MAS 4301 – Subgroups

Here is the sketch of how to prove something is a subgroup. We start with a group $(G, \cdot, e)$. You will be typically given a set in the following notation

$$H = \{g \in G : P\}$$

where is a property in the language of groups. Then you are asked to show that $H \leq G$.

Step 1. Show that $H \neq \emptyset$. The simplest way is to point out why the identity belongs to G. By this, you should that e satisfies the property P.

Step 2. Show that H is closed under products. To do this you take arbitrary elements in H and call them $a, b \in H$ (or if you prefer you could use h_1, h_2). You should point out that a, b satisfy property P. Then manipulate the product ab to show that this product satisfies P, and so $ab \in H$.

Step 3. Show that H is closed under inverse. To do this we take an arbitrary element $h \in H$, which means that h satisfies P, and then show h^{-1} satsifies P and so $h^{-1} \in H$.

(*Alternatively, sometime you can combine Step 2. and Step 3 together in one check and mention the One Step Test for subgroups.*

Your final statement should be something close "Therefore, $H \leq G$."

Examples of Subgroups

(1) The trivial ones: $\{e\}$ and G.
(2) If $H \leq G$, then the centralizer $C(H) = \{g \in G : gh = hg \text{ for all } h \in H\}$ is a subgroup of G. In particular, the center of G $(Z(G) = \{g \in G : xg = gx \text{ for all } x \in G\}$ is a subgroup of G.
(3) For any $g \in G$, the set $< g > \leq G$.
(4) If G is abelian, then the collection of elements of finite order is a subgroup of G.
(5) If $H \leq G$, then for any $g \in G$ the set $gHg^{-1} = \{x \in G : x = ghg^{-1} \text{ for some } h \in H\}$ is a subgroup of G.
(6) If $H \leq G$, then the normalizer of H $N(H) = \{g \in G : gHg^{-1} = H\}$ is a subgroup of G.
(7) If $K \leq H$ and $H \leq G$, then $K \leq G$.
(8) If $\{H_i\}$ is a collection of subgroups, then the intersection

$$\bigcap H_i = \{g \in G : \text{ for each } i, g \in H_i\}$$

is a subgroup of G.
(9) For any $s \in \{1, 2, \ldots, n\}$, the set

$$G_s = \{\sigma \in S_n : \sigma(s) = s\}.$$

is a subgroup of G.