Name: ____

Exam 1– MAS 4301H Fall 2013

Remark: Throughout the test n denotes a positive integer greater than 1. We consider the group $\mathbb{Z}/n\mathbb{Z}$ under addition. For $a \in \mathbb{Z}$ the congruence class containing a is denoted by $\bar{a}$ (instead of $[a]_n$).

1. Definitions:

   (a) State the definition of a group.

   (b) What does it mean for the group $(G, \cdot, e)$ to be an *abelian* group?

   (c) What does it mean for the group $(G, \cdot, e)$ to be a *cyclic* group?

   (d) Given a group $(G, \cdot, e_G)$ and subset $H \subseteq G$ define what it means for H to be a subgroup of G.

2. Give an example of an abelian group which is not cyclic.

3. Give an example of a non-abelian group.

4. In S_5 let $\sigma = (14)(235)$ and $\tau = (23)(45)$. Compute the following
   i) $\sigma \circ \tau$
   ii) σ^{-1}

5. Given a group $(G, \cdot, e_G)$ and a subgroup $H \leq G$ state what it means for H to be a normal subgroup of G.

6. Draw the lattice of subgroups of $\mathbb{Z}/20\mathbb{Z}$. Briefly explain.

7. State Lagrange's Theorem

8. In the cyclic group $\mathbb{Z}/n\mathbb{Z}$ characterize the generators.

9. Let $G = D_8 \times \mathbb{Z}_3$. What is the order of the element $(r, 2)$? (You might want to look at Problem 12.)

10. For each $1 \leq k \leq n$ consider the function $\Psi_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ defined by

$$\Psi_k(\bar{a}) = k\bar{a}.$$

a) Prove that Ψ_k is a group homomorphism.

b) Prove that Ψ_k is an automorphism if and only if $\gcd(k, n) = 1$.

c) Write out explicitly the function $\Psi_3 : \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$. (Tell me where $\bar{0}$ goes, where $\bar{1}$ goes, etc.)

11. Let $(G, \cdot, e_G)$ be a group and $H \leq G$. Consider the equivalence relation defined on G by $g_1 \sim g_2$ if and only if $g_2^{-1}g_1 \in H$. Prove that the equivalence class containing g is the set $gH = \{x \in G |$ there exists $h \in H$ such that $x = gh\}$.

[You may use that $\sim$ is an equivalence relation and that the equivalence class containing g is the set $[g] = \{y \in G | g \sim y\}$.]

12. Consider $G = D_8$ where r is the rotation by $90°$ and s is a flip through the vertical axis. Once we label the vertices of the square we can view the elemets of D_8 as permutations: $r = (1234)$, $s = (12)(34)$.

    (a) Show that $s \circ r \circ s = r^3$.

    (b) Compute r^2, $s \circ r^2$, and $r^2 \circ s$.

    (c) Let $H = \{e_G, r^2, s, s \circ r^2\}$. Prove that H is a normal subgroup of D_8.

    (d) Draw the lattice of subgroups of D_8.

13. Let $(G, \cdot, e_G)$ be a group and $a, b \in G$. Prove that $(ab)^{-1} = b^{-1}a^{-1}$.

14. Let $(G, \cdot, e_G)$ be a group. Define a relation on G as follows $g_1 \sim g_2$ if and only if there exists $h \in G$ such that $hg_1h^{-1} = g_2$. Prove that $\sim$ is an equivalence relation. [For symmetry use the previous problem.]

15. Consider $G = \mathbb{Z}_7 \times \mathbb{Z}_2$ and define a new multiplication on G as follows:

$$(a, b) * (c, d) = \begin{cases} (a + c, b + d), & \text{if } b = 0 \\ (a - c, b + d) & \text{if } b = 1. \end{cases}$$

Prove that G is a group under this multiplication. Identify the inverse $(a, 0)$ and $(a, 1)$. How many elements of order 2? of order 7?