Name: ____

Exam 1 – MAS 4301H Fall 2017

Remark: Throughout the test n denotes a positive integer greater than 1. Also, abstractly we consider a group $(G, \cdot, e)$. Recall that $\mathbb{Z}_n$ is the set $\{0, 1, \cdots, n-1\}$ and when equipped with addition is a group $(\mathbb{Z}_n, +, 0)$.

1. Definitions:
   (a) State the definition of a group.

   (b) What does it mean for the group $(G, \cdot, e)$ to be an *abelian* group?

   (c) Let $(G, \cdot, e)$ be a group? What is $< g >= \{ \qquad\qquad\qquad\qquad \}$?

   (d) Given a group $(G, \cdot, e_G)$ and a subset $H \subseteq G$ define what it means for H to be a subgroup of G.

2. Give an example of a cyclic group.

3. Give an example of an abelian group which is not cyclic.

4. Give an example of a non-abelian group.

5. In S_5, let $\sigma = (14)(235)$ and $\tau = (23)(45)$. Compute the following
   (a) $\sigma \circ \tau$
   (b) $\tau \circ \sigma$
   (c) σ^{-1}

6. State Lagrange's Theorem

7. In the cyclic group $\mathbb{Z}_n$ characterize the generators.

8. Find all the generators for $\mathbb{Z}_9$.

9. Let $G = D_8 \times \mathbb{Z}_3$. Recall that $r \in D_8$ is (clock-wise) rotation of the regular square by $90°$. Compute

$$< (r, 2) >= \{ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \}$$

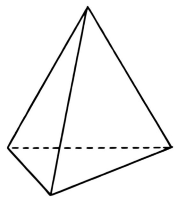
(a) What is the order of the element $(r, 2)$?
(b) What is $|G|$?
(c) Is G abelian?

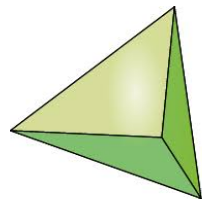
10. For each $1 \leq k \leq n$ consider the function $\Psi_k : \mathbb{Z}_n \to \mathbb{Z}_n$ defined by

$$\Psi_k(a) = ka.$$

   (a) Write out explicitly the function $\Psi_3 : \mathbb{Z}_7 \to \mathbb{Z}_7$. (Tell me where 0 goes, where 1 goes, etc.)
   (b) (Extra Credit:) Prove that Ψ_k is a group homomorphism.
   (c) (Extra Credit:) Characterize when Ψ_k is injective (and hence an isomorphism).

11. Let D_8 act on itself by conjugation. Compute the orbits of this action, i.e. list the sets of conjugates.

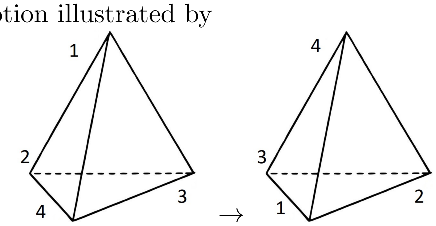




12.
Let G be the group of rigid motions in $\mathbb{R}^3$ of a tetrahedron.

(a) Explain why $|G| = 12$.

(b) Label the vertices by 1,2,3,4 and let G act on the vertices. Consider the permutation representation $\sigma : G \to S_4$ of G into S_4. Write out the elements of the group as permutations. For example, the rigid motion illustrated by



is the element (14)(23).

13. (Extra Credit:) Consider $G = \mathbb{Z}_7 \times \mathbb{Z}_2$ and define a new multiplication on G as follows:

$$(a,b) * (c,d) = \begin{cases} (a+c, b+d), & \text{if } b = 0 \\ (a-c, b+d) & \text{if } b = 1. \end{cases}$$

Prove that G is a group under this multiplication. Identify the inverse $(a,0)$ and $(a,1)$. How many elements of order 2? of order 7?