Remark: Throughout the test \( n \) denotes a positive integer greater than 1. We consider the group \( \mathbb{Z}/n\mathbb{Z} \) under addition. For \( a \in \mathbb{Z} \) the congruence class containing \( a \) is denoted by \( \bar{a} \) (instead of \( [a]_n \)).

1. Definitions:
   (a) State the definition of a group.

   (b) What does it mean for the group \((G, \cdot, e)\) to be an abelian group?

   (c) What does it mean for the group \((G, \cdot, e)\) to be a cyclic group?

   (d) Given a group \((G, \cdot, e_G)\) and subset \( H \subseteq G \) define what it means for \( H \) to be a subgroup of \( G \).

2. Give an example of an abelian group which is not cyclic.

3. Give an example of a non-abelian group.

4. In \( S_5 \) let \( \sigma = (14)(235) \) and \( \tau = (23)(45) \). Compute the following
   i) \( \sigma \circ \tau \)
   ii) \( \sigma^{-1} \)
5. Given a group \((G, \cdot, e_G)\) and a subgroup \(H \leq G\) state what it means for \(H\) to be a normal subgroup of \(G\).

6. Draw the lattice of subgroups of \(\mathbb{Z}/20\mathbb{Z}\). Briefly explain.

7. State Lagrange’s Theorem

8. In the cyclic group \(\mathbb{Z}/n\mathbb{Z}\) characterize the generators.

9. Let \(G = D_8 \times \mathbb{Z}_3\). What is the order of the element \((r, 2)\)? (You might want to look at Problem 12.)
10. For each $1 \leq k \leq n$ consider the function $\Psi_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ defined by

$$\Psi_k(\bar{a}) = k\bar{a}.$$ 

a) Prove that $\Psi_k$ is a group homomorphism.
b) Prove that $\Psi_k$ is an automorphism if and only if $\gcd(k, n) = 1$.
c) Write out explicitly the function $\Psi_3 : \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$. (Tell me where $\bar{0}$ goes, where $\bar{1}$ goes, etc.)

11. Let $(G, \cdot, e_G)$ be a group and $H \leq G$. Consider the equivalence relation defined on $G$ by $g_1 \sim g_2$ if and only if $g_2^{-1}g_1 \in H$. Prove that the equivalence class containing $g$ is the set

$$gH = \{x \in G \mid \text{there exists } h \in H \text{ such that } x = gh\}.$$ 

[You may use that $\sim$ is an equivalence relation and that the equivalence class containing $g$ is the set $[g] = \{y \in G \mid g \sim y\}$.]
12. Consider $G = D_8$ where $r$ is the rotation by $90^\circ$ and $s$ is a flip through the vertical axis. Once we label the vertices of the square we can view the elements of $D_8$ as permutations: $r = (1234), \ s = (12)(34)$.

(a) Show that $s \circ r \circ s = r^3$.
(b) Compute $r^2$, $s \circ r^2$, and $r^2 \circ s$.
(c) Let $H = \{e_G, r^2, s, s \circ r^2\}$. Prove that $H$ is a normal subgroup of $D_8$.
(d) Draw the lattice of subgroups of $D_8$.

13. Let $(G, \cdot, e_G)$ be a group and $a, b \in G$. Prove that $(ab)^{-1} = b^{-1}a^{-1}$.

14. Let $(G, \cdot, e_G)$ be a group. Define a relation on $G$ as follows $g_1 \sim g_2$ if and only if there exists $h \in G$ such that $hg_1h^{-1} = g_2$. Prove that $\sim$ is an equivalence relation. [For symmetry use the previous problem.]
15. Consider $G = \mathbb{Z}_7 \times \mathbb{Z}_2$ and define a new multiplication on $G$ as follows:

$$(a, b) \cdot (c, d) = \begin{cases} 
(a + c, b + d), & \text{if } b = 0 \\
(a - c, b + d) & \text{if } b = 1.
\end{cases}$$

Prove that $G$ is a group under this multiplication. Identify the inverse $(a, 0)$ and $(a, 1)$. How many elements of order 2? of order 7?