in G$.
    i. The centralizer of g, $C_G(g) = \{x \in G : xgx^{-1} = g\}$ is a subgroup of G.
    ii. If $\{H_i\}_{i \in I}$ is a family of subgroup of G, then $\bigcap H_i$ is a subgroup of G.
    iii. The cyclic subgroup generated by $g < g >= \{x \in G : \exists n \in \mathbb{Z}, x = g^n\}$, is a subgroup of G.
    iv. The set gHg^{-1} is a subgroup of G.
    v. for any $s \in \{1, \ldots, n\}$, the set $G_s = \{\sigma \in S_n : \sigma(s) = s\}$ is a subgroup of S_n.